Stabilization with feedback control in the Kaldor economic model

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Abstract. The state space exact linearization method - a strategy designed for a non-conventional control engineering of nonlinear differential systems is applied to provide a nonlinear feedback control law in a modified linearized version ([5], [6]), for the Kaldor SODE ([2]) used in macro-economic business cycles.

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1 Introduction.

The simplest control mechanisms are based on designing of external excitations, where the goal can be approached asymptotically ([5, 6]). In our work we use the parameter variation and the control duration, to analyze the controllability of a nonlinear system, by means of the modified state linearization method. The main idea of the employed method is to construct a coordinate transformation and a feedback control law, such that the input-state or the input-output relationship of the closed-loop system in the new coordinate is linear. The alternative used in this work is based on linearizing the control law ([6]). These methods is applied to the Kaldor's model of macro-economic business cycles ([2]).

2 The stabilization with nonlinear feedback control

Consider a single-input nonlinear control system of the form ([3]):

$$\frac{dx}{dt} = f(x) + g(x)u, \tag{2.1}$$

where $x = (x_1, x_2, \dots, x_n)^t \in \mathbf{R}^n$ are state variables and u is the control parameter, we assume that the vector fields $f, g : \mathbf{R}^n \to \mathbf{R}^n$ are both smooth enough. We denote

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(2.2) $$ad_f^k g(x) = [f, ad_f^{k-1} g](x), \quad (k \geq 1), \quad ad_f^0 g(x) = g(x),$$

where $[X, Y]$ represents the Lie bracket of the vector fields X and Y. Then we have the following

Theorem 1. (**[1]**) Let $f, g \in X(\mathbf{R}^n)$ such that the matrix

(2.3) $$C = [g(x_0), ad_f g(x_0), ad_f^2 g(x_0), \ldots, ad_f^{n-1} g(x_0)],$$

is nonsingular, and assume that there exists a neighborhood $U(x_0)$ of x_0 such that the distribution

(2.4) $$D = span\{g, ad_f g, ad_f^2 g, ..., ad_f^{n-1} g\}$$

is involutive on $U(x_0)$. Then there exists a real-valued function $\lambda(x)$ defined in $U(x_0)$ such that the following conditions hold true

(2.5) $$\begin{cases} L_g \lambda(x) = L_{ad_f g} \lambda(x) = ... = L_{ad_f^{n-2} g} \lambda(x) = 0, \\ L_{ad_f^{n-1} g} \lambda(x_0) \neq 0, \quad \forall x \in U(x_0), \end{cases}$$

where $L_X \lambda$ denotes the Lie derivative of the real-valued function $\lambda(x)$ relative to the vector field X. The converse holds true as well.

Remark. If the conditions of the Theorem are fulfilled, on $U(x_0)$, one can define the transformation:

(2.6) $$x \to z = (z_1, z_2, ..., z_n)^t = (\lambda(x), L_f \lambda(x), \ldots, L_f^{n-1} \lambda(x))^t,$$

which transforms the system (2.1) into the linear controllable SODE

(2.7) $$\frac{dz_1}{dt} = z_2, \quad \frac{dz_2}{dt} = z_3, \ldots, \frac{dz_{n-1}}{dt} = z_n, \quad \frac{dz_n}{dt} = v,$$

where

(2.8) $$\begin{cases} v = b(x) + a(x) u, \\ a(x) = L_g L_f^{n-1} \lambda(x), \quad b(x) = L_f^n \lambda(x), \end{cases}$$

Then the control law will have the following form ([5]):

(2.9) $$u = \frac{1}{a(x)} \left(-b(x) + \sum_{i=1}^{n} a_i z_i \right) = \frac{1}{L_g L_f^{n-1} \lambda} \left(-L_f^n \lambda + \sum_{i=1}^{n} a_i L_f^{i-1} \lambda \right).$$

If the previous theorem cannot be fulfilled, one possibility is to linearize the control law (2.9) in the neighborhood of the control goal. The control law will have the form ([6]):

(2.10) $$u_l = \frac{1}{a(x^g)} \left(-b(x^g) - \sum_{i=1}^{n} b'_{x_i}(x^g)(x_i - x_i^g) + \sum_{i=1}^{n} a_i z_i \right).$$

The equation (2.10) neatly aproximates the equation (2.9) when the control law is applied in a small neighborhood of x^g. The modified linearization control law has the form

$$u_m = (S(t - t_0)S(\varepsilon - \sum_{i=1}^{n} |x_i - x_i^g|)) \cdot u_l, \tag{2.11}$$

with the control started at $t = t_0$ and where we considered the switch function:

$$S(s) = \begin{cases} 1, & \text{for } s \geq 0 \\ 0, & \text{for } s \leq 0. \end{cases} \tag{2.12}$$

The equation (2.10) approximates well (2.9) if the control law is activated when the trajectory is close to the goal. Contrarily, if the control goal is far from the trajectory, the control law fails and intermediate goals should be introduced to achieve control ([5, 6]).

3 Linearized feedback control of Kaldor business cycle model

In this section we use the Chang–Smyth version of the Kaldor model of macro-economic business cycles [2]. Let x be the *gross product* and y the *capital stock*. Then the *net investment* I is assumed to depend on x and y. For y fixed, I is an s-shaped function of x. It is assumed that $\partial I/\partial y \leq 0$. The function

$$I(x, y) = 25 \cdot 2^{-(0.15x+0.00001)^{-2}} + 0.05x + 5 \cdot (320y^{-1})^3, \tag{3.1}$$

used by Lorenz ([7]), meets the requirements for the net investments as they depend on the gross product x and the capital stock y. Then $x(t)$ and $y(t)$ satisfy the SODE:

$$\begin{cases} \frac{dx}{dt} = \alpha(I(x, y) - sx) \\ \frac{dy}{dt} = I(x, y) - \delta y, \end{cases} \tag{3.2}$$

where α is the adjustment coefficient in the goods market, s represents the savings coefficient and δ the depreciation rate of the capital stock. For α sufficiently large, the system (3.2) is singularly perturbed.

Now we will apply the linearized feedback control to this macro-economic business cycle model. The vector fields from (2.1) are $g = (0, x)^t$ and

$$\begin{aligned} f = [&\alpha(25 \cdot 2^{-(0.15x+0.00001)^{-2}} + 0.05x + 5(320y^{-1})^3 - sx), \\ &25 \cdot 2^{-(0.15x+0.00001)^{-2}} + 0.05x + 5(320y^{-1})^3 - \delta y]^t, \end{aligned} \tag{3.3}$$

The determinant of the matrix C from (2.3) is $\det C = -\alpha \cdot 491520000 \cdot x^2 y^{-4}$, and for $\alpha x \neq 0$, C is non-singular. We note that $[g, ad_f g] \not\equiv (0, 0)^t$, and hence, the distribution D is not involutive. Thus, Theorem 1 cannot be applied, and *the modified method* of state space exact linearization will be used.

We construct the mapping $\lambda(x, y)$. Applying (2.5) to (3.3), and solving the obtained PDE, we get $\lambda = y + \mu$, where μ is a constant number determined by the control goal. Using λ determined previously, one gets the coordinate transformation (2.6) specified to our system:

$$(3.4) \qquad \begin{cases} z_1 = y + \mu \\ z_2 = 25 \cdot 2^{-(0.15x+0.00001)^{-2}} + 0.05x + 16384 \cdot 10^4 y^{-3} - \delta y, \end{cases}$$

with $v = a_1 z_1 + a_2 z_2$, $\sigma = 7.5 \cdot \tau \cdot \ln 2(0.15x + 10^{-5})^{-3} + 0.05$, $\rho = 25 \cdot \tau + 0.05x + \xi$, $\tau = 2^{-(0.15x+0.00001)^{-2}}$, $\xi = 16384 \cdot 10^4 y^{-3}$, $\psi = 49152 \cdot 10^4 y^{-4}$ and

$$(3.5) \qquad \begin{array}{l} a = x((\psi + \delta)^2 - \psi\alpha\sigma + \frac{196608 \cdot 10^4}{y^5}(\rho - \delta y)) \\ b = \alpha(\rho - sx)\left[\alpha(\sigma - s)\sigma + \sigma(\psi - \delta) + \right. \\ \left. +\alpha(\rho - sx)\left(\frac{2.25 \cdot \tau \cdot \ln^2(2)}{(0.15x+10^{-5})^6} - \frac{3.3750 \cdot \tau \cdot \ln(2)}{(0.15x+10^{-5})^4}\right)\right] + \\ +(\rho - \delta y)[(\psi + \delta)^2 - \psi\alpha\sigma + \frac{196608 \cdot 10^4(\rho - \delta x)}{y^5}]. \end{array}$$

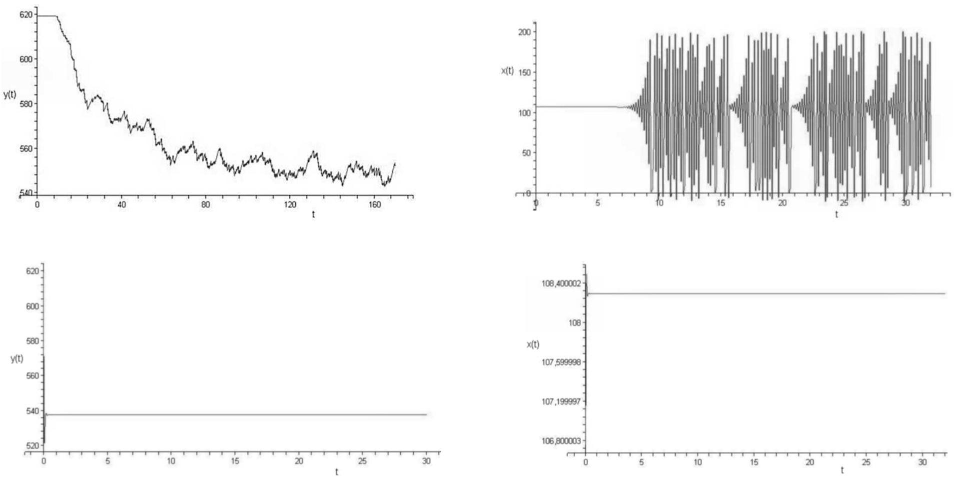
Replacing (3.4) and (3.5) into (2.10), and performing the linearization of the obtained relation, the modified form (2.11) of the control law can be built.

4 A numerical example

For $\alpha = 100$, $s = 0.29, \delta = 0.05$, the SODE (3.2) becomes

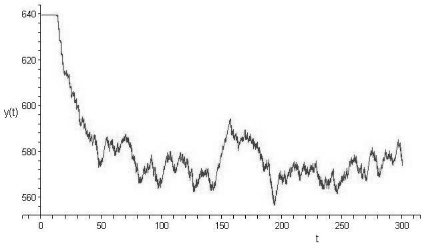
$$(4.1) \qquad \begin{cases} \frac{dx}{dt} = 100(25 \cdot 2^{-(0.15x+0.00001)^{-2}} + 0.05x + 5 \cdot (320y^{-1})^3 - 0.29x) \\ \frac{dy}{dt} = 25 \cdot 2^{-(0.15x+0.00001)^{-2}} + 0.05x + 5 \cdot (320y^{-1})^3 - 0.25y. \end{cases}$$

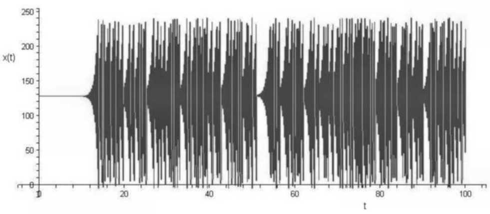
This SODE has one point of unstable equilibrium, $(106.7608394, 619.2128684)$. Using Maple techniques, one can provide (see the images below) the time history of y (left) and x (right) in the uncontrolled system (upper pair) and the controlled system with x-second equation control (lower pair).

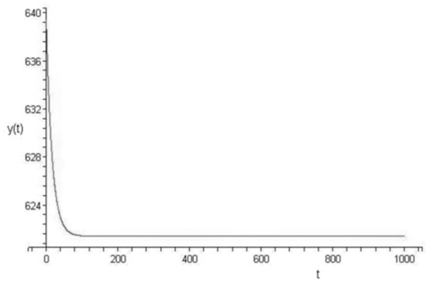


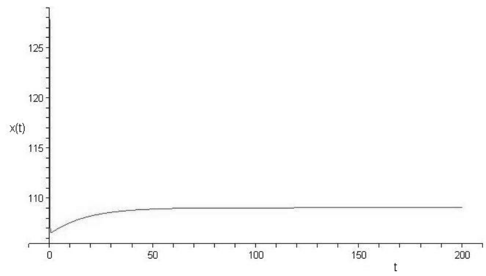
Here the control has been activated on x in the second equation of (2.1) at $t = 0$, with the goal $x^g = 108$ and $y^g = 550$. We have chosen, for the linearized feedback control law (2.11), the coefficients $a_1 = 1.5$ and $a_2 = 4.25$.

If one applies the control to x in the *first* equation considered with g in (3.5), the time history of x and y of the controlled system are described further, where we have chosen in (2.11) $a_1 = 1.5$ and $a_2 = 4.25$, and for $\alpha = 100$, $s = 0.29$, $\delta = 0.05$. Then one can provide (see the images from below) the time history of y (left) and x (right) in the uncontrolled system (upper pair) and the controlled system with x-first equation control (lower pair)









It should be noted that the modified method of linearization has been applied, while the goal ($x^g = 128$ and $y^g = 580$) cannot be approached in only one step, intermediate steps to obtain the desired level of system behavior being needed.

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References

[1] J. L. Breeden, A. Huber, *Reconstructing equations of motion from experimental data with unobserved variables*, Phys. Rev. A 42 (1990), 5817-5826.

[2] W. W. Chang, D. J. Smyth, *The existence and persistence of cycles in a nonlinear model: Kaldor's 1940 model re-examined*, Review of Economic Studies 38 (1971), 37-44.

[3] D. Gligorlski, D. Dimovski, V. Urumov, *Control in multidimensional chaotic systems*, Phys. Rev. E 51 (1995), 1-9.

[4] A. Isidori, *Nonlinear Control Systems*, Springer-Verlag (1989), 156-172.

[5] C. Liqun, L. Yan Zhu, *Control of the Lorentz chaos by the exact linearization*, App. Math. and Mec. 19 (1998), 67-73.

[6] C. Liqun, L. Yan Zhu, *A Parametric open-plus-closed-loop approach to control chaos in nonlinear oscillations*, Phys. Lett. A 262 (1999), 350-354.

[7] H. W. Lorenz, *Nonlinear dynamical economics and chaotic motion*, Lecture Notes in Economics and Mathematical Systems 334, Springer-Verlag, Berlin, 1989.

[8] C.S. Stamin and V. Balan, *Applications of the nonlinear feedback control in neural excitation processes*, Sci. Ann. of UASMV Iasi, to appear.

[9] H. R. Varian, *Catastrophe theory and the business cycle*, Economic Inquiry 17 (1979), 14-28.

[10] T. L. Vincent, J. Yu, *Control of a chaotic system*, Dyn. Contr. 1 (1991), 35-52.

[11] C. J. Wan, D. Bernstein, *Nonlinear feedback control with global stabilization*, Dyn. Contr. 5 (1995), 321-346.

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