$IS - LM$ model with pure money financing

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Abstract

Section 1 analyzes the Hopf bifurcation phenomena in an IS-LM flow with pure money financing. Section 2 studies the geometric dynamics associated to an IS-LM flow and to the Euclidean structure of the space.

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1 IS-LM flow with pure money financing

Let us consider the following ingredients:

(1) independent economic variables: the income Y, the rate of interest R, the money stock M;

(2) the economic functions: the investment $I(Y, R)$, the disposable income $Y^D = Y - T(Y)$, the saving $S(Y^D)$, the taxation $T(Y)$, the money demand $L(Y, R)$;

(3) economic constants: the adjustment speed of income reacting to excess demand $\alpha > 0$, the adjustment speed of excess demand for money $\beta > 0$, the economic barrier (government spending) G;

(4) economic constraints:

$$I_R < 0,\ S_{Y^D} > 0,\ I_Y > 0,\ 0 < T_Y < 1,\ L_Y > 0,\ L_R < 0.$$

The dynamic IS-LM with pure money financing of the government budget deficit is given by the flow

$$\frac{dY}{dt} = \alpha(I(Y, R) + G - S(Y^D) - T(Y))$$
$$\frac{dR}{dt} = \beta(L(Y, R) - M)$$
$$\frac{dM}{dt} = G - T(Y).$$

Suppose this ODEs system has a unique equilibrium point (Y_1, R_1, M_1) situated in the first orthant. To examine the possible emergence of cycles, we use the Jacobian matrix

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$$A = \begin{pmatrix} \alpha(I_Y - \sigma_Y) & \alpha I_R & 0 \\ \beta L_Y & \beta L_R & -\beta \\ -T_Y & 0 & 0 \end{pmatrix},$$

where $\sigma_Y = S_{Y^D}(1 - T_Y) + T_Y$, and all the partial derivatives are taken at the equilibrium point. The characteristic equation is

$$\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0,$$

where

$$a_1 = -(\alpha(I_Y - \sigma_Y) + \beta L_R)$$
$$a_2 = \alpha\beta((I_Y - \sigma_Y)L_R - I_R L_Y)$$
$$a_3 = -\alpha\beta T_Y I_R.$$

Taking into consideration the signs of derivatives we obtain $a_2, a_3 > 0$. The sign of a_1 remains uncertain. Supposing $I_Y - \sigma_Y > 0$, but sufficiently small, we ensure the condition $a_1 > 0$.

If we want to apply the Hopf bifurcation theorem, we need one negative real root and a pair of pure imaginary roots of characteristic equation. Since all the coefficients are positive, it remains to check that $a_1 a_2 - a_3 = 0$, i.e.,

$$\alpha\beta\{-[\alpha(I_Y - \sigma_Y) + \beta L_R][(I_Y - \sigma_Y)L_R - I_R L_Y] + T_Y I_R\} = 0.$$

Let us consider $\alpha = \text{const}$ and β parameter. Then the solution

$$\beta_0 = \frac{1}{L_R}\left\{\frac{T_Y I_R}{(I_Y - \sigma_Y)L_R - I_R L_Y} - \alpha(I_Y - \sigma_Y)\right\} > 0$$

is the critical value of β. Since $a_1 a_2 - a_3$ changes sign as β passes through β_0, the real part of the complex roots, $u(\beta)$, also changes sign as it passes through zero. Also, we can check

$$\frac{du}{d\beta}(\beta_0) \neq 0.$$

This completes the proof of the existence of a closed orbit for values of the parameter in a neighborhood of β_0.

2 IS-LM geometric dynamics

Introducing $x^1 = Y$, $x^2 = R$, $x^3 = M$, and $X^1 = \alpha(I(Y,R) + G - S(Y^D) - T(Y))$, $X^2 = \beta(L(Y,R) - M)$, $X^3 = G - T(Y)$, we obtain the Euler-Lagrange prolongation

$$\frac{d^2x^i}{dt^2} = \frac{\partial f}{\partial x^i} + \sum_j \left(\frac{\partial X^i}{\partial x^j} - \frac{\partial X^j}{\partial x^i}\right)\frac{dx^j}{dt}, \quad i,j = 1,2,3, \tag{1}$$

where

$$f = \frac{1}{2}\delta_{ij} X^i X^j.$$

is the density of energy determined by the Euclidean structure δ_{ij} and the vector field (X^1, X^2, X^3).

The economic constraints impose the signs of the coefficients in the differential system (1):

$$\frac{\partial X^1}{\partial x^2} - \frac{\partial X^2}{\partial x^1} = \alpha I_R - \beta L_Y < 0$$

$$\frac{\partial X^2}{\partial x^3} - \frac{\partial X^3}{\partial x^2} = -\beta < 0$$

$$\frac{\partial X^3}{\partial x^1} - \frac{\partial X^1}{\partial x^3} = -T_Y \in (-1, 0).$$

In this way, the gyroscopic force in the right hand side of (1) is specified.

The solutions of the second order differential system (1) are horizontal pregeodesics of the Riemann-Jacobi-Lagrange manifold $(R^3 \setminus \mathcal{E}, g_{ij}, N_j{}^i)$ determined by the following ingredients:

Lagrangian $L = \frac{1}{2}\delta_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt} - \delta_{ij}X^i\frac{dx^j}{dt} + f$

Hamiltonian $H = \frac{1}{2}\delta_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt} - f$

Jacobi metric $g_{ij} = (H + f)\delta_{ij}$

non-linear connection $N_j{}^i = \Gamma^i_{jk}y^k - F_j{}^i$

$$F_j{}^i = \delta^{ih}F_{jh}, \quad F_{ij} = \frac{\partial X_j}{\partial x^i} - \frac{\partial X_i}{\partial x^j}, \quad i, j, h, k = 1, 2, 3.$$

The geometric dynamics is like a geostrophic wind associated to the IS-LM flow.

Open problem. *Find the economic interpretation of trajectories in the IS-LM geometric dynamics that are different from IS-LM field lines.*

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