Dilations on Hilbert C^*- modules for C^*- dynamical systems

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Abstract. In this paper we investigate the dilations of a contractive covariant representation of a C^*- dynamical system on Hilbert C^*- modules. We prove that if $\alpha\colon A \to A$ is an injective C^*- morphism of C^*- algebras which has a strict positive transfer operator τ, then any covariant representation (φ, T, E) of the pair (A, α) on the Hilbert C^*-module E admits a coisometric dilation (Φ, V, F) adapted to τ and an isometric dilation (Ψ, W, G). These extend some results of Muhly and Solel ([3]) in the context of Hilbert C^*- modules and without assuming that the C^*- algebra A is unital.

M.S.C. 2000: 46L05, 46L08, 46L55, 47L55.
Key words: completely positive linear map, strict transfer operator, contractive covariant representation, dilation, defect operator.

1 Introduction

Hilbert C^*-modules are generalizations of Hilbert spaces by allowing the inner- product to take values in a C^*- algebra rather than in the field of complex numbers, but there are some differences between these two classes. For example, each closed submodule of a Hilbert space is complemented, but a closed submodule of a Hilbert C^*-module is not complemented in general [1, chapter 3].

Definition 1.1. A *pre-Hilbert A-module* is a complex vector space E which is also a right A-module, compatible with the complex algebra structure, equipped with an A-valued inner product $\langle \cdot, \cdot \rangle : E \times E \to A$ which is $\mathbb{C}$ -and A-linear in its second variable and satisfies the following relations:

1. $\langle \xi, \eta \rangle^* = \langle \eta, \xi \rangle$ for every $\xi, \eta \in E$;
2. $\langle \xi, \xi \rangle \geq 0$ for every $\xi \in E$;
3. $\langle \xi, \xi \rangle = 0$ if and only if $\xi = 0$.

Proceedings of The 4-th International Colloquium "Mathematics in Engineering and Numerical Physics" October 6-8 , 2006, Bucharest, Romania, pp. 81-86.
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We say that E is a *Hilbert A-module* if E is complete with respect to the topology determined by the norm $\|\cdot\|$ given by $\|\xi\| = \sqrt{\|\langle\xi,\xi\rangle\|}$.

Given two Hilbert C^*- modules E and F over a C^*- algebra B, the Banach space of all bounded module homomorphisms from E to F is denoted by $\mathcal{B}_B(E,F)$. The subset of $\mathcal{B}_B(E,F)$ consisting of all adjointable module homomorphisms from E to F (that is, $T \in \mathcal{B}_B(E,F)$ such that there is $T^* \in \mathcal{B}_B(F,E)$ satisfying $\langle\eta, T\xi\rangle = \langle T^*\eta, \xi\rangle$ for all $\xi \in E$ and for all $\eta \in F$) is denoted by $\mathcal{L}_B(E,F)$. We will write $\mathcal{B}_B(E)$ for $\mathcal{B}_B(E,E)$ and $\mathcal{L}_B(E)$ for $\mathcal{L}_B(E,E)$.

In general $\mathcal{B}_B(E,F) \neq \mathcal{L}_B(E,F)$.

Given a countable family $\{E_n\}_n$ of Hilbert B-modules, the vector space $\bigoplus\limits_n E_n = \Big\{(\xi_n)_n/\xi_n \in E_n, \sum\limits_n < \xi_n, \xi_n > \text{ converges in } B\Big\}$ becomes a Hilbert B- module with $(\xi_n)_n \cdot a = (\xi_n a)_n$ and $< (\xi_n)_n, (\eta_n)_n >= \sum\limits_n < \xi_n, \eta_n >$.

A submodule E_0 of E is complemented if $E_0 \oplus E_0^\perp = E$, where

$$E_0^\perp = \big\{\eta \in E / < \eta, \xi >= 0, \forall \xi \in E_0\big\}.$$

A representation of a C^*- algebra A on a Hilbert C^*-module E over B is a *-morphism Φ from A to $\mathcal{L}_B(E)$. The representation Φ is nondegenerate if $\Phi(A)E$ is dense in E.

Definition 1.2. Let A and B be two C^*-algebras and E a Hilbert B- module. Let $M_n(A)$ denote the $*$ -algebra of all $n \times n$ matrices over A with the algebraic operations and the topology obtained by regarding it as a direct sum of n^2 copies of A. A *completely positive linear map* is a linear map $\rho : A \to \mathcal{L}_B(E)$ such that the linear map $\rho^{(n)} : M_n(A) \to M_n(\mathcal{L}_B(E))$ defined by

$$\rho^{(n)}\left([a_{ij}]_{i,j=1}^n\right) = [\rho(a_{ij})]_{i,j=1}^n$$

is positive for any positive integer n.

Definition 1.3. A completely positive linear map is *strict* if $(\rho(e_\lambda))_\lambda$ is strictly Cauchy in $\mathcal{L}_B(E)$, for some approximate unit $(e_\lambda)_\lambda$ of A.

2 The main results

Definition 2.1. Let A be a C^*- algebra and let $\alpha\colon A \to A$ be an injective C^*-morphism. A *strict transfer operator for* α is a strict completely positive linear map $\tau\colon A \to A$ such that $\tau \circ \alpha = \mathrm{id}_A$.

Proposition 2.1. *Let A be a C^*- algebra, let $\varphi\colon A \to \mathcal{L}_B(E)$ be a nondegenerate representation of A on the Hilbert C^*- module E over a C^*- algebra B and let $\alpha\colon A \to A$ be an injective C^*- morphism which has a strict transfer operator τ.*

1. There is a Hilbert B- module E_τ, a representation Φ_τ of A on E_τ and an adjointable operator $V_\tau\colon E \to E_\tau$ such that

(a) $\varphi(a) = V_\tau^*\Phi_\tau(\alpha(a))V_\tau$, for all $a \in A$

(b) $\varphi(\tau(a)) = V_\tau^* \Phi_\tau(a) V_\tau$*, for all* $a \in A$

(c) $\Phi_\tau(A) V_\tau E$ *is dense in* E_τ

2. If Φ *is a representation of* A *on a Hilbert* B*- module* F *and* $V \colon E \to F$ *is an adjointable operator such that*

(a) $\varphi(a) = V^* \Phi(\alpha(a)) V$*, for all* $a \in A$

(b) $\varphi(\tau(a)) = V^* \Phi(a) V$*, for all* $a \in A$

(c) $\Phi(A) V E$ *is dense in* F

then there is a unitary operator $U \colon E_\tau \to F$ *such that*

(i) $U \Phi_\tau(a) = \Phi(a) U$*, for all* $a \in A$
and

(ii) $U V_\tau = V$.

Proof. 1. Let $\rho = \varphi \circ \tau$. Since φ is a nondegenerate representation of A, there is a morphism $\overline{\varphi} \colon M(A) \to \mathcal{L}_B(E)$ which is strictly continuous on the unit ball and $\overline{\varphi}|_A = \varphi$ [1, Proposition 2.5], and since τ is strictly there is $\overline{\tau} \colon M(A) \to M(A)$ a completely positive linear map which is strictly continuous on the unit ball and $\overline{\tau}|_A = \tau$ [1, Corollary 5.7]. Then $\overline{\varphi} \circ \overline{\tau} \colon M(A) \to \mathcal{L}_B(E)$ is a completely positive map which is continuous on the unit ball and $\overline{\varphi} \circ \overline{\tau}|_A = \rho$. From these facts and [1, Corollary 5.7] we conclude that ρ is a strict completely positive linear map from A to $\mathcal{L}_B(E)$.

Let $(\Phi_\tau, V_\tau, E_\tau)$ be the KSGNS representation of A associated with ρ [1, theorem 5.6]. Thus, we showed that there is a representation Φ_τ of A on a Hilbert B- module E_τ and an isometry $V_\tau \colon E \to E_\tau$ which verify the relations (b) and (c).

Let $a \in A$. Then

$$\varphi(a) = \varphi(\tau(\alpha(a))) = \rho(\alpha(a)) = V_\tau^* \Phi_\tau(\alpha(a)) V_\tau$$

and the assertion is proved.

2. By [1, Theorem 5.6], there is a unitary operator $U \colon E_\tau \to F$ which verifies the relations (i) and (ii). □

Remark 2.1. Taking into account the definitions of τ and Φ_τ [see proof of Theorem 5.6, 1] it is not difficult to check that $V_\tau V_\tau^*$ commutes with $\Phi_\tau(\alpha(a))$.

Definition 2.2. The representation $(\Phi_\tau, V_\tau, E_\tau)$ of A constructed above is called *the extension of* (φ, E) *adapted to* τ.

Remark 2.2. The extension of a representation φ adapted to τ is unique up to a unitary equivalence.

Definition 2.3. Let A be a C^*- algebra, let $\alpha \colon A \to A$ be an injective C^*- morphism. A *contractive* (resp. *isometric*, resp. *coisometric*, resp. *unitary*) *covariant representation* of the pair (A, α) on a Hilbert C^*- module is a triple (φ, T, E) consisting of a representation φ of A on a Hilbert C^*- module E over B and a contraction (resp. isometric, resp. coisometric, resp. unitary) operator T in $\mathcal{L}_B(E)$ such that

$$T\varphi(\alpha(a)) = \varphi(a)T$$

Definition 2.4. Let A be a C^*- algebra, let $\alpha\colon A \to A$ be an injective C^*- morphism and let (φ, T, E) be a contractive covariant representation of (A, α). A *coisometric* (resp. *isometric*, resp. *unitary*) *dilation* is a coisometric (resp. isometric, resp. unitary) covariant representation (Φ, V, F) on a Hilbert C^*- module F over B containing E as a complemented submodule such that $\Phi(a)E \subseteq E$ and $\Phi(a)|_E = \varphi(a)$ for all $a \in A$, $VE \subseteq E$ and $P_E V^n|_E = T^n$, where P_E is the projection of F on E .
Given a contractive covariant representation (φ, T, E) of a pair (A, α) such that α has a transfer operator τ, we construct a coisometric dilation of (φ, T, E) using the extension of φ adapted to τ.

The construction is done inductively and it is in the same manner as in [3, Theorem 1.2].

Given (φ, T, E) a contractive covariant representation of (A, α), we can choose $(\Phi_\tau, V_\tau, E_\tau)$ as in Proposition 2.1. and we will use the following notations:

$$\Delta_{T^*} = (I - TT^*)^{\frac{1}{2}}$$

$$\mathcal{D}_{T^*} = \overline{\Phi_\tau(A)V_\tau\Delta_{T^*}E} \subseteq E_\tau$$

$$D_{T^*} = \Delta_{T^*}V_\tau^*|_{\mathcal{D}_{T^*}}$$

Δ_{T^*} is called the *defect operator* of T^* and $\mathcal{D}_{T^*}$ is called the associated *defect space.*

Theorem 2.1. *Let A be a C^*- algebra, let $\alpha\colon A \to A$ be an injective C^*- morphism which has a transfer operator τ and let (φ, T, E) be a nondegenerate contractive covariant representation of (A, α) on a Hilbert C^*- module E over a C^*- algebra B. Then (φ, T, E) has a coisometric dilation adapted to τ, (Φ, V, F). Moreover, $VE \subseteq E$.*

Proof. Let $(\Phi_\tau, V_\tau, E_\tau)$ be the extension of φ adapted to τ constructed in Proposition 2.1. Clearly, $\Phi_\tau(A)\mathcal{D}_{T^*} \subseteq \mathcal{D}_{T^*}$ and $\widehat{\varphi}\colon A \to \mathcal{L}_B(\mathcal{D}_{T^*})$ defined by $\widehat{\varphi}(a) = \Phi_\tau(a)|_{\mathcal{D}_{T^*}}$ is a nondegenerate representation of A.

In the same manner as in the proof of [Theorem 1.2, 3], we conclude that $\left(\begin{bmatrix}\varphi & 0\\ 0 & \widehat{\varphi}\end{bmatrix}, \begin{bmatrix}T & D_{T^*}\\ 0 & 0\end{bmatrix}, E \oplus \mathcal{D}_{T^*}\right)$ is a contractive covariant representation of A such that $\begin{bmatrix}T & D_{T^*}\\ 0 & 0\end{bmatrix}$ is a partial isometry which restricted to E gives (φ, T, E).

This was the first step in the inductive construction.

Now, applying Proposition 2.1 to the representation $\widehat{\varphi}$ of A on $\mathcal{D}_{T^*}$, we obtain a Hilbert C^*- module E_1 over B, an adjointable operator $V_1\colon \mathcal{D}_{T^*} \to E_1$ and a representation $\Phi_1\colon A \to \mathcal{L}_B(E_1)$ such that $V_1^*\Phi_1(\alpha(a))V_1 = \widehat{\varphi}(a)$, for all $a \in A$.

Set :

$$\mathcal{D}_1 = \overline{\Phi_1(A)V_1\mathcal{D}_{T^*}} \subseteq E_1$$

$$D_1 = V_1^*|_{\mathcal{D}_1}$$

and

$$\widehat{\Phi}_1\colon A \to \mathcal{L}_B(\mathcal{D}_1),$$

defined by

$$\widehat{\Phi}_1(a) = \Phi_1(a)|_{\mathcal{D}_1}$$

Inductively, we obtain the sequences $(\mathcal{D}_k)_{k\geq 1}$, $(E_k)_{k\geq 1}$, $(V_k)_{k\geq 1}$, $(\Phi_k)_{k\geq 1}$, $(\widehat{\Phi}_k)_{k\geq 1}$, $(D_k)_{k\geq 1}$, where for $k > 1$:

$\mathcal{D}_k$ and E_k are Hilbert C^*- modules such that $\mathcal{D}_k = \overline{\Phi_k(A)V_k\mathcal{D}_{k-1}} \subseteq E_k$,
$V_k\colon \mathcal{D}_{k-1} \to E_k$ is an adjointable operator and $D_k = V_k^*|_{\mathcal{D}_k}$,
$\Phi_k\colon A \to \mathcal{L}_B(E_k)$ is a representation such that $V_k^*\Phi_k(\alpha(a))V_k = \Phi_{k-1}(a)$,
$\widehat{\Phi}_k\colon A \to \mathcal{L}_B(\mathcal{D}_k)$ is a representation such that $\widehat{\Phi}_k(a) = \Phi_k(a)|_{\mathcal{D}_k}$.

Set $F = E \oplus \mathcal{D}_{T^*} \oplus \mathcal{D}_1 \oplus \mathcal{D}_2 \oplus \dots$,

$$\Phi = \begin{bmatrix} \varphi & & & & \\ & \widehat{\varphi} & & & \\ & & \widehat{\Phi}_1 & & \\ & & & \widehat{\Phi}_2 & \\ & & & & \ddots \end{bmatrix} \text{ and } V = \begin{bmatrix} T & D_{T^*} & & & \\ & 0 & D_1 & & \\ & & 0 & D_2 & \\ & & & 0 & \ddots \\ & & & & \ddots \end{bmatrix}.$$

Clearly, F is a Hilbert B- module which contains E as a complemented submodule, Φ is a representation of A on F such that $\Phi(A)E \subseteq E$ and $\Phi(a)|_E = \varphi$ and V is an adjointable operator in $\mathcal{L}_B(F)$ such that $VE \subseteq E$ and $V|_E = T$.

Simple calculations show that $V\Phi(\alpha(a)) = \Phi(a)V$ for all $a \in A$, $VV^* = \mathrm{id}_F$ and $P_E V^n|_E = T^n$. Therefore (Φ, V, F) is a coisometric dilation adapted to τ of (φ, T, E). □

Proposition 2.2. *Let A be a C^*- algebra, let $\alpha\colon A \to A$ be an injective $\mathcal{C}^*$- morphism and let (φ, T, E) be a contractive covariant representation of (A, α) on a Hilbert C^*- module E over a C^*- algebra B. Then (φ, T, E) has an isometric dilation (Ψ, W, F). Further, if T is coisometric, then W is coisometric.*

Proof. Let $\Delta_T = (I - T^*T)^{\frac{1}{2}}$, the defect operator of T.

From $T\varphi(\alpha(a)) = \varphi(a)T$, for all $a \in A$ and taking into account that φ and α are C^*- morphisms, we deduce that $T^*T\varphi(\alpha(a)) = \varphi(\alpha(a))T^*T$ for all $a \in A$ and so $\Delta_T\varphi(\alpha(a)) = \varphi(\alpha(a))\Delta_T$ for all $a \in A$. Hence $(\varphi \circ \alpha)(A)\mathcal{D}_T \subseteq \mathcal{D}_T$, where $\mathcal{D}_T$ is the defect space of T.

Let $F = E \oplus \mathcal{D}_T \oplus \mathcal{D}_T \oplus \dots$. The map $\Psi\colon A \to \mathcal{L}_B(E)$ defined by

$$\Psi = \begin{bmatrix} \varphi & & & & \\ & \varphi \circ \alpha|_{\mathcal{D}_T} & & & \\ & & \varphi \circ \alpha^2|_{\mathcal{D}_T} & & \\ & & & \varphi \circ \alpha^3|_{\mathcal{D}_T} & \\ & & & & \ddots \end{bmatrix}$$

is a representation of A on F. Clearly, $\Psi(A)E \subseteq E$ and $\Psi(a)|_E = \varphi(a)$ for all $a \in A$.

Let

$$W = \begin{bmatrix} T & & & & \\ \Delta_T & 0 & & & \\ & I_{\mathcal{D}_T} & 0 & & \\ & & I_{\mathcal{D}_T} & & \\ & & & \ddots & \ddots \end{bmatrix}$$

Clearly, $P_E W^n|_E = T^n$ for all positive integer n.

A simple calculation shows that W is an isometry and moreover if T is a coisometry, W is a unitary.

It is not difficult to check that $W\Psi(\alpha(a)) = \Psi(a)W$ for all $a \in A$ and the proposition is proved. □

As in the case of dilation on Hilbert space [3, Corollary 1.3] we obtain the following corollary.

Corollary 2.1. *Let A be a C^*- algebra, let $\alpha\colon A \to A$ be an injective C^*- morphism which has a strict transfer operator τ and let (φ, T, E) be a contractive covariant representation of (A, α). Then (φ, T, E) has a unitary dilation (Ψ, W, F) adapted to τ.*

Proof. By Theorem 2.1, (φ, T, E) has a coisometric dilation (Φ, V, F) adapted to τ. Applying Proposition 2.2 to coisometric covariant representation (Φ, V, F) of (A, α), we obtain a unitary dilation (Ψ, U, G) of (Φ, V, F). It is not difficult to check that (Ψ, U, G) is a unitary dilation adapted to τ of (φ, T, E). □

Acknowledgement. This research was supported by grant CNCSIS code A 1065/2006.

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