

Fractional Euler - Lagrange and fractional Wong equations for Lie algebroids

Gheorghe Ivan, Mihai Ivan and Dumitru Opreş

Abstract. The main purpose of this paper is to describe the fractional Euler - Lagrange equations on Lie algebroids. In the case when the Lie algebroid is the direct product of the tangent bundle over a manifold with a Lie algebra, then the fractional Euler-Lagrange equations are exactly the fractional Wong equations.

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1 Introduction

The study of fractional problems of the calculus of variations and respective Euler - Lagrange equations is a subject of strong current research because of its numerous applications ([1],[3],[4],[5],[8]).

In 2002, O. Agrawal proved a formulation for variational problems with right and left fractional derivative in the Riemann- Liouville sense ([1],MR1930721). Then these Euler - Lagrange equations were used by D. Băleanu and T.Avkar to investigate problems with Lagrangians which are linear on the velocities ([1]). In [8] fractional problems of the calculus of variations with symmetric fractional derivatives are considered and correspondents Euler - Lagrange equations obtained, using both Lagrangian and Hamiltonian formalism. A extension of Noether's symmetry theorem to the fractional Riemann - Liouville integral functionals of the calculus of variations has recently introduced by El - Nabulsi in ([4],MR2191270).

The fundamental problem of the calculus of variations with Riemann - Liouville fractional integral, as considered by El- Nabulsi in [4], is the following: consider the action functional given by:

$$(1.1) \quad S^r(q) = \frac{1}{\Gamma(r)} \int_{t_0}^{t_1} L(q(t), \dot{q}(t))(\theta - t)^{r-1} dt,$$

under given boundary conditions $q(t_0) = q_0$, and $q(t_1) = q_1$, Γ is the Euler gamma function, $r \in (0, 1]$, θ is the observer time, t is the intrinsic time, $t \neq \theta$, and $L : TM \rightarrow M$ is a Lagrangian, where $\pi : TM \rightarrow M$ is the tangent bundle of the manifold M .

Using Theorem 1([4]), the critical points of $S^r(q)$ satisfies the following Euler - Lagrange equations:

$$(1.2) \quad \frac{\partial L}{\partial \bar{q}^i}(q(t), \dot{q}(t)) - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\bar{q}}^i}(q(t), \dot{q}(t)) \right) = \frac{1-r}{\theta-t} \frac{\partial L}{\partial \bar{q}^i}(q(t), \dot{q}(t)), \quad i = \overline{1, n}.$$

The equations (2) is fractional Euler - Lagrange equations.

The category of Lie algebroids has proved to be useful in the formulation and analysis of many problems in differential geometry and applied mathematics ([2],[6],[7],[9],[11]). In the context of Mechanics, a program was proposed by A. Weinstein ([11]) in order to develop a theory of Lagrangian and Hamiltonian systems on Lie algebroids and their discrete analog on Lie groupoids. In the last years, this program has been actively developed by many authors, and as a result, a powerful mathematical structure is emerging. It is therefore interesting to find a fractional action - like variational description of Lagrange's equations for a Lagrangian system defined on a more general Lie algebroid,

In variational description of Lagrange's equations for Lagrangian, the first steps in this direction were already done by A. Weinstein in the case of an integrable Lie algebroid (i.e. the Lie algebroid of a Lie groupoid) and by E. Martinez ([9]).

In this paper, we will prove that the fractional Euler - Lagrange equations on a Lie algebroid are precisely the equations for the critical points of the action functional of the form (1) defined on the set of admissible curve on a Lie algebroid with fixed endpoints, in the stronger sense; that is to say, we will prove that the set of such curve can be endowed with a structure of Banach manifold, that the action functional is continuously differential and that the equations for the critical points are precisely of the fractional Euler - Lagrange equations for the given Lagrange system as obtained in ([1]).

2 E - paths and E - homotopy on Lie algebroids

We start this section with the presentation of some concepts about Lie algebroids (see [4]).

Let M be a differentiable n - dimensional manifold and (TM, π, M) its tangent bundle. We consider (E, τ, M) be a vector bundle of rank m over M . Denote by $Sec(E)$ the $C^\infty(M)$ - module of sections of E .

A *Lie algebroid structure* on a vector bundle (E, τ, M) is given by a vector bundle morphism $\rho : E \rightarrow TM$ over the identity map in M (called the *anchor map*), together with a Lie algebra structure on the space $\Gamma(E)$ of sections determined by the Lie bracket $[\cdot, \cdot]_E$ which induces a Lie algebra homomorphism $\bar{\rho}$ from $Sec(E)$ into $Sec(TM) = \mathcal{X}(M)$, satisfying the compatibility condition (called *Leibniz identity*):

$$[\sigma, f\eta]_E = \bar{\rho}(\sigma)(f)\eta + f[\sigma, \eta]_E, \quad (\forall) f \in C^\infty(M), \sigma, \eta \in Sec(E).$$

A vector bundle (E, τ, M) endowed with a Lie algebroid structure $([\cdot, \cdot]_E, \rho)$ is called *Lie algebroid over M* and it is denoted by $(E, [\cdot, \cdot]_E, \rho)$.

We recall that the induced morphism $\bar{\rho} : Sec(E) \rightarrow \mathcal{X}(M)$ by the anchor map $\rho : E \rightarrow TM$ is given by $\sigma \in Sec(E) \rightarrow \bar{\rho}(\sigma) \in \mathcal{X}(M)$, where $\bar{\rho}(\sigma)(x) = \rho(s(x))$, $(\forall) x \in M$.

For a Lie algebroid $(E, [\cdot, \cdot]_E, \rho)$ over M , a local coordinate system $(x^\alpha), \alpha = \overline{1, n}$ on an open U included in the base manifold M and a local basis of sections $\{e_i | i = \overline{1, m}\}$ of the vector bundle $\tau^{-1}(U) \rightarrow U$, determines a local coordinate system $(x^\alpha, y^i), \alpha = \overline{1, n}, i = \overline{1, m}$ on E .

The anchor and the bracket of the Lie algebroid $(E, [\cdot, \cdot]_E, \rho)$ are locally determined by the local functions $\rho_i^\alpha \in C^\infty(M)$ and $C_{jk}^i \in C^\infty(M)$ on M given by:

$$(2.1) \quad \bar{\rho}(e_i) = \rho_i^\alpha \frac{\partial}{\partial x^\alpha}, \quad [e_i, e_j]_E = C_{ij}^k e_k \quad \text{for } \alpha = \overline{1, n}, i = \overline{1, m}.$$

The functions $\rho_i^\alpha, C_{ij}^k \in C^\infty(M)$ given by the relations (3) are called the *structure functions* of $(E, [\cdot, \cdot]_E, \rho)$ with respect to the choosen local coordinates system.

The structure functions of E verify the following relations, which result from the Leibniz identity and the Jacoby identity:

$$(2.2) \quad \rho_i^\alpha \frac{\partial \rho_j^\beta}{\partial x^\alpha} - \rho_j^\alpha \frac{\partial \rho_i^\beta}{\partial x^\alpha} = \rho_k^\beta C_{ij}^k, \quad \sum_{cyclic(i,j,k)} \left(\rho_i^\alpha \frac{\partial C_{jk}^l}{\partial x^\alpha} + C_{jk}^h C_{ih}^l \right) = 0.$$

Example 1. (i) Every Lie algebra of finite dimension $(A, [\cdot, \cdot]_A)$ over a singleton is a Lie algebroid $(A, [\cdot, \cdot]_A, \rho)$ with $\rho = 0$.

(ii) Let TM be tangent bundle to a manifold M . The pair (TM, Id_{TM}) endowed with the usually Lie bracket $[\cdot, \cdot]$ is a Lie algebroid over M , called the *canonical Lie algebroid associated to the tangent bundle*. \square

With the usual coordinates $(x^i, q^i), i = \overline{1, m}$ in TM induced from coordinates (x^i) in the base M , the structure functions of this Lie algebroid are $\rho_j^i = \delta_j^i$ and $C_{ij}^k = 0$.

One regards an element $a \in E$ as a generalized velocity, and the actual velocity v is obtained when applying the anchor to a , i.e. $v = \rho(a)$. A curve $a : [t_0, t_1] \rightarrow E$ is said to be *admissible* or an *E -path*, if $\dot{\gamma}(t) = \rho(a(t))$, where $\gamma(t) = \tau(a(t))$ is the base curve.

Given a Lie algebroid $\tau : E \rightarrow M$ we consider the vector bundle $\tau_1 : T^E E \rightarrow E$ where the total space of τ_1 is just $T^E E = \{(b, v) \in E \times TE \mid T\tau(v) = \rho(b)\}$ and projection τ_1 is given by $\tau_1(b, v) = \tau_E(v)$. We will use the redundant notation (a, b, v) for the element (b, v) where $a = \tau_E(v)$, so that τ_1 becomes the projection onto the first factor.

The vector bundle $\tau_1 : T^E E \rightarrow E$ can be endowed with a structure of Lie algebroid. The anchor $\rho^1 : T^E E \rightarrow TE$ is just the projection onto the third factor, i.e. $\rho^1(a, b, v) = v$. The local coordinate system (x^α, y^i) on E induces a local coordinate system $(x^\alpha, y^i, z^i, v^i)$ on $T^E E$, where z^i are the components of b in the basis $\{e_i\}$ and v^i are given by the coordinate expression of v , i.e. $b = z^i e_i$ and $v = \rho_i^\alpha z^i \frac{\partial}{\partial x^\alpha} + v^i \frac{\partial}{\partial y^i}$.

Every section $\eta \in Sec(E)$ can be lifted to a section η^c of $T^E E$ given by $\eta^c(a) = (a, \eta(m), v)$ where $v \in T_a E$ is the vector that projects to $\rho(\eta(m))$ and satisfies the condition $v\hat{\theta} = \widehat{d_\eta \theta}$, for every section θ of the dual bundle E^* .

In this expression, $\widehat{\theta} \in C^\infty(E)$ is the linear function associated to the section $\theta \in Sec(E^*)$, d_η is the Lie derivative with respect to a section η of E , see [9]. There exists a canonical map $\chi_E : \mathcal{T}^E E \rightarrow \mathcal{T}^E E$ such that $\chi_E^2 = Id$, called the *canonical involution*. It is defined by $\chi_E(a, b, v) = (b, a, v)$ for every $(a, b, v) \in \mathcal{T}^E E$.

In terms of the canonical involution, the complete lift of a section $\eta \in Sec(E)$ is given by

$$(2.3) \quad \eta^c(a) = \chi_E(\eta(m), a, T_m \eta(\rho(a))).$$

We will make extensive use of the following map. Given an admissible curve a in E over $\gamma = \tau \circ a$, we consider the map $\Xi_a : Sec_\gamma(E) \rightarrow Sec_a(TE)$ given by

$$(2.4) \quad \Xi_a(\sigma) = \rho^1(\chi_E(\sigma, a, \dot{\sigma})).$$

In other words, it is determined by $\chi_E(\sigma, a, \dot{\sigma}) = (a, v, \Sigma_a(\sigma))$. From the definition, it is easy to prove the following property

$$(2.5) \quad \Xi_a(f\sigma) = f\Xi_a(\sigma) + \dot{f}\sigma_a^v$$

for every function $f \in C^\infty(R)$, and σ_a^v denotes the canonical vertical lift of σ .

In local coordinates, if $\eta = \eta^i e_i$ is a local section of E , then the vector field associated to its complete lift has the local expression

$$(2.6) \quad \rho^1(\eta^c) = \rho_i^\alpha \eta^i \frac{\partial}{\partial x^\alpha} + (\rho_i^\alpha y^i \frac{\partial \eta^j}{\partial x^\alpha} + C_{kl}^j y^k \eta^l) \frac{\partial}{\partial y^i}.$$

The canonical involution is locally given by

$$(2.7) \quad \chi_E(x^\alpha, y^i, z^i, v^i) = (x^\alpha, z^i, y^i, v^i + C_{jk}^i z^j y^k).$$

The local expression of the map Ξ_a is

$$(2.8) \quad \Xi_a(\sigma)(t) = \rho_i^\alpha(\gamma(t)) \sigma^i(t) \frac{\partial}{\partial x^\alpha} |_{(\gamma^\alpha, a^i)} + (\dot{\sigma}^i(t) + C_{jk}^i(\gamma(t)) a^j(t) \sigma^k(t)) \frac{\partial}{\partial y^i} |_{(\gamma^\alpha, a^i)},$$

where a and σ has the local expressions $a(t) = (\gamma^\alpha(t), a^i(t))$ and $\sigma(t) = (\gamma^\alpha(t), \sigma^i(t))$.

Let $I = [0, 1]$ and $J = [t_0, t_1]$ and we denote the coordinates in R^2 by (s, t) . When we write $TI \times TJ$ we mean $TR_{I \times J}^2$ the restriction of the tangent bundle of R^2 to $I \times J$.

Two E -paths a_0 and a_1 are said to be *homotopic*, if there exists a morphism of Lie algebroids $\Phi : TI \times TJ \rightarrow E$ with the following properties:

$$(2.9) \quad \begin{cases} \Phi(\frac{\partial}{\partial t})|_{(0,t)} = a_0(t), & \Phi(\frac{\partial}{\partial s})|_{(s,t_0)} = 0 \\ \Phi(\frac{\partial}{\partial t})|_{(1,t)} = a_1(t), & \Phi(\frac{\partial}{\partial s})|_{(s,t_1)} = 0 \end{cases}$$

We will say that Φ is an E -homotopy from the E -path a_0 to the E -path a_1 .

The set of all E -paths defined on the interval J will be denoted by $\mathcal{A}(J, E)$.

It is clear that E -homotopy is an equivalence relation on $\mathcal{A}(J, E)$. It was proved in [Martinez] that every E -homotopy class is a Banach submanifold of $\mathcal{A}(J, E)$ with dimension equal to the dimension of E . The partition into E -homotopy classes defines a foliation on the set $\mathcal{A}(J, E)$. We consider in the set $\mathcal{A}(J, E)$ the structure of differentiable Banach manifold induced by the above foliation. The manifold obtained in this way is denoted by $\mathcal{P}(J, E)$ and is called the *space of E -paths on the Lie algebroid E* .

3 Fractional Euler - Lagrange equations on the space $\mathcal{P}(J, E)$

With the manifold structure defined on the space $\mathcal{P}(J, E)$ we can formulate the fraction action variation principle.

Let us fix two points $m_0, m_1 \in M$, and consider the set $\mathcal{P}(J, E)_{m_0}^{m_1}$ of those E -paths with fixed base endpoints to m_0 and m_1 , that is:

$$(3.1) \quad \mathcal{P}(J, E)_{m_0}^{m_1} = \{\mathcal{P}(J, E) | \tau(a(t_0)) = m_0 \text{ and } \tau(a(t_1)) = m_1\}.$$

We remark that $\mathcal{P}(J, E)_{m_0}^{m_1}$ is a Banach submanifold of $\mathcal{P}(J, E)$.

Theorem 1. *Let $L \in C^\infty(E)$ be a Lagrangian function on the Lie algebroid E and fix two points $m_0, m_1 \in M$. Consider the action functional $S^r : \mathcal{P}(J, E) \rightarrow R$ given by:*

$$(3.2) \quad S^r(a) = \frac{1}{\Gamma(r)} \int_{t_0}^{t_1} L(a(t))(\theta - t)^{r-1} dt,$$

where $r \in (0, 1]$, θ is the observer time, Γ is the Euler gamma function. The critical points of S^r on the Banach manifold $\mathcal{P}(J, E)_{m_0}^{m_1}$ are exactly those elements of that space which satisfy the following fractional Euler - Lagrange equations:

$$(3.3) \quad \delta^r L(\dot{a}(t)) = 0,$$

where

$$(3.4) \quad \langle \delta^r L(\dot{a}(t)), \sigma(t) \rangle = \langle dL, \Sigma_a(\sigma) \rangle - \frac{d}{dt} \langle \theta_L \circ a, \sigma \rangle - \frac{1-r}{\theta-t} \langle \theta_L \circ a, \sigma \rangle.$$

Proof. The action functional S^r is a smooth function on $\mathcal{P}(J, E)_{m_0}^{m_1}$. The tangent space to such manifold at $a \in \mathcal{P}(J, E)_{m_0}^{m_1}$ is F_a , i.e. the set of vector fields along a of the form $\Xi_a(\sigma)$ for $\sigma \in Sec_\gamma(E)$ with $\sigma(t_0) = \sigma(t_1) = 0$. Taking account that $\Xi_a(f\sigma) = f\Sigma_a(\sigma) + \dot{f}\sigma_a^v$, for every function $f : J \rightarrow R$, and following the steps in [], we get (here follows that d is the usual (Fréchet) differential of a function on a manifold):

$$\begin{aligned} 0 &= \langle dS^r(a), \Xi_a(f\sigma) \rangle = \frac{1}{\Gamma(r)} \int_{t_0}^{t_1} \langle dL, \Xi_a(f\sigma) \rangle (\theta - t)^{r-1} dt = \\ &= \frac{1}{\Gamma(r)} \int_{t_0}^{t_1} [f(t) \langle dL, \Xi_a(\sigma) \rangle + \dot{f}(t) \langle dL, \sigma_a^v \rangle] (\theta - t)^{r-1} dt = \\ &= \frac{1}{\Gamma(r)} \left(\int_{t_0}^{t_1} [f(t) \langle dL, \Xi_a(\sigma) \rangle - \frac{d}{dt} \langle \theta_L \circ a, \sigma \rangle (\theta - t)^{r-1} + (r-1) \langle \theta_L \circ a, \sigma \rangle \right. \\ &\quad \left. (\theta - t)^{r-2} \right] dt + \end{aligned}$$

$$+f(t) \langle \theta_L \circ a, \sigma \rangle (\theta - t)^{r-1} \Big|_{t_0}^{t_1} = \frac{1}{\Gamma(r)} \int_{t_0}^{t_1} f(t) \langle \delta^r L(\dot{a}(t)), \sigma(t) \rangle dt,$$

where $\delta^r L$ is given by (15). Since this holds for every function f and every section $\sigma \in \text{Sec}_\gamma(E)$ it follows that the critical points are determined by the equation $\delta^r L(\dot{a}(t)) = 0$, that is, the fractional Euler - Lagrange equation. \square .

In local coordinates the θ_L is given by $\theta_L = \frac{\partial L}{\partial y^i} dy^i$, and the fractional Euler - Lagrange equation is:

$$(3.5) \quad \begin{cases} \frac{d}{dt} \left(\frac{\partial L}{\partial y^i} \right) + \frac{\partial L}{\partial y^k} C_{ij}^k y^j - \rho_i^\alpha \frac{\partial L}{\partial x^\alpha} - \frac{1-r}{\theta-t} \frac{\partial L}{\partial y^i} = 0 \\ \dot{x}^\alpha = \rho_i^\alpha y^i \end{cases}$$

If we take $r = 1$ in the Theorem 1, we obtain Theorem 4 from [1].

If the Lagrangian L is hyperregular, then from (16) result the *fractional Hamilton equations*:

$$(3.6) \quad \begin{cases} \dot{x}^\alpha = \rho_i^\alpha \frac{\partial H}{\partial p_i} \\ \dot{p}_i = \frac{r-1}{\theta-t} p_i - \rho_i^\alpha \frac{\partial H}{\partial x^\alpha} - C_{ij}^k p_k \frac{\partial H}{\partial p_j}, \end{cases}$$

where $p_i = \frac{\partial L}{\partial y^i}$, and $H = p_i y^i - L$.

The *fractional Hamilton dynamics* D^r on E^* is represented by the vector field:

$$(3.7) \quad D^r(x, \xi) = \rho_i^\alpha \frac{\partial H}{\partial p_i} \frac{\partial}{\partial x^\alpha} + \left(\frac{r-1}{\theta-t} p_i - \rho_i^\alpha \frac{\partial H}{\partial x^\alpha} - C_{ij}^k p_k \frac{\partial H}{\partial p_j} \right) \frac{\partial}{\partial \xi_i}.$$

Example 2. (a) *r-geodesics*. The simplest hyperregular Lagrangian on E is given by a symmetric positive - definite metric g in E . The metric induces an isomorphism of the vector bundles $\tilde{g} : E \rightarrow E^*$ which, in turn, induces an isomorphism of corresponding tensor bundles. With respect to this isomorphism the metric g corresponds to a contravariant metric G . In local coordinates have

$$g = g_{ij}(x) e^i \otimes e^j, \quad G = g^{ij}(x) e_i \otimes e_j, \quad g_{ij} g^{jk} = \delta_i^k, \quad L(x, y) = \frac{1}{2} g_{ij}(x) y^i y^j.$$

The equations (16) read:

$$(3.8) \quad \begin{cases} \frac{d}{dt} (g_{ik} y^i) = \frac{1-r}{\theta-t} g_{ik} y^i + (C_{ik}^j g_{sj} + \frac{1}{2} \rho_k^\alpha \frac{\partial g_{ij}}{\partial x^\alpha}) y^i y^j, k = \overline{1, n} \\ \dot{x}^\alpha = \rho_i^\alpha(x) y^i \end{cases}$$

The last equation can be rewritten in the form:

$$(3.9) \quad \dot{y}^l + \Gamma_{ij}^l(x, \theta, r) y^i y^j - \frac{1-r}{\theta-t} y^l = 0,$$

where

$$(3.10) \quad \Gamma_{ij}^l = \frac{1}{2}g^{kl}(\rho_j^\alpha \frac{\partial g_{ik}}{\partial x^\alpha} + \rho_i^\alpha \frac{\partial g_{jk}}{\partial x^\alpha} - \rho_k^\alpha \frac{\partial g_{ij}}{\partial x^\alpha} - C_{ik}^s g_{sj} - C_{jk}^s g_{si}.$$

The equations

$$(3.11) \quad \begin{cases} \dot{x}^\alpha = \rho_i^\alpha(x)y^i \\ \dot{y}^l + \Gamma_{ij}^l y^i y^j - \frac{1-r}{\theta-t} y^l = 0, \end{cases}$$

with $\Gamma_{ij}^l = \Gamma_{ij}^l(x, \theta, r)$ as in (21) are r - geodesics for $\tau : E \rightarrow M$. $E = TM$ in adapted coordinates we have $C_{ij}^l = 0, \rho_i^k = \delta_i^k$, the r - - geodesic equations read:

$$(3.12) \quad \ddot{x}^l + \Gamma_{ij}^l \dot{x}^i \dot{x}^j - \frac{1-r}{\theta-t} \dot{x}^l = 0,$$

where $\Gamma_{ij}^l = \Gamma_{ij}^l(x, \theta, r)$ being the Christoffel symbols of the Levi - Civita connection associated with the metric g .

(b) **Wong r - equations.** Consider an algebroid E which is the direct product $E = TM \times \mathcal{A}$ of the canonical Lie algebroid TM over a manifold M of dimension m and an arbitrary Lie R - algebra \mathcal{A} of dimension n . Both anchors coincide with the projection pr_1 on the first factor, i.e. on TM . For local coordinates (x^α) in M and a basis (v_i) of \mathcal{A} we have the adapted coordinates (x^a, \dot{x}^b, v^i) in E , where (v^i) is the basis in \mathcal{A}^* dual to (v_i) . Here we understand TM and $M \times \mathcal{A}$ as subbundles of E according to the natural immersions I_1 and I_2 . Let us assume additionally that we have a Riemannian metric g on M and a metric h on \mathcal{A} .

These metrics induce a hyperregular quadratic Lagrangian L on E given by:

$$(3.13) \quad L(x, \dot{x}, \bar{v}) = \frac{1}{2}(h_{ij}\bar{v}^i \bar{v}^j + g_{\alpha\beta} \dot{x}^\alpha \dot{x}^\beta).$$

The fraction Euler - Lagrange for (16) is given by:

$$(3.14) \quad \begin{cases} \ddot{x}^\beta = \Gamma_{\lambda\epsilon}^\beta \dot{x}^\lambda \dot{x}^\epsilon + \frac{1-r}{\theta-t} \dot{x}^\beta \\ \dot{\bar{v}}^k = C_{ji}^l h^{ik} h_{ls} \bar{v}^j \bar{v}^s + \frac{1-r}{\theta-t} \bar{v}^k. \end{cases}$$

The equations (25) are exactly the fraction Wong equations.

Remark The concept of Leibniz algebroid is more generally that Lie algebroid (see [7], [10]). We can also describe the fractional Euler-Lagrange equations and fractional Wong equations for Leibniz algebroids. These results will be presented in an other paper.

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Authors' address:

Gheorghe Ivan, Mihai Ivan and Dumitru Opreş
West University of Timișoara, Department of Mathematics,
4 Bd. V. Pârvan, 300223, Timișoara , Romania.
e-mail : ivan@math.uvt.ro, opris@math.uvt.ro