(α, β) - complex Finsler metrics

Nicoleta Aldea and Gheorghe Munteanu

Abstract. In this paper we introduce in study a new class of complex Finsler spaces, namely complex Finsler spaces with (α, β) - metric. Some examples of (α, β) - complex Finsler metrics are given by complex Randers metric $L := (\alpha + |\beta|)^2$, complex Kropina metric $L := (\frac{\alpha^2}{|\beta|})^2$, $|\beta| \neq 0$, complex Matsumoto metric $L := (\frac{\alpha^2}{\alpha - |\beta|})^2$, $|\beta| \neq 0$ and purely Hermitian metric $L := \alpha^2 + |\beta|^2$.

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1 Introduction

As compared to the real case, in complex Finsler geometry are not known so many classes of complex Finsler metrics. Besides the significant Kobayashi and Caratheodory metrics (see [1]), which quickened the study of such Finsler geometry, we know two rather trivial classes of complex Finsler metrics: the complex Finsler metrics which come from Hermitian metrics on the base manifold (the purely Hermitian metrics in [11]), and the locally Minkowski complex metrics. Therefore, any new class of complex Finsler spaces with some presence in both theory and applications is welcomed.

In the previous paper, [4], we initiated the study of complex Randers metric. As in the real case [5, 6, 16, 9, 10], a direct generalization of the complex Randers spaces leads to complex Finsler spaces with $(\alpha, \beta)-$ metric. This problem set up the subject of the present paper in which we introduce the complex $(\alpha, \beta)-$ metrics, i.e. complex metrics constructed from just two pieces of familiar data: a purely Hermitian metric and a differential 1-form both globally defined on an underlying complex manifold. We hope that this class of metrics will offer a geometrical model, especially for quantum physics theories.

In the following, let us briefly set the basic notions which are needed; for more information see [1, 11].

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Let M be a complex manifold, $\dim_C M = n$. The complexified of the real tangent bundle $T_C M$ splits into the sum of holomorphic tangent bundle $T'M$ and its conjugate $T''M$. The bundle $T'M$ is in its turn a complex manifold, the local coordinates in a chart will be denoted by (z^k, η^k) and these are changed by the rules: $z'^k = z'^k(z)$, $\eta'^k = \frac{\partial z'^k}{\partial z^j}\eta^j$. The complexified tangent bundle of $T'M$ is decomposed as $T_C(T'M) = T'(T'M) \oplus T''(T'M)$. A natural local frame for $T'_u(T'M)$ is $\{\frac{\partial}{\partial z^k}, \frac{\partial}{\partial \eta^k}\}$, which is changes by the rules obtained with Jacobi matrix of above transformations. Note that the change rule of $\frac{\partial}{\partial z^k}$ contains the second order partial derivatives.

Let $V(T'M) = \ker \pi_* \subset T'(T'M)$ be the vertical bundle, spanned locally by $\{\frac{\partial}{\partial \eta^k}\}$. A complex nonlinear connection, briefly (*c.n.c.*), determines a supplementary complex subbundle to $V(T'M)$ in $T'(T'M)$, i.e. $T'(T'M) = H(T'M) \oplus V(T'M)$. It determines an adapted frame $\{\frac{\delta}{\delta z^k} = \frac{\partial}{\partial z^k} - N_k^j \frac{\partial}{\partial \eta^j}\}$, where $N_k^j(z, \eta)$ are the coefficients of the (*c.n.c.*), ([1], [2], [11]).

A continuous function $F : T'M \to \mathbb{R}^+$ is called complex Finsler metric on M if it satisfies the conditions:

i) $L := F^2$ is smooth on $\widetilde{T'M} := T'M \backslash \{0\}$;

ii) $F(z, \eta) \geq 0$, the equality holds if and only if $\eta = 0$;

iii) $F(z, \lambda\eta) = |\lambda| F(z, \eta)$ for $\forall \lambda \in \mathbb{C}$;

iv) the Hermitian matrix $\left(g_{i\bar{j}}(z, \eta)\right)$, with $g_{i\bar{j}} = \frac{\partial^2 L}{\partial \eta^i \partial \bar{\eta}^j}$, called the fundamental metric tensor, is positive definite.

The pair (M, F) is called a *complex Finsler space.* The iv)-th assumption involves the strongly pseudoconvexity of the Finsler metric F on complex indicatrix $I_{F,z} = \{\eta \in T'_z M \mid F(z, \eta) < 1\}$.

2 Complex $(\alpha, \beta)-$ metrics

Following the ideas from the real case, [5, 6, 16, 9, 10], we shall introduce a new class of complex Finsler metrics. We consider $z \in M$, and $\eta \in T'_z M$, $\eta = \eta^i \frac{\partial}{\partial z^i}$. On M, let

- $a := a_{i\bar{j}}(z) dz^i \otimes d\bar{z}^j$ be a purely Hermitian positive metric and
- $b = b_i(z) dz^i$ be a differential $1-$form.

By these objects we define the function F on $T'M$

$$F(z, \eta) := F(\alpha(z, \eta), |\beta(z, \eta)|), \tag{2.1}$$

where

$$\begin{aligned} \alpha(z, \eta) \quad &: \quad = \sqrt{a_{i\bar{j}}(z)\eta^i \bar{\eta}^j}; \\ |\beta(z, \eta)| \quad &= \quad \sqrt{\beta(z, \eta)\overline{\beta(z, \eta)}} \text{ with } \beta(z, \eta) = b_i(z)\eta^i. \end{aligned} \tag{2.2}$$

By analogy with real case, we call the function from (2.1) the *complex* $(\alpha, \beta)-$ *metric* and the pair $(M, F(\alpha(z, \eta), |\beta(z, \eta)|))$ a *complex Finsler space with* $(\alpha, \beta)-$ *metric* with assumption that it satisfies the conditions of complex Finsler metric.

Obviously, the function $L := F^2(\alpha(z, \eta), |\beta(z, \eta)|)$ depends on z and η by means of the real valued functions $\alpha := \alpha(z, \eta)$ and $\beta := \beta(z, \eta)$. Moreover α and β are homogeneous with respect to η, i.e. $\alpha(z, \lambda\eta) = |\lambda|\alpha(z, \eta)$, $\beta(z, \lambda\eta) = \lambda\beta(z, \eta)$ for any

$\lambda \in \mathbb{C}$, indeed $L(z,\lambda\eta) = \lambda\bar{\lambda}L(z,\eta)$, for any $\lambda \in \mathbb{C}$, and so this homogeneity property implies

$\frac{\partial\alpha}{\partial\eta^i}\eta^i = \frac{1}{2}\alpha$; $\frac{\partial|\beta|}{\partial\eta^i}\eta^i = \frac{1}{2}|\beta|$; $\alpha L_\alpha + |\beta|L_{|\beta|} = 2L$;
$\alpha L_{\alpha\alpha} + |\beta|L_{\alpha|\beta|} = L_\alpha$; $\alpha L_{\alpha|\beta|} + |\beta|L_{|\beta||\beta|} = L_{|\beta|}$;
$\alpha^2 L_{\alpha\alpha} + 2\alpha|\beta|L_{\alpha|\beta|} + |\beta|^2 L_{|\beta||\beta|} = 2L$,
where $L_\alpha := \frac{\partial L}{\partial\alpha}$, $L_{|\beta|} := \frac{\partial L}{\partial|\beta|}$, $L_{\alpha\alpha} := \frac{\partial^2 L}{\partial\alpha^2}$, etc.

First we shall determine the fundamental tensor of the complex $(\alpha,\beta)-$ metric $F(\alpha(z,\eta), |\beta(z,\eta)|)$, i.e. $g_{i\bar{j}} = \partial^2 L(\alpha,|\beta|) \;/\; \partial\eta^i\partial\bar{\eta}^j$. For this, let us consider

$$\begin{aligned} \frac{\partial\alpha}{\partial\eta^i} &= \frac{1}{2\alpha}l_i;\ \frac{\partial|\beta|}{\partial\eta^i} = \frac{\bar{\beta}}{2|\beta|}b_i;\ \eta_i := \frac{\partial L}{\partial\eta^i} = \rho_0 l_i + \mu_0\bar{\beta}b_i \\ b^i &: = a^{\bar{j}i}b_{\bar{j}};\ ||b||^2 := a^{\bar{j}i}b_i b_{\bar{j}}; \\ \frac{\partial\rho_0}{\partial\bar{\eta}^j} &: = \rho_{-2}l_{\bar{j}} + \mu_{-2}\beta b_{\bar{j}};\ \frac{\partial\mu_0}{\partial\bar{\eta}^j} := \mu_{-2}l_{\bar{j}} + \mu'_{-2}\beta b_{\bar{j}}, \end{aligned} \tag{2.3}$$

where

$$\begin{aligned} l_i &: = a_{i\bar{j}}\bar{\eta}^j;\ \left(a^{\bar{j}i}\right) \text{ is the inverse of } \left(a_{i\bar{j}}\right); \\ \rho_0 &: = \frac{L_\alpha}{2\alpha};\ \mu_0 := \frac{L_{|\beta|}}{2|\beta|}; \\ \rho_{-2} &: = -\frac{|\beta|L_{\alpha|\beta|}}{4\alpha^3};\ \mu_{-2} := \frac{L_{\alpha|\beta|}}{4\alpha|\beta|};\ \mu'_{-2} := -\frac{\alpha L_{\alpha|\beta|}}{4|\beta|^3} \end{aligned}$$

and their conjugates.

Note that $\eta_i = \rho_0 l_i + \mu_0\bar{\beta}b_i$ is uniquely represented in this form. Indeed, if $f(z,\eta)l_i + g(z,\eta)b_i = 0$, contracting it by η^i, we obtain $f(z,\eta)\alpha^2 + g(z,\eta)\beta = 0$. Deriving the last relation with respect to β, it results $g(z,\eta) = 0$ and from here $f(z,\eta)\alpha^2 = 0$. $\alpha \neq 0$ leads to $f(z,\eta) = 0$. Another remark is that the functions ρ_0, μ_0, ρ_{-2}, μ_{-2}, μ'_{-2} are real valued. We called it, as real case the invariants of the complex Finsler space with $(\alpha,\beta)-$ metric. Moreover, the subscripts 0 and -2 give the degree of homogeneity with respect to η of this invariants.

Taking into account (2.3), we have:
$g_{i\bar{j}} = \frac{\partial^2 L(\alpha,|\beta|)}{\partial\eta^i\partial\bar{\eta}^j} = \frac{\partial}{\partial\bar{\eta}^j}\left(\frac{\partial L}{\partial\eta^i}\right) = \frac{\partial}{\partial\bar{\eta}^j}\left(\rho_0 l_i + \mu_0\bar{\beta}b_i\right) = \frac{\partial\rho_0}{\partial\bar{\eta}^j}l_i + \rho_0\frac{\partial l_i}{\partial\bar{\eta}^j} + \frac{\partial\mu_0}{\partial\bar{\eta}^j}\bar{\beta}b_i + \mu_0 b_i b_{\bar{j}} = \left(\rho_{-2}l_{\bar{j}} + \mu_{-2}\beta b_{\bar{j}}\right)l_i + \rho_0 a_{i\bar{j}} + \left(\mu_{-2}l_{\bar{j}} + \mu'_{-2}\beta b_{\bar{j}}\right)\bar{\beta}b_i + \mu_0 b_i b_{\bar{j}}$.

So, we have proved

Proposition 2.1. *The fundamental metric tensor of the complex $(\alpha,\beta)-$ metric is given by*

$$g_{i\bar{j}} = \rho_0 a_{i\bar{j}} + \rho_{-2}l_i l_{\bar{j}} + \mu_{-2}\left(\beta l_i b_{\bar{j}} + \bar{\beta}b_i l_{\bar{j}}\right) + \mu'_0 b_i b_{\bar{j}}, \tag{2.4}$$

where $\mu'_0 := \mu_0 + |\beta|^2\mu'_{-2}$.

A immediately computation give
$\mu_{-2}\left(\beta l_i b_{\bar{j}} + \bar{\beta}b_i l_{\bar{j}}\right) = \frac{\mu_{-2}}{\mu_0\rho_0}\eta_i\eta_{\bar{j}} - \frac{\mu_0\mu_{-2}}{\rho_0}|\beta|^2 b_i b_{\bar{j}} - \frac{\rho_0\mu_{-2}}{\mu_0}l_i l_{\bar{j}}$. Plugging it into (2.4), we obtain

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$$g_{i\bar{j}} = \rho_0 a_{i\bar{j}} + \rho'_{-2} l_i l_{\bar{j}} + \mu''_0 b_i b_{\bar{j}} + \mu''_{-2}\eta_i\eta_{\bar{j}}, \tag{2.5}$$

where $\rho'_{-2} := \rho_{-2} - \frac{\rho_0\mu_{-2}}{\mu_0}$, $\mu''_0 := \mu'_0 - \frac{\mu_0\mu_{-2}}{\rho_0}|\beta|^2$, $\mu''_{-2} := \frac{\mu_{-2}}{\mu_0\rho_0}$.

For to find the formulas of the determinant and the inverse of the fundamental tensor $g_{i\bar{j}}$ from (2.5) we can apply the Proposition 2.2 from [4], in a recursive algorithm in at the most three steps. The sign of the real function ρ'_{-2}, μ''_0 and μ''_{-2} depends of the form of the complex $(\alpha,\beta)-$ metric. Therefore, in the next section we study some particular classes of complex $(\alpha,\beta)-$ metrics for which we find the concrete formulas of these invariants and the fundamental tensor $g_{i\bar{j}}$

3 Examples of complex Finsler spaces with $(\alpha,\beta)-$ metric

The real case suggested us to consider the following ([5, 6, 16, 9, 10]) complex $(\alpha,\beta)-$ metrics: *complex Randers metric* $L := (\alpha+|\beta|)^2$, *complex Kropina metric* $L := (\frac{\alpha^2}{|\beta|})^2$, $|\beta| \neq 0$, *complex Matsumoto metric* $L := (\frac{\alpha^2}{\alpha-|\beta|})^2$, $|\beta| \neq 0$ and *purely Hermitian metric* $L := \alpha^2 + |\beta|^2$. It is natural to inquire us when this classes of complex $(\alpha,\beta)-$ metrics are complex Finsler metrics. Further we give attention the first two classes of complex $(\alpha,\beta)-$ metrics.

3.1. Complex Randers metric $L := (\alpha+|\beta|)^2$

The invariants of a complex Randers metric are: $\rho_0 := \frac{F}{\alpha}$; $\mu_0 := \frac{F}{|\beta|}$; $\rho_{-2} := -\frac{|\beta|}{2\alpha^3}$; $\mu_{-2} := \frac{1}{2\alpha|\beta|}$; $\mu'_{-2} := -\frac{\alpha}{2|\beta|^3}$. So, we obtain as in [4]:

$$g_{i\bar{j}} = \frac{F}{\alpha} a_{i\bar{j}} - \frac{F}{2\alpha^3} l_i l_{\bar{j}} + \frac{F}{2|\beta|} b_i b_{\bar{j}} + \frac{1}{2L}\eta_i\eta_{\bar{j}}. \tag{3.1}$$

Examples 1. If $\alpha(z,\eta) = mc\sqrt{\gamma_{i\bar{j}}(z)\eta^i\bar{\eta}^j}$ and $\beta(z,\eta) = \frac{e}{m}A_i(z)\eta^i$, a model for complex electrodynamics is obtain.

2. We consider α given by

$$\alpha^2(z,\eta) := \frac{|\eta|^2 + \varepsilon\left(|z|^2|\eta|^2 - |< z,\eta >|^2\right)}{(1+\varepsilon|z|^2)^2}, \tag{3.2}$$

where $|z|^2 := \sum_{k=1}^n z^k\bar{z}^k$, $< z,\eta > := \sum_{k=1}^n z^k\bar{\eta}^k$, $|< z,\eta >|^2 = < z,\eta > \overline{< z,\eta >}$, defined over the disk $\Delta^n_r = \left\{z \in \mathbf{C}^n,\ |z| < r,\ \ r := \sqrt{\frac{1}{|\varepsilon|}}\right\}$ if $\varepsilon < 0$, on $\mathbf{C}^n$ if $\varepsilon = 0$ and on the complex projective space $P^n(\mathbf{C})$ if $\varepsilon > 0$. Note that $\alpha^2(z,\eta) = a_{i\bar{j}}(z)\eta^i\bar{\eta}^j$ and thus are purely Hermitian metrics which have special properties. They are Kähler with constant holomorphic curvature $\mathcal{K}_\alpha = 4\varepsilon$. Particularly, for $\varepsilon = -1$ we obtain the *Bergman metric* on the unit disk $\Delta^n := \Delta^n_1$; for $\varepsilon = 0$ the *Euclidean metric* on $\mathbf{C}^n$, and for $\varepsilon = 1$ the *Fubini-Study metric* on $P^n(\mathbf{C})$. By deformation of (3.2) metrics, taking $|\beta(z,\eta)| = \frac{|<z,\eta>|}{1+\varepsilon|z|^2}$ we obtain some examples of complex Randers metrics:

$$F_\varepsilon := \frac{\sqrt{|\eta|^2 + \varepsilon\left(|z|^2|\eta|^2 - |< z,\eta >|^2\right)}}{1+\varepsilon|z|^2} + \frac{|< z,\eta >|}{1+\varepsilon|z|^2}. \tag{3.3}$$

For example, F_{-1} is of negative holomorphic curvature $\mathcal{K}_{F_{-1}} = \frac{-2\alpha F_{-1}}{\gamma}$, $\gamma := L_{-1} - \alpha^2(1-|z|^2)$. $\Box$

A special approach of complex Randers spaces is made in [4].

3.2. Complex Kropina metric $L := (\frac{\alpha^2}{|\beta|})^2$, $|\beta| \neq 0$

For a complex Kropina metric, the invariants have the form: $\rho_0 := 2q^2$; $\mu_0 := -q^4$; $\rho_{-2} := \frac{2}{|\beta|^2}$; $\mu_{-2} := -\frac{2q^2}{|\beta|^2}$; $\mu'_{-2} := \frac{2q^4}{|\beta|^2}$, $\rho'_{-2} := -\frac{2}{|\beta|^2}$, $\mu''_0 := 0$, $\mu''_{-2} := \frac{1}{q^4|\beta|^2}$, where $q := \frac{\alpha}{|\beta|}$. By Corollary 2.1 we obtain

Proposition 3.1. *For the complex Kropina metric* $F := (\frac{\alpha^2}{|\beta|})$, $|\beta| \neq 0$ *we have*

$$g_{i\bar{j}} = 2q^2 a_{i\bar{j}} - \frac{2}{|\beta|^2} l_i l_{\bar{j}} + \frac{1}{q^4|\beta|^2}\eta_i\eta_{\bar{j}}. \tag{3.4}$$

Example 3. With $\alpha^2(z,\eta)$ from (3.2) and $|\beta(z,\eta)| = \frac{|<z,\eta>|}{1+\varepsilon|z|^2}$ we can built some examples of complex Kropina metrics:

$$\tilde{F}_\varepsilon := \frac{|\eta|^2 + \varepsilon\left(|z|^2|\eta|^2 - |< z,\eta >|^2\right)}{(1+\varepsilon|z|^2)\,|< z,\eta >|} = \frac{|\eta|^2}{|< z,\eta >|} - \frac{\varepsilon\,|< z,\eta >|}{1+\varepsilon|z|^2}. \tag{3.5}$$

A detailed study of the class of complex Kropina spaces make up the subject a forthcoming paper.

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Authors' address:

Nicoleta Aldea and Gheorghe Munteanu
Transilvania Univ., Faculty of Mathematics and Informatics,
Iuliu Maniu 50, Braşov 500091, Romania.
e-mail: nicoleta.aldea@lycos.com, gh.munteanu@info.unitbv.ro