Extrema of some curvature quadratic forms

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Abstract

Having in mind that any geometric inequality reflects a free or constrained optimum problem, we apply the techniques of nonlinear programming to justify some well-known Chen inequalities or to obtain new inequalities (see also [15]). In this sense, we study the extrema of curvature quadratic forms of an n-dimensional slant submanifold in a $(2m+1)$-dimensional Sasakian spatial form, using proper values and proper vectors of curvature operator.

Section 1 reproduces a key Lemma regarding extrema of a quadratic form restricted to the unit sphere. Section 2 studies the constrained critical points of curvature quadratic forms of an n-dimensional slant submanifold in a $(2m+1)$-dimensional Sasakian spatial form, underlying their connection to a suitable frame. Section 3 gives bounds for curvature quadratic forms.

Mathematics Subject Classification: 53C25, 53C21.
Key words: Sasakian space form, slant submanifold, curvature bounds.

1 Smallest and the bigest proper values of a symmetric matrix

We recall a well-known theorem, necessary to our study:

Lemma 1. *The critical values of the restriction of quadratic form* $f(x) = \sum\limits_{i,j=1}^{m} a_{ij}x_ix_j$ *to the sphere* $x_1^2 + ... + x_m^2 = 1$ *are proper values of the matrix* $A = (a_{ij})$.

Proof. A point x of the unit sphere is a critical point for the restriction of the function f if and only if there exists $\lambda \in \mathbf{R}$ such that

$$\nabla({}^txAx) = \lambda\nabla({}^txx),$$

i.e., $Ax = \lambda x$, which shows that critical points are in fact orthonormal proper vectors of the matrix A. If we know the proper versor x, then multiplying with ${}^tx$ we find the proper value ${}^txAx = \lambda{}^txx = \lambda$.

Consequently, the numbers $\min\{{}^txAx \mid {}^txx = 1\}$, $\max\{{}^txAx \mid {}^txx = 1\}$ are respectively the smallest and the bigest of the proper values of the symmetric matrix A.

Proceedings of The 3-rd International Colloquium "Mathematics in Engineering and Numerical Physics" October 7-9 , 2004, Bucharest, Romania, pp. 304-312.

2 Constrained critical points of curvature quadratic forms of an n-dimensional slant submanifold in a $(2m+1)$-dimensional Sasakian spatial form

Let $\widetilde{M}(c)$ be a Sasakian space form of dimension $2m+1$ and M be a slant submanifold of dimension n. If X, Y, Z, W are vector fields tangent to M, then the curvature tensor field of the Sasakian space form $\widetilde{M}(c)$ has the restriction

$$\begin{aligned}\widetilde{R}(X,Y,Z,W) \ = \ & \frac{c+3}{4}\{-g(Y,Z)g(X,W)+g(X,Z)g(Y,W)\}\\ & +\frac{c-1}{4}\{-\eta(X)\eta(Z)g(Y,W)+\eta(Y)\eta(Z)g(X,W)\\ & -g(X,Z)\eta(Y)g(\xi,W)+g(Y,Z)\eta(X)g(\xi,W)\\ & -g(\phi Y,Z)g(\phi X,W)+g(\phi X,Z)g(\phi Y,W)\\ & +2g(\phi X,Y)g(\phi Z,W)\}, \quad \forall X,Y,Z,W\in\Gamma(TM).\end{aligned}$$

We use the Gauss equation

$$R(X,Y,Z,W)=\widetilde{R}(X,Y,Z,W)-g(h(X,W),h(Y,Z))+g(h(X,Z),h(Y,W)).$$

The second vectorial fundamental forms

$$h(X,W)=\sum_{T=n+1}^{2m+1}h^T(X,W)\xi_T \quad \text{şi} \quad h(Y,Z)=\sum_{S=n+1}^{2m+1}h^S(Y,Z)\xi_S$$

lead to

$$g(h(X,W),h(Y,Z))=g\left(\sum_{T=n+1}^{2m+1}\sum_{S=n+1}^{2m+1}h^T(X,W)h^S(Y,Z)\xi_T\xi_S\right).$$

Since $g(\xi_T,\xi_S)=\delta_{TS}$, it follows

$$g(h(X,W),h(Y,Z))=\sum_{T=n+1}^{2m+1}h^T(X,W)h^T(Y,Z).$$

In these conditions,

$$\begin{aligned}(2.1)\qquad R(X,Y,Z,W) \ = \ & \widetilde{R}(X,Y,Z,W)\\ & +\sum_{T=n+1}^{2m+1}[h^T(X,Z)h^T(Y,W)-h^T(X,W)h^T(Y,Z)].\end{aligned}$$

Let $\{e_1,...,e_n\}$ be an orthonormal frame in T_pM and $\{e_{n+1},...,e_{2m},e_{2m+1}\}$ an orthonormal frame in $T_p^{\perp}M$. We built the 2-planes

$$\Pi_{ij}^{pq}=\frac{1}{2}(e_i^pe_j^q-e_i^qe_j^p).$$

The set $\{\Pi_{ij}^{pq}\}$ is orthonormal with respect to G_{pqrs} since

$$G_{pqrs}\Pi_{ij}^{pq}\Pi_{ij}^{rs} = \frac{1}{2}(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}), \quad p,q,r,s = 1,2,...,n, \quad i,j,k,l = 1,2,...,n.$$

We consider the curvature quadratic forms

$$\rho_{ij} = R_{pqrs}\Pi_{ij}^{pq}\Pi_{ij}^{rs},$$

where $i < j$ (for i,j = fixed, the scalar curvature determined by the orthonormal plane Π_{ij}^{pq}), and we look for their extrema with the constraint

$$G_{pqrs}\Pi_{ij}^{pq}\Pi_{ij}^{rs} = \frac{\delta_{ii}\delta_{,jj}}{2},$$

which is a consequence of the orthonormalization condition. Applying Lemma 1, there exist λ_{ij} such that

$$\nabla\rho_{ij} = \lambda_{ij}\nabla(G_{pqrs}\Pi_{ij}^{pq}\Pi_{ij}^{rs}).$$

The gradient $\nabla(G_{pqrs}\Pi_{ij}^{pq}\Pi_{ij}^{rs})$ has the components (partial derivatives)

$$\begin{aligned}\frac{\partial\rho_{ij}}{\partial\Pi_{kl}^{uv}} &= R_{pqrs}\Pi_{ij}^{rs}(\delta_u^p\delta_v^q - \delta_u^q\delta_v^p)(\delta_i^k\delta_j^l - \delta_i^l\delta_j^k)\\ &= 2R_{uvrs}\Pi_{ij}^{rs}(\delta_i^k\delta_j^l - \delta_i^l\delta_j^k).\end{aligned}$$

Let us show that the frame $\{e_1, ..., e_n = \xi\}$ of T_pM and the frame $\{e_{n+1}, ..., e_{2m}, e_{2m+1}\}$ of $T_p^{\perp}M$, used in [4, Theorem 1, Ch.3], produce the 2-planes which are solutions of the system that gives the critical points of the Lagrange functions

$$L_{ij} = \rho_{ij} - \lambda_{ij}\left(G_{pqrs}\Pi_{ij}^{pq}\Pi_{ij}^{rs} - \frac{1}{2}\delta_{ii}\delta_{jj}\right),$$

i.e., λ_{ij} are proper values of curvature operator.

The contribution of $\widetilde{R}$ to the scalar curvature determined by the orthonormal planes Π_{ij}^{pq}, reduces to the quadratic forms

$$\widetilde{\rho}_{ij} = \widetilde{R}_{pqrs}\Pi_{ij}^{pq}\Pi_{ij}^{rs}.$$

Computing the partial derivatives we find

$$\frac{\partial\widetilde{\rho}_{ij}}{\partial\Pi_{kl}^{uv}} = 2\widetilde{R}_{uvrs}\Pi_{ij}^{rs}(\delta_i^k\delta_j^l - \delta_i^l\delta_j^k).$$

The expression of $\widetilde{R}_{uvrs}\Pi_{ij}^{rs}$ is

$$\begin{aligned}\widetilde{R}_{uvrs}\Pi_{ij}^{rs} &= \left\{\frac{c+3}{4}\left[-g_{vi}g_{uj} + g_{ui}g_{vj}\right] + \frac{c-1}{4}\left[-\eta_u\eta_i g_{vj} + \eta_v\eta_i g_{uj}\right.\right.\\ &\qquad \left.\left.-g_{ui}\eta_v\eta_j + g_{vi}\eta_u\eta_j - \phi_{iv}\phi_{ju} + \phi_{iu}\phi_{jv} + 2\phi_{vu}\phi_{ji}\right]\right\}\\ &= \left\{\frac{c+3}{4}\left[-g_{vi}g_{uj} + g_{ui}g_{vj}\right] + \frac{c-1}{4}\left[-\eta_u\eta_i g_{vj} + \eta_v\eta_j g_{uj}\right.\right.\\ &\qquad \left.\left.-g_{ui}\eta_v\eta_i + g_{vi}\eta_u\eta_j - \phi_{iv}\phi_{ju} + \phi_{iu}\phi_{jv} + 2\phi_{vu}\phi_{ji}\right]\right\}.\end{aligned}$$

On the selected frame, $\widetilde{R}_{uvrs}\Pi^{rs}_{ij}$ is

$\frac{c+3}{4} + \frac{c-1}{4}\cos^2\theta$, for $i=1,\ j=2,\ u=1,\ v=2$;
$-\frac{c+3}{4} + \frac{c-1}{4}3\cos^2\theta$, for $i=1,\ j=2,\ u=2,\ v=1$
$\frac{c-1}{2}\cos^2\theta$, for $i=1,\ j=2,\ u=3,\ v=4$;
$\frac{c-1}{2}\cos^2\theta$, for $i=1,\ j=2,\ u=4,\ v=3$;
$\dots$
$\frac{c-1}{2}\cos^2\theta$, for $i=1,\ j=2,\ u=n,\ v=n-1$;
$\frac{c-1}{2}\cos^2\theta$, for $i=1,\ j=2,\ u=n-1,\ v=n$;
0, for $i=1,\ j=x,u,v$;
$\frac{c+3}{4} + \frac{c-1}{4}$, for $i=1,\ j=n,\ u=1,\ v=n$;
$-\frac{c+3}{4} + \frac{c-1}{4}$, for $i=1,\ j=n,\ u=n,\ v=1$;
$\frac{c-1}{4}$, for $i=1,\ j=n,\ u=n,\ v=n$;
$\dots$
$\frac{c+3}{4} + \frac{c-1}{4}\cos^2\theta$, for $i=3,\ j=4,\ u=3,\ v=4$;
$-\frac{c+3}{4} + \frac{c-1}{4}3\cos^2\theta$, for $i=3,\ j=4,\ u=4,\ v=3$;
$\frac{c+3}{4}$, for $i=3,\ j=x,\ u=3,\ v=x$;
$-\frac{c+3}{4}$, for $i=3,\ j=x,\ u=x,\ v=3$;
$\frac{c+3}{4} + \frac{c-1}{4}$, for $i=3,\ j=n,\ u=3,\ v=n$;
$-\frac{c+3}{4} + \frac{c-1}{4}$, for $i=3,\ j=n,\ u=n,\ v=3$;
$\frac{c-1}{4}$, for $i=3,\ j=n,\ u=n,\ v=n$;
$\dots$
$\frac{c+3}{4} + \frac{c-1}{4}\cos^2\theta$, for $i=n-1,\ j=n,\ u=n-1,\ v=n$;
$-\frac{c+3}{4} + \frac{c-1}{4}3\cos^2\theta$, for $i=n-1,\ j=n,\ u=n,\ v=n-1$,
with $x=\overline{3,n-1}$.

It rests to analyze the contribution of quadratic forms

$$D_{ij} = \sum_{T=n+1}^{2m+1} (h^T_{ps}h^T_{qr} - h^T_{pq}h^T_{rs})\Pi^{pq}_{ij}\Pi^{rs}_{ij}.$$

For that we have

$$C^{kl}_{uvij} = \frac{\partial D_{ij}}{\partial \Pi^{uv}_{kl}} = 2\sum_{T=n+1}^{2m+1} (h^T_{us}h^T_{vr} - h^T_{uv}h^T_{rs})\Pi^{rs}_{ij}(\delta^k_i\delta^l_j - \delta^l_i\delta^k_j).$$

On the other hand

$$2\sum_{T=n+1}^{2m+1}(h^T_{us}h^T_{vr}-h^T_{uv}h^T_{rs})\Pi^{rs}_{ij}=\sum_{T=n+1}^{2m+1}(h^T_{us}h^T_{vr}-h^T_{uv}h^T_{rs})e^r_ie^s_j.$$

Consequently

$$C^{kl}_{uvij}=\sum_{T=n+1}^{2m+1}(h^T_{us}h^T_{vr}-h^T_{uv}h^T_{rs})(\delta^k_i\delta^l_j-\delta^l_i\delta^k_j)e^r_ie^s_j.$$

For our problem, we consider the coefficient of C^{kl}_{uvij}, i.e.,

$$C_{uvij}=\sum_{T=n+1}^{2m+1}(h^T_{us}h^T_{vr}-h^T_{uv}h^T_{rs})e^r_ie^s_j=\sum_{T=n+1}^{2m+1}(h^T_{uj}h^T_{vi}-h^T_{uv}h^T_{ij})=$$

$$=(h^{n+1}_{uj}h^{n+1}_{vi}-h^{n+1}_{uv}h^{n+1}_{ij})+\sum_{T=n+2}^{2m+1}(h^T_{uj}h^T_{vi}-h^T_{uv}h^T_{ij}).$$

Using the frame in the proof of [4, Theorem 1, Ch.3], for $x=\overline{3,n-1}$, we find $(h^{n+1}_{uj}h^{n+1}_{vi}-h^{n+1}_{uv}h^{n+1}_{ij})$ equal with

$$\begin{array}{cll}
0, & \text{for} & i=1,\ j=2,\ u=1,\ v=2;\\
-h^{n+1}_{11}h^{n+1}_{22}, & \text{for} & i=1,\ j=2,\ u=2,\ v=1;\\
0, & \text{for} & i=1,\ j=2,\ u=3,\ v=4;\\
0, & \text{for} & i=1,\ j=2,\ u=4,\ v=3;\\
... & & \\
0, & \text{for} & i=1,\ j=2,\ u=n, v=n-1;\\
0, & \text{for} & i=1,\ j=2,\ u=n-1, v=n;\\
0, & \text{for} & i=1,\ j=x,\ u,v;\\
0, & \text{for} & i=1,\ j=n,\ u=1, v=n;\\
h^{n+1}_{nn}h^{n+1}_{11}, & \text{for} & i=1,\ j=n,\ u=n,\ v=1;\\
-h^{n+1}_{nn}h^{n+1}_{11}, & \text{for} & i=1,\ j=n,\ u=n,\ v=n;\\
... & & \\
0, & \text{for} & i=3,\ j=4,\ u=3,\ v=4;\\
h^{n+1}_{44}h^{n+1}_{33}, & \text{for} & i=3,\ j=4,\ u=4,\ v=3;\\
0, & \text{for} & i=3,\ j=x,\ u=3,\ v=x;\\
h^{n+1}_{xx}h^{n+1}_{33}, & \text{for} & i=3,\ j=x,\ u=x,\ v=3;\\
0, & \text{for} & i=3, j=n, u=3, v=n;\\
h^{n+1}_{nn}h^{n+1}_{33}, & \text{for} & i=3,\ j=n,\ u=n,\ v=3;\\
-h^{n+1}_{nn}h^{n+1}_{33}, & \text{for} & i=3,\ j=n,\ u=n,\ v=n;\\
... & & \\
0, & \text{for} & i=n-1,\ j=n,\ u=n-1,\ v=n;\\
h^{n+1}_{nn}h^{n+1}_{n-1n-1}, & \text{for} & i=n-1,\ j=n,\ u=n,\ v=n-1.
\end{array}$$

On the other hand, by the proof of [4, Theorem 1, Ch.3], we have $a=h^{n+1}_{11}, b=h^{n+1}_{22}$ şi $\mu=h^{n+1}_{33}=...=h^{n+1}_{nn}$, where $r=n+1,...,2m+1$. Consequently the previous components are combinations of products formed by a, b and μ.

Remark. We shall compute the proper values of submatrix

$$\begin{pmatrix} h^r_{11} & h^r_{12} \\ h^r_{12} & -h^r_{11} \end{pmatrix},$$

where $r = n+2, ..., 2m+1$. We have

$$\begin{vmatrix} h^r_{11} - \lambda^r & h^r_{12} \\ h^r_{12} & -h^r_{11} - \lambda^r \end{vmatrix} = 0,$$

i.e., $(\lambda^r)^2 = (h^r_{11})^2 + (h^r_{12})^2$, with the solutions $\lambda^r_{1,2} = \pm\sqrt{(h^r_{11})^2 + (h^r_{12})^2}$. Obviously

$$\lambda^r_1 \cdot \lambda^r_2 = -[(h^r_{11})^2 + (h^r_{12})^2].$$

Coming back to our proof, we compute the terms

$$F_{uvij} = \sum_{T=n+2}^{2m+1} (h^T_{uj} h^T_{vi} - h^T_{uv} h^T_{ij}).$$

If $h^T_{11} = -h^T_{12}$, we obtain

$$F_{uvij} = \begin{cases} 0, & \text{for} \quad i=1,\ j=2,\ u=1,\ v=2, \\ -\displaystyle\sum_{T=n+2}^{2m+1} [(h^T_{11})^2 + (h^T_{12})^2], & \text{for} \quad i=1,\ j=2,\ u=2,\ v=1, \\ 0, & \text{for all } i<j, u, v. \end{cases}$$

It follows that Π^{uv}_{kl}, $k<l$ are proper vectors with proper values λ_{ij} obtained totalizing the previous data. The proper values λ_{ij} can be computed taking into account the Lemma 1. Indeed, knowledge of the proper vectors Π^{uv}_{kl}, $k<l$, impose

$$\frac{\partial D_{ij}}{\partial \Pi^{uv}_{kl}} \Pi^{uv}_{kl} + \frac{\partial \widetilde{\rho}_{ij}}{\partial \Pi^{uv}_{kl}} \Pi^{uv}_{kl} - \lambda_{ij} \frac{\partial E_{ij}}{\partial \Pi^{uv}_{kl}} \Pi^{uv}_{kl} = 0,$$

(with summation after u, v şi $k<l$), where

$$E_{ij} = G_{pqrs} \Pi^{pq}_{ij} \Pi^{rs}_{ij} - \frac{1}{2} \delta_{ii} \delta_{jj}.$$

Consequently λ_{ij} is:

$$\begin{array}{ll} \dfrac{c+3}{4} + \dfrac{c-1}{4}\cos^2\theta, & \text{for } i=1,\ j=2,\ u=1,\ v=2 \\ \omega, & \text{for } i=1,\ j=2,\ u=2,\ v=1 \\ \dfrac{c-1}{2}\cos^2\theta, & \text{for } i=1,\ j=2,\ u=3,\ v=4 \\ \dfrac{c-1}{2}\cos^2\theta, & \text{for } i=1,\ j=2,\ u=4,\ v=3 \\ \dots & \\ \dfrac{c-1}{2}\cos^2\theta, & \text{for } i=1,\ j=2,\ u=n,\ v=n-1 \\ \dfrac{c-1}{2}\cos^2\theta, & \text{for } i=1,\ j=2,\ u=n-1,\ v=n \end{array}$$

$$\begin{array}{ll}
0, & \text{pentru } i=1,\ j=x,u,v \\
\dfrac{c+3}{4}+\dfrac{c-1}{4}, & \text{for } i=1,\ j=n,\ u=1,\ v=n \\
-a\mu-\dfrac{c+3}{4}+\dfrac{c-1}{4}, & \text{for } i=1,\ j=n,\ u=n,\ v=1 \\
-a\mu+\dfrac{c-1}{4}, & \text{for } i=1,\ j=n,\ u=n,\ v=n \\
\dots & \\
\dfrac{c+3}{4}+\dfrac{c-1}{4}\cos^2\theta, & \text{for } i=3, j=4, u=3, v=4 \\
\mu^2-\dfrac{c+3}{4}+\dfrac{c-1}{4}3\cos^2\theta, & \text{for } i=3, j=4, u=4, v=3
\end{array}$$

$$\begin{array}{ll}
\dfrac{c+3}{4}, & \text{for } i=3,\ j=x,\ u=3,\ v=x \\
-\mu^2-\dfrac{c+3}{4}, & \text{for } i=3,\ j=x,\ u=x,\ v=3 \\
\dfrac{c+3}{4}+\dfrac{c-1}{4}, & \text{for } i=3,\ j=n,\ u=3,\ v=n \\
\mu^2-\dfrac{c+3}{4}+\dfrac{c-1}{4}, & \text{for } i=3,\ j=n,\ u=n,\ v=3 \\
-\mu^2+\dfrac{c-1}{4}, & \text{for } i=3,\ j=n,\ u=n,\ v=n \\
\dots & \\
\dfrac{c+3}{4}+\dfrac{c-1}{4}\cos^2\theta, & \text{for } i=n-1,\ j=n,\ u=n-1,\ v=n \\
\mu^2-\dfrac{c+3}{4}+\dfrac{c-1}{4}3\cos^2\theta, & \text{for } i=n-1,\ j=n,\ u=n,\ v=n-1,
\end{array}$$

where

$$\omega=-ab+\sum_{T=n+2}^{2m+1}[(h_{11}^T)^2+(h_{12}^T)^2]-\frac{c+3}{4}+\frac{c-1}{4}3\cos^2\theta$$

and $x=\overline{3,n}$.

3 Constrained extrema of curvature quadratic forms of an n-dimensional slant submanifold in a $(2m+1)$-dimensional Sasakian spatial form

Taking into account the results from the previous section, we obtain

- $\dfrac{c+3}{4}+\dfrac{c-1}{4}\cos^2\theta$ is a proper value multiple of order $\dfrac{n}{2}$,
- ω is a simple proper value,
- $\dfrac{c-1}{2}\cos^2\theta$ is a proper value multiple of order $n-2$,
- $-\dfrac{c+3}{4}+\dfrac{c-1}{4}$ is a simple proper value,

- $-a\mu - \frac{c+3}{4} + \frac{c-1}{4}$ is a proper value multiple of order n,
- $-b\mu - \frac{c+3}{4} + \frac{c-1}{4}$ is a proper value multiple of order n,
- $-\mu^2 - \frac{c+3}{4} + \frac{c-1}{4}$ is a simple proper value,
- $-a\mu + \frac{c-1}{4}$ is a simple proper value,
- $-b\mu + \frac{c-1}{4}$ is a simple proper value,
- $\mu^2 - \frac{c+3}{4} + \frac{c-1}{4} 3\cos^2\theta$ is a proper value multiple of order $\frac{n-2}{2}$,
- $\frac{c+3}{4}$ is a proper value multiple of order $1+(n-3)(n-1)$,
- $\mu^2 - \frac{c+3}{4}$ is a proper value multiple of order $1+(n-3)(n-1)$,
- $-\mu^2 + \frac{c-1}{4}$ is a proper value multiple of order $n-2$.

All these proper values correspond to the previous proper vectors.
The summ of all these proper values is the scalar curvature.

In fact, we have proved

Theorem. *The extrema of the quadratic curvature forms* $R_{ABCD}\Pi_{ij}^{AB}\Pi_{ij}^{CD}$ *of an* n*-dimensional slant submanifold in a* $(2m+1)$*-dimensional Sasakian spatial form, constrained by orthonormal frames, are respectively the biggest and the smallest proper values of the associated linear curvature operator. The specification of this extreme values depends on the order relation between the numbers* a, b *and* $a+b=\mu$.

Acknowledgement. The present work is developed within the Grant CNCSIS MEN 1107/2005.

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