New results on homogeneous systems

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Abstract. A universal enveloping algebra for a homogeneous system is defined. After proving the uniqueness and the existence, the usual results concerning such a universal enveloping algebra are proved in a similar way as the corresponding results on the universal enveloping algebra of a Lie algebra.

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1 Introduction

Homogeneous systems (shortly, hs) were introduced by K. YAMAGUTI in 1958 (see [5]). Their name comes from the fact that the tangent space at the identity class of a homogeneous space G/H can be organized as a homogeneous system.

LIE algebras are particular classes of hs. That is why, in the study of hs we try to follow the ways used in the studies on Lie algebras.

The aim of this paper is to construct a universal enveloping algebra (shortly, UEA) for a hs. To this end we analyze the construction of the UEA for a Lie algebra. It must be remarked that the construction of such a UEA imposes the existence of a covariant functor $\mathcal{L} : \mathcal{A}ss_K \to Lie_K$, where $\mathcal{A}ss_K$ is the category of all associative K-algebras, and Lie_K - the category of all LIE K-algebras. Actually, the construction of the UEA for a Lie algebra assures the existence of a left adjoint functor for $\mathcal{L}$. Therefore, the construction of the UEA for a hs impose, necessarily, the construction of a covariant functor from the category of all binary algebras to the category of all hs, whose restriction to $\mathcal{A}ss_K$ must be coincident with $\mathcal{L}$.

2 Preliminaries

All vector spaces used in this paper are considered over a field K with characteristic $\neq 2$.

Following Yamaguti [5] we give the following definition.

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Definition 2.1. A homogeneous system (briefly hs) is a vector space V over the field K endowed with a family of multilinear compositions,

$$V \times V \to V, (x_1, x_2) \to x_1 \circ x_2,$$

$$\underbrace{V \times V \times ... \times V}_{n\ times} \to V, (x_1, x_2, ..., x_n) \to [x_1, x_2, ..., x_n], n \geq 3$$

satisfying the following properties:

$(hs.1)$ $x \circ x = 0, \ \ \forall x \in V$

$(hs.2)$ $[x, x, x_1, ..., x_k] = 0, \forall k \in \mathbb{N}^*, \ \ \forall x, x_1, ..., x_k \in V,$

$(hs.3)$ $(x \circ y) \circ z + (y \circ z) \circ x + (z \circ x) \circ y + [x, y, z] + [y, z, x] + [z, x, y] = 0,$
$\forall x, y, z \in V,$

$(hs.4)$ $[x \circ y, z, w] + [y \circ z, x, w] + [z \circ x, y, w] + [x, y, z, w] + [y, z, x, w]+$
$+[z, x, y, w] = 0, \ \ \forall x, y, z, w \in V,$

$(hs.5)$ $[x_1, ..., x_k, y_1 \circ y_2] = [x_1, ..., x_k, y_1] \circ y_2 + y_1 \circ [x_1, ..., x_k, y_2]+$
$+[x_1, ..., x_k, y_1, y_2] - [x_1, ..., x_k, y_2, y_1],$
$\forall x_1, ..., x_k, y_1, y_2 \in V, k \geq 2,$

$(hs.6)$ $[D_{(x_1,...,x_k)}, D_{(y_1,y_2)}] = D_{([x_1,...,x_k,y_1],y_2)} + D_{(y_1,[x_1,...,x_k,y_2])}+$
$+D_{(x_1,...,x_k,y_1,y_2)} - D_{(x_1,...,x_k,y_2,y_1)} - D_{(x_1,...,x_k,y_1 \circ y_2)},$
$\forall y_1, ..., y_k, y_1, y_2 \in V$

$(hs.7)$ $[D_{(x_1,...,x_k)}, D_{(y_1,...,y_\ell,y_{\ell+1})}] = \widetilde{\mathcal{D}}([D_{(x_1,...,x_k)}, D_{(y_1,...,y_\ell)}], y_{\ell+1})-$
$-[D_{(x_1,...,x_k,y_{\ell+1})}, D_{(y_1,...,y_\ell)}] - D_{(x_1,...,x_k,[y_1,...,y_\ell,y_{\ell+1}])}+$
$+D_{(y_1,...,y_\ell,[x_1,...,x_k,y_{\ell+1}])}, \forall x_1, ..., x_k, y_1, ..., y_\ell, y_{\ell+1} \in V;$

here the endomorphism $D_{(a_1,...,a_k)}$ is defined by $D_{(a_1,...,a_k)}(x) = [a_1, ..., a_k, x]$, $\forall x \in V$, the family $\mathcal{D} = \{D_{(a_1,...,a_k)} | \forall a_1, ..., a_k \in V, \forall n \geq 2\}$ is a Lie subalgebra of $g\ell(V)$, and $\widetilde{\mathcal{D}} : \mathcal{D} \times V \to \mathcal{D}$ is the bilinear mapping defined by $\widetilde{\mathcal{D}}(D_{(x_1,...,x_k)}, x) = D_{(x_1,...,x_k,x)}$.

We shall denote by $\mathbb{HS}$ the category with hs as objects and the hs-homomorphisms as morphisms.

First example of hs was provided by K. Yamaguti in connection with any Lie algebra $L = V \oplus A$, where V is a vector space and A - a Lie algebra. If we denote by $Lie_{\otimes}$ the category of all Lie algebras of the form $L = V \oplus A$ (where V is a vector space and A - a Lie algebra), then the construction of Yamaguti assures the existence of a covariant functor $\mathcal{Y} : Lie_{\oplus} \to \mathbb{HS}$.

Another important example is connected with any nonassociative algebra $A(\cdot)$. Denote by L_a the left multiplication of $A(\cdot)$ by $a \in A$, namely the endomorphism defied by $L_a(x) = a \cdot x$, for all $x \in A$. The following result was proved in [1].

Theorem 2.1. *Let $A(\cdot)$ be a non-associative algebra over K. The vector space A, endowed with following operations,*

$$x \circ y = x \cdot y - y \cdot x,$$

$$[x_1, x_2, ..., x_n, x_{n+1}] = D_{(x_1,x_2,...,x_n)}(x_{n+1}), \forall x_1, x_2, ..., x_n, x_{n+1} \in A, \forall n \geq 2,$$

where

$$D_{(x_1,x_2)} = [L_{x_1}, L_{x_2}] - L_{x_1 \circ x_2},$$

$$D_{(x_1,x_2,...,x_n,x_{n+1})} = [D_{(x_1,x_2,...,x_n)}, L_{x_{n+1}}] - L_{[x_1,x_2,...,x_n,x_{n+1}]},$$

becomes an hs.

Then we have obtained a covariant functor $\mathcal{HS} : \mathcal{A} \to \mathbb{HS}$, which extends the functor $\mathcal{L}$.

3 Free algebras

Following closely [6] Cap.I, we recall some constructions and results on free nonassociative K-algebras. Let $X = \{x_j\}_{j\in J}$, where J is a totally ordered set, and let $X^* = X \cup \{), (\}$. Consider the set of all finite sequences with elements from X^*. It is said that two sequences $y_1y_2...y_n$ and $z_1z_2...z_m$, with $y_i, z_j \in X^*$ are equal if $n = m$ and $y_i = z_i, \forall i \in \{1, 2, ..., n\}$.

We define recurrently the set $W[X]$ by following two rules:

1°. $x_j \in W[X], \forall i \in J,$

2°. $if\ \ x_1, x_2 \in X, u, v \in W[X] \setminus X,\ \ then\ \ x_1x_2, x_1(u), (v)x_2, (u)(v) \in W[X].$

The elements of $W[X]$ are called *nonassociative words*. Taking into account of the recurrence from the definition of $W[X]$ we can define a binary operation on $W[X]$ by:

$$\begin{aligned}&x_1 \cdot x_2 = x_1x_2,\ \ x_1 \cdot u = x_1(u),\ \ v \cdot x_2 = (v)x_2,\ \ u \cdot v = (u)(v)\\ &\forall x_1, x_2 \in X, \forall u, v \in W[X] \setminus X.\end{aligned}$$

Let $K[X]$ denotes the vector space over K with basis $W[X]$. By endowing $K[X]$ with the binary operation obtained extending - by bilinearity - the before defined binary operation on $W[X]$, it becomes a binary non-associative algebra called the *free algebra* over K generated by X. The algebra $K[X]$ has a universal property (see Theorem 1, [6], p.11).

According with Theorem 2.1, $K[X]$ can be organized as a hs; we denote by $[x_1, x_2, ..., x_k]_w, k \geq 3$, the multilinear compositions defining the hs-structure of $\mathcal{HS}(K[X])$.

4 The nonassociative tensor algebra

Recall the definition and some properties of the nonassociative tensor algebra associated with any K-module.

Consider the nonassociative tensor algebra $T\{V\}$ of a K-module V as being

$$T\{V\} = V^{\otimes 0} \oplus V^{\otimes 1} \oplus V^{\otimes 2} \oplus ... \oplus V^{\otimes n} \oplus ...$$

where $V^{\otimes 0} = K$, $V^{\otimes 1} = V$, $V^{\otimes n} = \sum_{i=1}^{n-1} V^{\otimes i} \otimes V^{\otimes n-i}$, for $n > 1$ and the product of $v \in V^{\otimes i}$ and $w \in V^{\otimes j}$ is defined as $v \cdot w = (v) \otimes (w) \in V^{\otimes i+j}$. The absence of associativity compels us to make use of pairs of parenthesis indicating the order of operations. We shall use the notation $]x_1 \otimes x_2 \otimes ... \otimes x_n[$ for a generic monomial of degree n with a specific settlement of the $n-2$ pairs of parenthesis; such an element is formally obtained substituting everywhere in a monomial of $K[V]$ the binary operation symbol " $\cdot$ " by " $\otimes$ ".

5 Universal enveloping algebra

We shall associate to each hs A over K a (nonassociative) algebra which is generated as freely as possible by A subject to a family of monomial relations in A. Throughout this section we shall be concerned with not necessarily associative algebras containing identity elements, only; homomorphism for algebras means homomorphisms in the usual sense mapping 1 into 1.

Definition 5.1. *Let A be a hs over K. A universal enveloping algebra of A is a pair (U, i), where U is a (not necessarily associative) algebra over K, $i : A \rightarrow U$ is a hs-homomorphism of A in $\mathcal{HS}(U)$, such that for any (not necessarily associative) algebra B and any hs-homomorphism $j : A \rightarrow \mathcal{HS}(B)$, there exists a unique homomorphism of algebras $\Phi : U \rightarrow B$ such that $\Phi \circ i = j$.*

Proposition 5.1. *If (U, i) is the universal enveloping algebra of the hs A, then U is generated by $i(A) \cup \{1\}$.*

Proposition 5.2. *The universal enveloping algebra (U, i) of the hs A is unique up to an isomorphism.*

The **existence** of a suitable pair (U, i) is also not difficult to establish. To this end, we consider the nonassociative tensor algebra $T\{A\}$ on A (see Section 4) and the two sided ideal J in $T\{A\}$ generated by $\{x \otimes y - y \otimes x - x \circ y, [x_1, x_2, ..., x_n]^{\otimes} - [x_1, x_2, ..., x_n] \mid x, y, x_1, x_2, ..., x_n \in A, n \geq 3\}$; here A is considered naturally imbedded in $T\{A\}$ and $[x_1, x_2, ..., x_n]^{\otimes}$ is formally obtained from $[x_1, x_2, ..., x_n]_w$ by changing everywhere " $\cdot$ " with " $\otimes$ ". Define $\mathcal{U}(A) = T\{A\}/J$ and consider the canonical homomorphism $\pi : T\{A\} \rightarrow \mathcal{U}(A)$. Let $i : A \rightarrow \mathcal{U}(A)$ be the restriction to A of the natural projection π.

We can prove now the following existence result.

Proposition 5.3. *$(\mathcal{U}(A), i)$ is a universal enveloping algebra of A.*

Proof. Since $\mathcal{U}(A)$ is a binary algebra over K, according with Section 2, $\mathcal{U}(A)$ carries a natural hs-structure such that i is a hs-homomorphism. Let $j : A \rightarrow B$ be as in Definition 5.1. The universal property of $T\{A\}$ yields an algebra homomorphism $\Phi' : T\{A\} \rightarrow B$ which extends j. The mapping j forces all $x \otimes y - y \otimes x - x \circ y, [x_1, x_2, ..., x_n]^{\otimes} - [x_1, x_2, ..., x_n] (n \geq 3)$ to lie in $Ker\ \Phi'$ so Φ' induces an algebra homomorphism $\Phi : U \rightarrow B$ such that $\Phi \circ i = j$.The uniqueness of Φ is evident since $Im\ i \cup \{1\}$ generate $\mathcal{U}(A)$.

Consequently, the covariant functors $\mathcal{HS} : \mathcal{ALG}_K \rightarrow \mathbb{HS}_K$ and $\mathcal{U} : \mathcal{ALG}_K \rightarrow \mathbb{HS}_K$ are consisting in an adjoint pair of functors; more exactly, the functor $\mathcal{U}$ is a left adjoint to the functor $\mathcal{HS}$.

Theorem 5.4. 1. *Let A_1 and A_2 be two hs, $(U_1, i_1), (U_2, i_2)$ their corresponding universal enveloping algebras, and $\alpha : A_1 \rightarrow A_2$ be a hs-homomorphism. Then, there exists a unique hs-homomorphism $\alpha' : U_1 \rightarrow U_2$ such that $\alpha' \circ i_1 = i_2 \circ \alpha$.*
2. *Let $(\mathcal{U}, i)$ be the UEA of the hs A, B - a two-sided ideal of A and $\mathcal{R}$ - the two-sided ideal of $\mathcal{U}$ generated by $i(B)$. Then $j : a+B \rightarrow i(a)+\mathcal{R}$, $\forall a \in A$ is a hs-homomorphism of A/B and $\mathcal{HS}(\mathcal{V})$, where $\mathcal{V} = \mathcal{U}/\mathcal{R}$; moreover, $(\mathcal{V}, j)$ is the universal enveloping algebra for A/B.*

3. *Let D be a derivation of the hs A, then there exists a uniquely defined derivation D' for the algebra $\mathcal{U}$ such that $D' \circ i = i \circ D$..*

Proof. 1. If α is a hs-homomorphism of A_1 and A_2 , then $i_2 \circ \alpha$ is a hs-homomorphism of A_1 and $\mathcal{HS}(\mathcal{U}_2)$. Then, there exists a uniquely defined hs-homomorphism α' from $\mathcal{U}_1$ to $\mathcal{U}_2$ such that $\alpha' \circ i_1 = i_2 \circ \alpha$.
2. Let us denote that the mapping $a \to i(a) + \mathcal{R}$ from A to $\mathcal{V} = \mathcal{U}/\mathcal{R}$ is a hs-homomorphism of A into $\mathcal{HS}(\mathcal{V})$. Since $i(B) \subset \mathcal{R}$, B is carried in 0 by this homomorphism. Consequently, we obtain the induced hs-homomorphism $a+B \to i(a)+\mathcal{R}$ from hs A/B to $\mathcal{HS}(\mathcal{V})$. This is just the mapping j. Let now θ be a hs-homomorphism of the hs A/B and $\mathcal{HS}(U)$, where U is any algebra. Then, the mapping $\eta : A \to \mathcal{HS}(U)$, defined by $\eta(a) = \theta(a + B), \forall a \in A$, is a hs-homomorphism. Consequently, there exists a hs-homomorphism $\eta' : \mathcal{U} \to U$ such that $\eta' \circ i = \eta$. If $b \in B$, then $\eta(b) = 0$, so that $i(b) \in ker\ \eta'$. Then, $\mathcal{R} \subseteq ker\ \eta'$ and, consequently, we obtain the induced hs-homomorphism $\theta' : u + \mathcal{R} \to \eta'(u)$ from $\mathcal{V} = \mathcal{U}/\mathcal{R}$ into U. So, it results $\theta = \theta' \circ j$. We must prove now that θ' is uniquely defined. For this, it is enough to prove that $j(A/B)$ generates $\mathcal{V}$. According to Proposition 5.1, $\mathcal{U}$ is generated by $i(A) \cup \{1\}$, so it follows that $\mathcal{V}$ is generated by elements of the form $j(a + B)$, i.e., by the set $j(A/B)$.
3. Let D be a derivation of the hs A. According to **1** there exists a unique algebra endomorphism $D' : \mathcal{U} \to \mathcal{U}$ such that $D' \circ i = i \circ D$. Consequently, $D'(i(x)) = i(D(x))$, $\forall x \in A$ and $D'([i(x), i(y)]) = D'(i([x, y])) = i(D([x, y])) = i([D(x), y] + [x, D(y)]) = [D'(i(x)), i(y)] + [i(x), D'(i(y))]$, $\forall x$,
$y \in A$. Since $i(A) \cup \{1\}$ generates $\mathcal{U} = \mathcal{U}(A)$, it results that D' is a derivation of $\mathcal{U}$.

Remark 5.5. *All definitions and results given in this section can be restated without supposing that the used algebras have identity elements.*

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