Variational study of an elliptic boundary problem

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Abstract. Using the Lax-Milgram theorem and the techniques of the abstract functional analysis, we prove the existence and uniqueness of the solution of a boundary value problem for a non-homogeneous Helmholtz equation.

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Let us consider the Helmholtz's equation rewritten in the form

$$\frac{\partial^2 \varphi(x,y)}{\partial x^2} + \frac{\partial^2 \varphi(x,y)}{\partial y^2} + k^2 \varphi(x,y) = -f(x,y) \tag{1}$$

with an homogeneous boundary value problem

$$\varphi|_{\Gamma} = 0, \tag{2}$$

where the boundary Γ of D is a smooth contour, f is a given function in $\forall(x,y) \in D$ and k is a constant.

In order to get the solution φ of the problem (1)-(2), we define the following spaces:

(a) the Hilbert space $H = L_2(D)$ (the quadratically integrable functions) with the scalar product defined by the formula

$$(u,v) = \iint_D u(x,y)v(x,y)dxdy \tag{3}$$

(b) the Sobolev space defined by

$$W^{1,2}(D) = \{\tilde{u} \in L_2(D) | \exists g_1, g_2 \in L_2(D) \text{ such that}$$

$$\iint_D \tilde{u}\frac{\partial v}{\partial x} = -\iint_D g_1 v \quad \text{and} \quad \iint_D \tilde{u}\frac{\partial v}{\partial y} = -\iint_D g_2 v, \ \forall v \in C_c(D)\} \tag{4}$$

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where $C_c(D)$ is the space of the continuous functions with compact support.

Now, we define in the space $W^{1,2}(D)$ the scalar product

$$(5)\qquad (u,v)=(u,v)_{L_2}+\sum_{i=1}^{2}\left(\frac{\partial u}{\partial x_i},\frac{\partial v}{\partial x_i}\right)_{L_2}=\iint_D\left(uv+\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\frac{\partial v}{\partial y}\right)dxdy.$$

Here, the $W^{1,2}(D)$ is the prolongation of the space $C^1(D)$ with the limit points of the Cauchy sequences from this space, hence a Banach space, [1]. The space $W^{1,2}(D)$ becomes a Hilbert space $H^1(D)$ with the scalar product (5).

If the functions $\tilde{u}$ satisfy the boundary condition (2), then $H^1(D)$ becomes the Hilbert space $H_0^1(D)$.

Lemma 1. *Every classical solution of the problem* (1)-(2) *is a weak solution.*

Proof. A classical solution of the problem is a function $\varphi\in C^2(\bar{D})$, which verifies (1)-(2).

A weak solution of the same problem is a function $\varphi\in H_0^1(D)$ under the following condition:

$$(6)\qquad \iint_D \nabla_\varphi\cdot\nabla v-k^2\iint_D\varphi v=\iint_D fv,\quad \forall v\in H_0^1(D)$$

where

$$\nabla\varphi\cdot\nabla v=\frac{\partial\varphi}{\partial x}\frac{\partial v}{\partial x}+\frac{\partial\varphi}{\partial y}\frac{\partial v}{\partial y}\ \text{and}\ v|_\Gamma=0.$$

Let us consider $\varphi\in H^1(D)\cap C(\bar{D})$ and $\varphi=0$ on the boundary Γ. Then $\varphi\in H_0^1(D)$ and we shall prove that (6) is verified for every $v\in H_0^1(D)$.

For this, we multiply (1) by $v\in C_c^1(D)$ and integrate over the domain D. We have

$$(7)\qquad \iint_D\left(\frac{\partial^2\varphi}{\partial x^2}v+\frac{\partial^2\varphi}{\partial y^2}v\right)+k^2\iint_D\varphi v=-\iint_D fv.$$

Applying the Green's formula to the first term we get

$$(8)\qquad \begin{gathered}\iint_D v\Delta\varphi=\int_\Gamma\left(-\frac{\partial\varphi}{\partial y}vdx+\frac{\partial\varphi}{\partial x}vdy\right)-\iint_D\nabla v\cdot\nabla\varphi\Rightarrow\\ \iint_D v\Delta\varphi=-\iint_D\nabla v\nabla\varphi.\end{gathered}$$

Since $C_c^1(D)$ is dense in $W^{1,2}(D)$, we obtain from (7) and (8) for every $v\in H_0^1(D)(v|_\Gamma=0)$, the following equality

$$(9)\qquad -\iint_D\nabla\varphi\nabla v+k^2\iint_D\varphi v=-\iint_D fv$$

and lemma is proved. □

Definition. A bilinear form $a(u,\varphi) : H \times H \to \mathbf{R}$ is called:

1. *continuous*, if there exists a constant K_1 such that

$$|a(\varphi, v)| \le K_1 |\varphi|\, |v|, \quad \forall \varphi, v \in H; \tag{10}$$

2. *coercive*, if there exists a constant $\gamma > 0$ such that

$$a(\varphi, \varphi) \ge \gamma |\varphi|^2, \quad \forall \varphi \in H(D) \tag{11}$$

Lemma 2. *The bilinear form*

$$a(\varphi, v) = \iint_D \nabla\varphi \nabla v - k^2 \iint_D \varphi v \tag{12}$$

is continuous in $H^1(D) \times H^1(D)$.

Proof. Let us consider $K_1 = \max(1; k^2)$. Using the Cauchy-Schwarz inequality we get

$$|a(\varphi, v)| = \left| \iint_D \nabla\varphi\nabla v - k^2 \iint_D \varphi v \right| \le \iint_D |\nabla\varphi\nabla v - k^2 \varphi v| \le$$

$$\le K_1 \left(\iint_D [\varphi^2 + (\nabla\varphi)^2] \right)^{1/2} \left(\iint_D [v^2 + (\nabla v)^2] \right)^{1/2} = K_1 ||\varphi||_{H^1(D)} ||v||_{H^1(D)}.$$

It follows from (10) that the bilinear form (12) is continuous in $H^1(D) \times H^1(D)$. □

Lemma 3. *The bilinear form*

$$a(\varphi, v) = \iint_D \nabla\varphi \cdot \nabla v - k^2 \iint_D \varphi v \tag{13}$$

is coercive in $H^1_0(D)$.

Proof. In accordance with (13) we have

$$a(\varphi, \varphi) = \iint_D \left[\left(\frac{\partial \varphi}{\partial x} \right)^2 + \left(\frac{\partial v}{\partial y} \right)^2 \right] - k^2 \iint_D \varphi^2 \tag{14}$$

where $\varphi \in H^1_0$ (vanish on Γ).

We enclose the domain D in a rectangle D_1 (with sides α and β), whose two sides are the coordinate axes. Equating this function to zero, we extend it over the entire rectangle D_1. If (x_1, y_1) is an arbitrary point, we get

$$\varphi(x_1, y_1) = \int_0^{x_1} \frac{\partial \varphi(x, y_1)}{\partial x} dx + \varphi(0, y_1) = \int_0^{x_1} \frac{\partial \varphi(x, y_1)}{\partial x} dx.$$

Using the Cauchy-Schwarz inequality we get

$$\varphi^2(x_1, y_1) = \left(\int_0^{x_1} 1 \cdot \frac{\partial \varphi(x, y_1)}{\partial x} dx \right)^2 \le \int_0^{x_1} 1^2 dx \int_0^{x_1} \left(\frac{\partial \varphi(x, y_1)}{\partial x} \right)^2 dx =$$

$$= x_1 \int_0^{x_1} \left(\frac{\partial \varphi(x, y_1)}{\partial x} \right)^2 dx \le \alpha \int_0^{\alpha} \left(\frac{\partial \varphi(x, y_1)}{\partial x} \right)^2 dx = \alpha \cdot F(y_1).$$

Integrating over the entire rectangle D_1 we obtain

$$\iint_{D_1} \varphi^2(x_1, y_1) dx_1 dy_1 \leq \int_0^{\alpha} \alpha dx_1 \int_0^{\beta} F(y_1) dy_1 \leq \alpha^2 \iint_{D_1} \left(\frac{\partial \varphi(x, y)}{\partial x}\right)^2 dxdy.$$

Analogously,

$$\iint_{D_1} \varphi^2(x_1, y_1) dx_1 dy_1 \leq \beta^2 \iint_{D_1} \left(\frac{\partial \varphi(x, y)}{\partial y}\right)^2 dxdy.$$

Therefore

$$\begin{aligned} \iint_D \left[\left(\frac{\partial \varphi}{\partial x}\right)^2 + \left(\frac{\partial \varphi}{\partial y}\right)^2\right] dxdy &\geq \frac{\alpha^2 + \beta^2}{\alpha^2 \beta^2} \iint_D \varphi^2(x, y) dxdy \geq \\ &\geq \frac{2}{\alpha\beta} \iint_D \varphi^2(x, y) dxdy \end{aligned} \tag{15}$$

Thus, we obtained the Friedrichs inequality for our problem. If $A = \max(\alpha, \beta)$, we find from (14) and (15) the following inequlity

$$a(\varphi, \varphi) \geq \left(\frac{2}{A^2} - k^2\right) \iint_D \varphi^2 = \gamma \iint_D \varphi^2, \quad \forall \varphi \in H_0^1(D) \tag{16}$$

In view of the definition, the bilinear form is coercive if $k \in \left(0, \dfrac{\sqrt{2}}{A}\right)$. □

Theorem (Lax-Milgram, [1]). *Let* $a(\varphi, v)$ *be a bilinear form,* $a : H_0^1 \times H_0^1 \to \mathbf{R}$, *which is continuous and coercive. Then, for every* $f \in L_2(D)$ *exists an unique* $\varphi \in H_0^1$ *such that*

$$a(\varphi, v) = \iint_D fv, \quad \forall v \in H_0^1(D). \tag{17}$$

Moreover, if $a(\varphi, v)$ *is symmetric we find* $\varphi \in H_0^1$ *by*

$$\frac{1}{2} a(\varphi, \varphi) - (f, \varphi) = \min_{v \in H_0^1} \left\{\frac{1}{2} \iint_D (|\nabla v|^2 - k^2 v^2) - \iint_D fv\right\}. \tag{18}$$

Theorem 1. *If* $f \in L_2(D)$, *then the weak solution of the problem*

$$\Delta \varphi(x, y) + k^2 \varphi(x, y) = -f(x, y) \tag{19}$$

$$\varphi|_{\Gamma} = 0 \tag{20}$$

exists and is unique.

Proof. It follows from Lemma 2 and Lemma 3 that $a(\varphi, v)$ is continuous and coercive in D. Applying the theorem Lax-Milgram's for $f \in L_2(D)$, we get an unique weak solution of our boundary problem. □

Theorem 2. *If* $\varphi \in H_0^1(D) \subset L_2(D)$ *is a weak solution of* (19)-(20), *then* $\varphi \in H^2(D)$.

Proof. Let us consider the function v of the form

$$v(x,y) = \begin{cases} e^{-\varphi}, & \forall (x,y) \in D \\ 0, & \forall (x,y) \notin D \end{cases}$$

where $\varphi \in H_0^1(D) \subset L_2(D)$.

If the boundary Γ of D is a smooth contour, it is sufficient that $v \in H^1(D)$ and it is not necessary that $v \in C(\bar{D})$, [1]. We shall show that starting in (4) with the function v and $\tilde{u} = \dfrac{\partial \varphi}{\partial x} \in L_2(D)$, we obtain

$$\iint_D \frac{\partial \varphi}{\partial x}\frac{\partial v}{\partial x} = -\iint_D \left(\frac{\partial \varphi}{\partial x}\right)^2 e^{-\varphi}.$$

Hence, there exits a function

$$g_1 = \left(\frac{\partial \varphi}{\partial x}\right)^2 \in L_2(D) \quad \text{such that} \quad \iint_D \frac{\partial \varphi}{\partial x}\frac{\partial v}{\partial x} = -\iint_D g_1 v. \tag{21}$$

Analogously, there exists

$$g_2 = \left(\frac{\partial \varphi}{\partial y}\right)^2 \in L_2(D) \quad \text{such that} \quad \iint_D \frac{\partial \varphi}{\partial y}\frac{\partial v}{\partial y} = -\iint_D g_2 v. \tag{22}$$

According to (4), (21) and (22), it should be observed that $\dfrac{\partial \varphi}{\partial x} \in H^1(D)$ and $\dfrac{\partial \varphi}{\partial y} \in H^1(D)$ when $f \in L_2(D)$. Consequently, $f \in H^2(D)$.

Theorem 3. *Let the mild solution of* (19)-(20) *be* $\varphi \in C^2(D)$. *If* $f \in L_2(D)$, *then* φ *is a classical solution of the problem.* □

Proof. Let us consider $\varphi \in C^2(D)$, which verifies (6)

$$\iint_D \left(\frac{\partial \varphi}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial \varphi}{\partial y}\frac{\partial v}{\partial y}\right) - k^2 \iint_D \varphi v = \iint_D f v$$

$\forall v \in C_c^1(D)$ and $v|_\Gamma = 0$. Integrating by parts we get

$$\iint_D \left[\left(-\frac{\partial^2 \varphi}{\partial x^2} - \frac{\partial^2 \varphi}{\partial y^2}\right) - k^2 \varphi - f\right] v dx dy = 0.$$

Since $C_c^1(D)$ is dense in $L_2(D)$, we obtain that

$$\Delta \varphi(x,y) + k^2 \varphi(x,y) = -f(x,y) \tag{23}$$

$$\varphi|_\Gamma = 0 \tag{24}$$

everywhere in D. Since $\varphi \in C^2(D)$, the equation (23) is verified in $\bar{D}$. □

Conclusion. Many authors paid attention to the abstract variational formulation of the boundary problem for partial differential equations, [6], [9], [12], [13]. An entertaining and complete survey of the results obtained in this field of functional analysis appears in [1]. Applications of the theory of semi-groups of linear operators to differential equations are presented by Pazy in [13].

An exact solution for the problem (1)-(2) has been presented in [15]. If $D = [0,a] \times [0,b]$, it is of the form

$$\varphi(x,y) = \int_0^a \int_0^b f(x,y) G(x,y,\xi,\eta) d\xi d\eta \tag{25}$$

where

$$G(x,y,\xi,\eta) = \frac{2}{a} \sum_{n=1}^{\infty} \frac{\sin(p_x x)(\sin(p_n \xi)}{\beta_n \sinh(\beta_n b)} \cdot H_n(y,\eta), p_n = \frac{\pi n}{a},$$

$$\beta_n = \sqrt{p_n^2 - k^2},\ a \geq b$$

$$H_n(y,\eta) = \begin{cases} \sinh(\beta_n \eta) \sinh \beta_n (b-y),\ b \geq y > \eta \geq 0 \\ \sinh(\beta_n y) \sinh \beta_n (b-\eta),\ b \geq \eta > y \geq 0 \end{cases},\ n = 1,2,\ldots \tag{26}$$

It should be observed that in our case, when we study the solution of a boundary problem using the techniques of the abstract functional analysis, the natural oscillating frequencies $k > 0$, belong to

$$k \in \left(0, \frac{\sqrt{2}}{A}\right) \subset \left(0, \frac{\pi}{A}\right),$$

the interval which was obtained from (26). Here A is the greatest side of the rectangle D_1.

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