

Generalized concircular recurrent Weyl spaces

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Abstract. In [2], De and Guha introduced the generalized recurrent Riemannian manifolds and in [4], Maralabhavi and Rathnamma studied the generalized recurrent and concircular recurrent Riemannian manifolds. In this work, the generalized concircular recurrent Weyl spaces are considered and it is proved that a Weyl space which is either generalized concircular recurrent or generalized recurrent is a concircular recurrent Weyl space. In addition, it is shown that every concircular recurrent Weyl space is a conformally recurrent.

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1 Introduction

An n -dimensional differentiable manifold W_n is said to be a Weyl space if it has a conformal metric tensor g and a symmetric connection ∇ satisfying the compatibility condition given by the equation

$$(1.1) \quad \nabla_k g_{ij} - 2T_k g_{ij} = 0 ,$$

where T_k denotes a covariant vector field. Under the renormalization

$$(1.2) \quad \tilde{g} = \lambda^2 g$$

of the metric tensor g , T is transformed by the law

$$(1.3) \quad \tilde{T}_k = T_k + \partial_k \ln \lambda$$

where λ is a function defined on W_n .

An object A defined on $W_n(g, T)$ is called a satellite of weight $\{p\}$ of the tensor g_{ij} , if it admits a transformation of the form

$$(1.4) \quad \tilde{A} = \lambda^p A$$

under the renormalization of the metric tensor g_{ij} ([3],[6]).

The prolonged covariant derivative of a satellite A is defined by

$$(1.5) \quad \dot{\nabla}_k A = \nabla_k A - p T_k A .$$

We note that the prolonged covariant derivative preserves the weight.

Writing (1.1) in local coordinates and expanding it we find that

$$(1.6) \quad \partial_k g_{ij} - g_{hj} \Gamma_{ik}^h - g_{ih} \Gamma_{jk}^h - 2 T_k g_{ij} = 0$$

where Γ_{kl}^i are the connection coefficients of the form

$$(1.7) \quad \Gamma_{kl}^i = \left\{ \begin{matrix} i \\ kl \end{matrix} \right\} - (\delta_k^i T_l + \delta_l^i T_k - g^{im} g_{kl} T_m).$$

Let $R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z$ denote the curvature tensor associated with the connection ∇ .

A non-flat Weyl space is called recurrent, if its curvature tensor R satisfies

$$(1.8) \quad (\dot{\nabla}_U R)(X, Y, Z) = \phi(U) R(X, Y, Z)$$

where ϕ is a nonzero 1-form of weight zero and $R(X, Y, Z)$ is the curvature tensor of type (1,3) [1].

In [2], De and Guha have introduced a non-flat Riemannian space whose curvature tensor R of type (1,3) satisfies the condition

$$(1.9) \quad (\nabla_U R)(X, Y, Z) = A(U)R(X, Y, Z) + B(U)(g(Y, Z)X - g(X, Z)Y)$$

where A and B are 1-forms.

A non-flat Weyl space is called generalized recurrent whose curvature tensor R satisfies the condition

$$(1.10) \quad (\dot{\nabla}_U R)(X, Y, Z) = A(U)R(X, Y, Z) + B(U)(g(Y, Z)X - g(X, Z)Y)$$

where A and B are 1-forms of weight 0 and -2, respectively. Note that when $B = 0$ a generalized recurrent Weyl space becomes a recurrent Weyl space.

2 Generalized Concircular Recurrent Weyl Spaces

The concircular curvature tensor of a Weyl space is given by [7]

$$z(X, Y, Z) = R(X, Y, Z) - \frac{R}{n(n-1)}[g(Y, Z)X - g(X, Z)Y].$$

A non-flat Weyl space is called generalized concircular recurrent, if its concircular curvature tensor satisfies the condition

$$(2.1) \quad (\dot{\nabla}_U z)(X, Y, Z) = A(U)z(X, Y, Z) + B(U)(g(Y, Z)X - g(X, Z)Y) .$$

Theorem 2.1 The Weyl space $W_n(g, T)$ which is either generalized concircular recurrent or generalized recurrent is concircular recurrent .

Proof. We first note that, in local coordinates the concircular curvature tensor of W_n is given by [7]

$$z_{jkh}^i = R_{jkh}^i - \frac{R}{n(n-1)} (g_{jk} \delta_h^i - g_{jh} \delta_k^i) .$$

Suppose now that $W_n(g, T)$ is a generalized concircular Weyl space. Then, by expressing (2.1) in local coordinates we have

$$(2.2) \quad \dot{\nabla}_l z_{jkh}^i = A_l z_{jkh}^i + B_l (\delta_h^i g_{jk} - \delta_k^i g_{jh}) .$$

By contracting (2.2) with respect to i and h , we get

$$(2.3) \quad \dot{\nabla}_l z_{jki}^i = \dot{\nabla}_l z_{jk} = A_l z_{jk} + B_l (n-1) g_{jk}$$

where

$$(2.4) \quad z_{jk} = R_{jk} - \frac{R}{n} g_{jk} .$$

Transvecting (2.4) by g^{jk} and using $g^{jk} z_{jk} = 0$, we get $B_l = 0$. Thus, from (2.2) we find

$$\dot{\nabla}_l z_{jkh}^i = A_l z_{jkh}^i ,$$

showing that $W_n(g, T)$ is a concircularly recurrent.

Next, suppose that $W_n(g, T)$ is a generalized recurrent Weyl space. By virtue of (1.10), we can write

$$(2.5) \quad \dot{\nabla}_l R_{jkh}^i = A_l R_{jkh}^i + B_l (\delta_h^i g_{jk} - \delta_k^i g_{jh}) .$$

By contracting (2.5) with respect to i and h , we get

$$(2.6) \quad \dot{\nabla}_l R_{jk} = A_l R_{jk} + (n-1) B_l g_{jk} .$$

Transvecting (2.6) by g^{jk} we have

$$(2.7) \quad \dot{\nabla}_l R = A_l R + n(n-1) B_l .$$

On the other hand, taking the prolonged covariant derivative of the concircular curvature tensor of W_n we have

$$(2.8) \quad \dot{\nabla}_l z_{jkh}^i = \dot{\nabla}_l R_{jkh}^i - \frac{\dot{\nabla}_l R}{n(n-1)} (\delta_h^i g_{jk} - \delta_k^i g_{jh}) .$$

Substituting (2.7) in (2.8) and using (2.5) we find

$$\dot{\nabla}_l z_{jkh}^i = A_l z_{jkh}^i ,$$

which implies that the generalized recurrent Weyl space $W_n(g, T)$ is concircularly recurrent.

3 Conformally recurrent Weyl spaces

The conformal curvature tensor C_{jkh}^i of W_n is given by [5]

$$(3.1) \quad \begin{aligned} C_{jkh}^i &= R_{jkh}^i + \frac{2}{n(n-2)} (\delta_h^i R_{[jk]} - \delta_k^i R_{[jh]} - g_{jh} g^{im} R_{[mk]} + g_{jk} g^{im} R_{[mh]} - \\ &\quad - (n-2) \delta_j^i R_{[hk]}) - \frac{1}{n-2} (\delta_h^i R_{jk} - \delta_k^i R_{jh} - g_{jh} g^{im} R_{mk} + g_{jk} g^{im} R_{mh}) \\ &\quad + \frac{R}{(n-1)(n-2)} (g_{jk} \delta_h^i - g_{jh} \delta_k^i) \end{aligned}$$

where the bracket denotes antisymmetrization.

An n -dimensional Weyl space is said to be a conformally recurrent Weyl space if its conformal curvature tensor C_{jkh}^i , of weight 0, satisfies the condition

$$(3.2) \quad \dot{\nabla}_l C_{jkh}^i = \lambda_l C_{jkh}^i$$

where λ_l ($\neq T_l$) is a non-zero covariant vector field of weight 0.

C_{jkh}^i may be expressed in terms of z_{jkh}^i and $z_{jk} = z_{jki}^i$ as follows:

$$(3.3) \quad \begin{aligned} C_{jkh}^i &= z_{jkh}^i + \frac{2}{n(n-2)} (\delta_h^i z_{[jk]} - \delta_k^i z_{[jh]} - g_{jh} g^{im} z_{[mk]} + g_{jk} g^{im} z_{[mh]} - (n-2) \delta_j^i z_{[hk]}) \\ &\quad - \frac{1}{n-2} (\delta_h^i z_{jk} - \delta_k^i z_{jh} - g_{jh} g^{im} z_{mk} + g_{jk} g^{im} z_{mh}) . \end{aligned}$$

Theorem 3.1 Every concircular recurrent Weyl space is a conformally recurrent Weyl space.

Proof. A concircular recurrent Weyl space is characterized by

$$(3.4) \quad \dot{\nabla}_l z_{tjkh} = \lambda_l z_{tjkh}$$

where λ_l ($\neq T_l$) is a non-zero covariant vector field of weight 0.

As an immediate consequence of (3.4), we get

$$(3.5) \quad \dot{\nabla}_l z_{jk} = \lambda_l z_{jk} .$$

Changing the order j and k in (3.5) and subtracting the equation so obtained from (3.5), we get

$$(3.6) \quad \dot{\nabla}_l z_{[jk]} = \lambda_l z_{[jk]} .$$

Therefore, by using (3.3),(3.4),(3.5) and (3.6), we get

$$\dot{\nabla}_l C_{jkh}^i = \lambda_l C_{jkh}^i$$

from which the desired result follows.

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