Interactions of nonholonomic economic systems

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Abstract

Section 1 describes a nonholonomic economic system as a Gibbs-Pfaff distribution on R^5. Section 2 interprets the economical equilibrium after interaction states as the set of all constrained critical points of an objective function, and gives an example of totally degenerate interaction.

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1 Nonholonomic economic system

A nonholonomic economic system means the nonholonomic hypersurface (the Gibbs-Pfaff distribution) $\{R^5, \Omega = 0\}$, where $R^5 = \{(G, I, E, P, Q)|G =$ potential growth, $I =$ internal politic stability, $E =$ entropy, $P =$ price, $Q =$ production quantity $\}$, and $\Omega = dG - IdE + PdQ = 0$ is the Gibbs-Pfaff equation describing the "mobility" of economic variables.

Since the Gibbs-Pfaff form Ω is a contact form, the Gibbs-Pfaff equation is not completely integrable and its maximal integral manifolds are 2-dimensional.

Let us fix E and Q as states variables of the system. Then the maximal integral surfaces of the Gibbs-Pfaff equation are of the form

$$G = f(E, Q), \quad I = \frac{\partial f}{\partial E},\ P = -\frac{\partial f}{\partial Q},$$

where f is an arbitrary C^2 function.

The Gibbs-Pfaff economic distribution can be expressed contravariantly by

$$GPD = sp\,\{X_1 = \partial_I,\ X_2 = I\partial_G + \partial_E,\ X_3 = \partial_P,\ X_4 = \partial_Q - P\partial_G\}.$$

The vector fields X_i, $i = 1, 2, 3, 4$ satisfy

$$[X_1, X_2] = \partial_G,\ [X_i, X_j] = 0,\ i \neq 1,\ j \neq 2,\ i \neq j.$$

Consequently they determine a Lie algebra whose constants of structure are $C^1_{12} = 1$ and $C^i_{jk} = 0$ otherwise (with regards to indices). Since the vector fields $X_1, X_2, X_3, X_4,$

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$[X_1, X_2] = \partial_G$ are linearly independent at each point of R^5, any two points of R^5 are joined by a concatenation of field lines of X_1, X_2, X_3, X_4 [3]. On the other hand, we have:

1) Orbits of $X_1 : G = c_1,\ E = c_2,\ P = c_3,\ Q = c_4$ (straight lines);
2) Orbits of $X_2 : G = c_1 E + c_2,\ I = c_1,\ P = c_3,\ Q = c_4$ (straight lines);
3) Orbits of $X_3 : G = c_1,\ I = c_2,\ E = c_3,\ Q = c_4$ (straight lines);
4) Orbits of $X_4 := G = -c_4 Q + c_1,\ I = c_2,\ E = c_3,\ P = c_4$ (straight lines).

A complementary distribution to GPD (1-dimensional and even orthogonal to GPD with respect to the Euclidean metric in R^5) is

$$GPD' = sp\,\{N = \partial_G - I\partial_E + P\partial_Q\}.$$

The field lines of the vector field N are

$$I = c_1,\ E = c_1 G + d_1,\ P = c_2,\ Q = c_2 G + d_2,$$

where c_1, c_2, d_1, d_2 are arbitrary constants (family of straight lines).

Let $c(t) = (G(t), I(t), E(t), P(t), Q(t)),\ t \in I$, be an integral curve of the Gibbs-Pfaff equation $\Omega = 0$, with the starting point $(G(0), I(0), E(0), P(0), Q(0))$. Then

$$G(t) - G(0) = \int_c IdE - PdQ.$$

2 An example of totally degenerate interaction of two nonholonomic systems

Our point of view is that two nonholonomic economic systems

$$\{R^5, \Omega_1 = dG_1 - I_1 dE_1 + P_1 dQ_1 = 0\}$$

$$\{R^5, \Omega_2 = dG_2 - I_2 dE_2 + P_2 dQ_2 = 0\}$$

interact on the product nonholonomic economic system $\{R^5 \times R^5, \Omega_1 = 0, \Omega_2 = 0\}$, if they have a common objective function whose constrained critical points are of interest.

In other words, the "economical equilibrium after interaction states" is described by the set of all critical points of an objective function f constrained by the constant level sets of other functions, say g, h, and by the Gibbs-Pfaff equations $\Omega_1 = 0, \Omega_2 = 0$.

These constrained critical points are zeros of the Lagrange 1-form

$$L = df + \lambda_1 dg + \lambda_2 dh + \lambda_3 \Omega_1 + \lambda_4 \Omega_2$$

which belong to the set $g = c_1$, $h = c_2$. Such a critical point is called *degenerate* if the restriction of the quadratic form

$$dL = d^2 f + \lambda_1 d^2 g + \lambda_2 d^2 h + \lambda_3 d\Omega_1 + \lambda_4 d\Omega_2$$

to the subspace $dg = 0$, $dh = 0$, $\Omega_1 = 0$, $\Omega_2 = 0$ is degenerate (the operator d means usual differentiation). The interaction is called *totally degenerate* if all critical points are degenerate.

Now let us accept that we are interested in an objective function $f(G_1, G_2)$ and two holonomic constraints $g(E_1, E_2) = c_1$, $h(Q_1, Q_2) = c_2$.

Theorem. *The critical points of the function* $f(G_1, G_2)$ *constrained by* $g(E_1, E_2) = c_1$, $h(Q_1, Q_2) = c_2$, $\Omega_1 = 0$, $\Omega_2 = 0$ *are degenerate.*

Proof. We can use MAPLE.

> with(linalg):

$$> A := hessian(f(G1, G2) + lambda1 * g(E1, E2) + lambda2 * h(Q1, Q2),$$
$$[G1, I1, E1, P1, Q1, G2, I2, E2, P2, Q2]);$$

$$> L1 := matrix([[0,0,0,0,0,0,0,0,0,0], [0,0,-1/2,0,0,0,0,0,0,0],$$
$$[0,-1/2,0,0,0,0,0,0,0,0], [0,0,0,0,1/2,0,0,0,0,0], [0,0,0,1/2,0,0,0,0,0,0],$$
$$[0,0,0,0,0,0,0,0,0,0], [0,0,0,0,0,0,0,0,0,0], [0,0,0,0,0,0,0,0,0,0],$$
$$[0,0,0,0,0,0,0,0,0,0], [0,0,0,0,0,0,0,0,0,0]]);$$

$$> L2 := matrix([[0,0,0,0,0,0,0,0,0,0], [0,0,0,0,0,0,0,0,0,0],$$
$$[0,0,0,0,0,0,0,0,0,0], [0,0,0,0,0,0,0,0,0,0], [0,0,0,0,0,0,0,0,0,0],$$
$$[0,0,0,0,0,0,0,0,0,0], [0,0,0,0,0,0,0,-1/2,0,0], [0,0,0,0,0,0,-1/2,0,0,0],$$
$$[0,0,0,0,0,0,0,0,0,1/2], [0,0,0,0,0,0,0,0,1/2,0]]);$$

$> B := matadd(A, lambda3 * L1);$
$> C := matadd(B, lambda4 * L2);$

$$C := \begin{bmatrix} \frac{\partial^2}{\partial G1^2} f(G1, G2), 0, 0, 0, 0, \frac{\partial^2}{\partial G2\, \partial G1} f(G1, G2), 0, 0, 0, 0 \\ 0, 0, -\frac{\lambda 3}{2}, 0, 0, 0, 0, 0, 0, 0 \\ 0, -\frac{\lambda 3}{2}, \lambda 1 \left(\frac{\partial^2}{\partial E1^2} g(E1, E2)\right), 0, 0, 0, 0, \lambda 1 \left(\frac{\partial^2}{\partial E2\, \partial E1} g(E1, E2)\right), 0, 0 \\ 0, 0, 0, 0, \frac{\lambda 3}{2}, 0, 0, 0, 0, 0 \\ 0, 0, 0, \frac{\lambda 3}{2}, \lambda 2 \left(\frac{\partial^2}{\partial Q1^2} h(Q1, Q2)\right), 0, 0, 0, 0, \lambda 2 \left(\frac{\partial^2}{\partial Q2\, \partial Q1} h(Q1, Q2)\right) \\ \frac{\partial^2}{\partial G2\, \partial G1} f(G1, G2), 0, 0, 0, 0, \frac{\partial^2}{\partial G2^2} f(G1, G2), 0, 0, 0, 0 \\ 0, 0, 0, 0, 0, 0, 0, -\frac{\lambda 4}{2}, 0, 0 \\ 0, 0, \lambda 1 \left(\frac{\partial^2}{\partial E2\, \partial E1} g(E1, E2)\right), 0, 0, 0, -\frac{\lambda 4}{2}, \lambda 1 \left(\frac{\partial^2}{\partial E2^2} g(E1, E2)\right), 0, 0 \\ 0, 0, 0, 0, 0, 0, 0, 0, 0, \frac{\lambda 4}{2} \\ 0, 0, 0, 0, \lambda 2 \left(\frac{\partial^2}{\partial Q2\, \partial Q1} h(Q1, Q2)\right), 0, 0, 0, \frac{\lambda 4}{2}, \lambda 2 \left(\frac{\partial^2}{\partial Q2^2} h(Q1, Q2)\right) \end{bmatrix}$$

$> definite(C,' positive_d ef');$

$$false$$

$> det(C);$

$$\frac{\lambda 3^4 \lambda 4^4}{256}\left(\left(\frac{\partial^2}{\partial G1^2}\,\mathrm{f}(G1,\ G2)\right)\left(\frac{\partial^2}{\partial G2^2}\,\mathrm{f}(G1,\ G2)\right) - \left(\frac{\partial^2}{\partial G2\,\partial G1}\,\mathrm{f}(G1,\ G2)\right)^2\right)$$

$$> U := matrix([[0,1,0,0,0,0,0,0,0,0],[0,0,0,1,0,0,0,0,0,0],$$
$$[0,0,0,0,0,0,1,0,0,0],[-I1*diff(g(E1,E2),E2)/diff(g(E1,E2),E1),$$
$$0,-diff(g(E1,E2),E2)/diff(g(E1,E2),E1),0,0,I2,0,1,0,0],[0,0,0,0,0,$$
$$0,0,0,1,0],[P1*diff(h(Q1,Q2),Q2)/diff(h(Q1,Q2),Q1),0,$$
$$0,0,-diff(h(Q1,Q2),Q2)/diff(h(Q1,Q2),Q1),-P2,0,0,0,1]]);$$

$> V := transpose(U);$
$> C1 := multiply(U,C,V);$
$> definite(C1,' positive_d ef');$

$$false$$

$> det(C1);$

$$0$$

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