On a Gauss-Kuzmin-type problem for a new continued fraction expansion with explicit invariant measure

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Abstract

We study a new continued fraction expansion of reals in the unit interval. Using the ergodic behaviour of a homogeneous random system with complete connections associated with this expansion we obtain a Gauss-Kuzmin-type theorem.

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1. Introduction

The Gauss-Kuzmin theorem is one of the most important results in the metrical theory of regular continued fractions (see [5], [6]).

From the time of Gauss, a great number of Gauss-Kuzmin-type theorems followed. The Gauss-Kuzmin problem has been generalized in various directions for other continued fraction expansions. We remark that the Gauss transformation has strong ties with chaos theory [1], [2].

In this paper we consider another expansion of reals in the unit interval I different from the regular continued fraction-expansion. Using the approach of dependence with complete connections, our aim is to prove a Gauss-Kuzmin-type theorem for this new expansion. In Section 2 we describe this expansion and the associated transformation. In Section 3 we introduce a homogeneous random system with complete connections associated with this expansion. In Section 4 the ergodic behaviour of this random system allows us to obtain a convergence rate result.

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2. Another expansion and some examples

Write $x \in [0,1)$ as

$$\frac{2^{-a_1}}{1+\frac{2^{-a_2}}{1+\ddots}} = [a_1, a_2, \ldots], \tag{1}$$

where a_n are natural numbers. First, it is clear that every irrational $x \in [0,1)$ has a unique expansion of the type of (1). Second, we note that some particular cases of this type of continued fractions have been studied before. For example, by setting $q = 1/2$ and $a_n = n$, the right-hand side of (1) gives the well-known continued fraction of Rogers and Ramanujan

$$\frac{q}{1+\frac{q^2}{1+\frac{q^3}{1+\ddots}}}.$$

Another example is the result due to Davison [3]. Let $a_n = F_n$, where F_n is the n-th Fibonacci number. Davison showed that

$$\frac{2^{-F_1}}{1+\frac{2^{-F_2}}{1+\frac{2^{-F_3}}{1+\ddots}}} = \frac{1}{2}\sum_{n\geq 1} 2^{-[n\phi]},$$

where ϕ is the Golden Ratio and $[\cdot]$ denotes the entire function.

Define the transformation $T : [0,1) \to [0,1)$ as follows

$$T(x) = \begin{cases} 0, & x = 0 \\ [a_2, a_3, \ldots], & x = [a_1, a_2, \ldots]. \end{cases} \tag{2}$$

It follows from (1) and (2) that for $x \neq 0$ we have

$$x = \frac{2^{-a_1}}{1+T(x)}. \tag{3}$$

Consequently, using (3) we can write the transformation T of [0,1) as

$$T(x) = 2^{\{\log(1/x)/\log 2\}} - 1, \quad x \neq 0,$$

where $\{u\}$ denotes the fractionary part of a real u.

One should think of a_n as the incomplete quotients or digits of x. For $x \neq 0$, we get

$$a_1(x) = [\log(1/x)/\log 2],$$
$$a_n(x) = a_1(T^{(n-1)}(x)),\ n \in \mathbf{N}^*,\ n \geq 2.$$

3. Preliminary results

Let μ be a non-atomic probability measure on $\mathcal{B}$ (= the σ-algebra of Borel subsets of $I = [0,1]$) and define

$$F_n(x) = \mu(T^n < x),\ x \in I,\ n \in \mathbf{N},$$

with $F_0(x) = \mu([0,x))$.

Proposition 1. *For each $n \in \mathbf{N}$, F_n satisfies the following Gauss-type relation*

$$F_{n+1}(x) = \sum_{k\geq 0}\left(F_n(\alpha^k) - F_n\left(\frac{\alpha^k}{x+1}\right)\right),\ x \in I, \tag{4}$$

where $\alpha = 1/2$.

Proof. Since $T^n = \dfrac{2^{-a_{n+1}}}{T^{n+1}+1}$ it follows that

$$\begin{aligned} F_{n+1}(x) = \mu(T^{n+1} < x) &= \sum_{k\geq 0}\mu\left(\frac{\alpha^k}{x+1} < T^n < \alpha^k\right) = \\ &= \sum_{k\geq 0}\left(F_n(\alpha^k) - F_n\left(\frac{\alpha^k}{x+1}\right)\right). \end{aligned}$$

□

We now assume that F_0' exists and is bounded (μ has bounded density). By induction we have that F_n', $n \in \mathbf{N}^*$, exist and are bounded as well. This allows us to differentiate (4) term by term, obtaining

$$F_{n+1}'(x) = \sum_{k\geq 0}\frac{\alpha^k}{(x+1)^2}F_n'\left(\frac{\alpha^k}{x+1}\right). \tag{5}$$

Let us introduce the function f_n in (5) where

$$f_n(x) = \frac{F_n'(x)}{\rho(x)}, \quad n \in \mathbf{N},$$

with $\rho(x) = \dfrac{1}{(x+1)(x+2)}$, $x \in I$. Then (5) becomes

$$f_{n+1}(x) = \sum_{k\geq 0}p_k(x)f_n\left(\frac{\alpha^k}{x+1}\right), \tag{6}$$

where

$$p_k(x) = \frac{\alpha^{k+1}(x+1)(x+2)}{(\alpha^k+x+1)(\alpha^{k+1}+x+1)}.$$

Proposition 2. *The function $P(x,k) = p_k(x)$ defines a transition probability function from $(I,\mathcal{B})$ to $(\mathbf{N},\mathcal{P}(\mathbf{N}))$.*

Proof. We have to verify that $\sum_{k\geq 0}P(x,k) = 1$ for all $x \in I$. Since

$$P(x,k) = \alpha^{k+1} + \frac{\Delta_k}{\alpha^k + x + 1} - \frac{\Delta_{k+1}}{\alpha^{k+1} + x + 1}$$

with $\Delta_k = \alpha^k - \alpha^{2k}$, it follows that

$$\sum_{k \geq 0} P(x,k) = \lim_{k \to \infty} \left(\frac{\alpha(1-\alpha^{k+1})}{1-\alpha} - \frac{\Delta_{k+1}}{\alpha^{k+1} + x + 1} \right) = 1.$$

□

Proposition 2 allows us to consider the random system with complete connections (RSCC) (see [4] Section 1.1)

$$((I, \mathcal{B}), (\mathbf{N}, \mathcal{P}(\mathbf{N})), u, P) \tag{7}$$

where $I, \mathcal{B}, \mathcal{P}(\mathbf{N})$ and P are defined previously, while $u : I \times \mathbf{N} \to I$ is given by

$$u(x,k) = u_k(x) = \frac{\alpha^k}{x+1}.$$

Further, we denote by U the associated Markov operator of the RSCC (7) with the transition probability function

$$Q(x,B) = \sum_{\{k \in \mathbf{N} | u_k(x) \in B\}} p_k(x), \ x \in I, \ B \in \mathcal{B}.$$

Then $Q^n(\cdot,\cdot)$ will denote the n-step transition probability function of the same Markov chain (see [4]).

The ergodic behaviour of the RSCC (7) allows us to find the limiting distribution function $F = F_\infty$ and the invariant measure Q^∞ induced by F. To study the ergodicity of the RSCC (7), let us consider the norm $|| \cdot ||_L$ defined on $L(I)$ (= the space of Lipschitz real-valued functions defined on I) by

$$||f||_L = \sup_{x \in I} |f(x)| + \sup_{x' \neq x''} \frac{|f(x') - f(x'')|}{|x' - x''|}, \ f \in L(I).$$

Proposition 3. *The RSCC (7) is uniformly ergodic.*

Proof. We have

$$\frac{d}{dx} u(x,k) = -\frac{\alpha^k}{(x+1)^2},$$

$$\frac{d}{dx} P(x,k) = -\frac{\Delta_k}{(\alpha^k + x + 1)^2} + \frac{\Delta_{k+1}}{(\alpha^{k+1} + x + 1)^2},$$

for all $x \in I$ and $k \in \mathbf{N}$, so that $\sup_{x \in I} \left| \frac{d}{dx} u(x,k) \right| = \alpha^k$ and $\sup_{x \in I} \left| \frac{d}{dx} P(x,k) \right| < \infty$. Hence the requirements of definition of an RSCC with contraction are fulfiled (see Definition 3.1.15 in [4]). By Theorem 3.1.6 in [4]), it follows that the Markov chain associated with this RSCC with contraction is a Doeblin-Fortet chain and its transition operator is a Doeblin-Fortet operator. To prove the regularity of U w.r.t. $L(I)$ let us define recursively $x_{n+1} = (x_n + 2)^{-1}$, $n \in \mathbf{N}$, with $x_0 = x$. A criterion of regularity is expressed in Theorem 3.2.13 in [4], in terms of the supports $\sum_n(x)$ of the n-step transition probability functions $Q^n(x,\cdot)$, $n \in \mathbf{N}^*$. Clearly $x_{n+1} \in \sum_1(x_n)$ and therefore Lemma 3.2.14 in [4] and an induction argument lead to the conclusion that $x_n \in \sum_n(x)$, $n \in \mathbf{N}^*$. But $\lim_{n\to\infty} x_n = \sqrt{2}-1$ for any $x \in I$. Hence

$$d\left(\sum\nolimits_n(x), \sqrt{2}+1\right) \le |x_n - \sqrt{2} - 1| \to 0 \quad \text{as} \quad n \to \infty,$$

where $d(x,y) = |x-y|$, $\forall x, y \in I$. Finally, the regularity of U w.r.t. $L(I)$ follows from Theorem 3.2.13 in [4]. Moreover, $Q^n(\cdot,\cdot)$ converges uniformly to a probability measure Q^∞ and that there exist two positive constants $q < 1$ and K such that

$$(8) \qquad ||U^n f - U^\infty f||_L \le K q^n ||f||_L, \quad \forall n \in \mathbf{N}^*, \quad \forall f \in L(I),$$

where

$$(9) \qquad U^n f(\cdot) = \int_I f(y) Q^n(\cdot, dy), \quad U^\infty f(\cdot) = \int_I f(y) Q^\infty(dy).$$

□

Proposition 4. *The invariant probability measure Q^∞ of the transformation T has the density $\rho(x) = \dfrac{1}{(x+1)(x+2)}$, $x \in I$, with the normalizing factor $\dfrac{1}{\log(4/3)}$.*

Proof. On account of the uniqueness of Q^∞ we have to show that $\int_I Q(x,B)Q^\infty(dx) = Q^\infty(B)$ for any $B \in \mathcal{B}$. Since the intervals $[0,u) \subset I$ generate $\mathcal{B}$ it is sufficient to check the above equation just for $B = [0,u)$, $0 < u \le 1$. We have

$$Q(x,[0,u)) = \sum_{\{k | 0 \le u_k(x) < u\}} P(x,k).$$

Then

$$\begin{aligned}
&\frac{1}{\log(4/3)} \int_I Q(x,[0,u))\rho(x)dx = \\
&= \frac{1}{\log(4/3)} \int_I \sum_{k \ge \left[\frac{\log u(x+1)}{\log \alpha}\right]+1} \left(\frac{1}{\alpha^{k+1}+x+1} - \frac{1}{\alpha^k + x + 1} \right) dx = \\
&= \frac{1}{\log(4/3)} \log \frac{2(u+1)}{u+2} = Q^\infty([0,u)).
\end{aligned}$$

Similarly we can treat the case $\alpha < u \le 1$ and obtain the desired result.

□

4. Main result

Now we are able to find the limiting distribution function

$$F(x) = F_\infty(x) = \lim_{n\to\infty} \mu(T^n < x)$$

and obtain a convergence rate result.

Theorem 1 (A Gauss-Kuzmin-Type Theorem). *If μ has a Riemann integrable density, then*

$$F(x) = \frac{1}{\log(4/3)} \log \frac{2(x+1)}{x+2}, \quad x \in I.$$

If the density of μ is a Lipschitz function, then there exist two positive constants $q < 1$ and K such that for all $x \in I$, $n \in \mathbf{N}^*$

$$\mu(T^n < x) = \frac{1}{\log(4/3)}(1 + \theta q^n) \log \frac{2(x+1)}{x+2},$$

where $\theta = \theta(\mu, n, x)$, with $|\theta| \leq K$.

Proof. By virtue of (9) we have

$$U^\infty f_0 = \int_I f_0(x) Q^\infty(dx) = \frac{1}{\log(4/3)}, \quad f_0 \in L(I).$$

Taking into account (8) there exist two constants $q < 1$ and K such that

$$||U^n f_0 - U^\infty f_0|| \leq K q^n ||f_0||_L, \quad \forall n \in \mathbf{N}^*.$$

As $L(I)$ is a dense subset of $C(I)$ (= the metric space of real continuous functions defined on I with the supremum norm $||f|| = \sup_{x \in I} |f(x)|$) we have

$$\lim_{n \to \infty} ||(U^n - U^\infty) f_0|| = 0 \tag{10}$$

for all $f_0 \in C(I)$. Therefore (10) remains valid for measurable f_0 that are Q^∞-almost surely continuous, that is, for Riemann-integrable f_0. We have

$$\begin{aligned} F(x) &= \lim_{n \to \infty} \mu(T^n < x) = \lim_{n \to \infty} \int_0^x U^n f_0(u) \rho(u) du = \\ &= \frac{1}{\log(4/3)} \int_0^x \rho(u) du = \frac{1}{\log(4/3)} \log \frac{2(x+1)}{x+2}. \end{aligned}$$

□

Remark. The study of optimality of the convergence rate remains an open question. A Wirsing-type approach [8] to the Perron-Frobenius operator of the associated transformation under its invariant measure, allows us to obtain a near-optimal solution to this problem and will appear in [7].

References

[1] R.M. Corless, *Continued fractions and chaos*, Amer. Math. Monthly 99 (1992), no.3, 203-215.

[2] P. Cvitanović, *Circle maps: irrationally winding*, From Number Theory to Physics (Les Houches, 1989) (M. Waldschmidt, P. Moussa, J.M. Luck, and C. Itzykson, eds.), Springer-Verlag, Berlin, 1992, 631-658.

[3] J.L. Davison, *A series and its associated continued fraction*, Proc. Amer. Math. Soc. 63 (1977), no.1, 29-32.

[4] M. Iosifescu and Ş. Grigorescu, *Dependence with Complete Connections and Its Applications*, Cambridge University Press, Cambridge, 1990.

[5] M. Iosifescu and C. Kraaikamp, *Metrical Theory of Continued Fractions*, Kluwer Academic Publishers, Dordrecht, 2002.

[6] F. Schweiger, *Ergodic Theory of Fibred Systems and Metric Number Theory*, Clarendon Press, Oxford Univ. Press, New York, 1995.

[7] G.I. Sebe, *A Wirsing-type approach to some continued fraction expansion*, Int. J. Math. Math. Sci. (to appear).

[8] E. Wirsing, *On the theorem of Gauss-Kuzmin-Lévy and a Frobenius-type theorem for function spaces*, Acta Arith. **24** (1974), 507-528.

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