

Theoretical research regarding the Floquet stability theorem with applications

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Abstract. The research focuses on the study of the motion stability for the dynamical systems described by differential equation systems with periodical coefficients, using the Floquet stability theorem. In the paper is related an original method for identification, in the plane of principal parameters of the mathematical model of the dynamical system, the stabilities and instabilities regions of the dynamical system motion. The original mathematical results are used for justify the method described here. The method is applied to study the motion stability of the couple pantograph – contact wire of the electrical locomotive. The parameters of the system consist of two concentrated masses, the bending stiffness, the horizontal tension, the viscous damping and the mass per unit length of the wire, the other damping coefficients and stiffness elements of the system and any constant speed specified in the model. We study the stability of the system using these parameters and our original method.

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1 Introduction

Firstly we describe some results about the differential linear equations and systems. Consider a linear differential equation of order for the unknown function:

$$(1.1) \quad y^{(n)} + a_{n-1}y^{(n-1)} + \dots + a_1y' + a_0y = f$$

where $a_{n-1}, \dots, a_1, a_0, f$ are functions defined on an interval $J \subset \mathbb{R}$ with complex values, and initial conditions $y(x_0) = y_0, y'(x_0) = y_1, \dots, y^{(n-1)}(x_0) = y_{n-1}$.

Using the notation $w_{n+1} = y^{(n)}$, $w_n = y^{(n-1)}$, ..., $w_2 = y'$, $w_1 = y$, the equation (1.1) can be written in a matrix form as

$$(1.2) \quad W' = AW + g$$

We present without proof [10], the following theorem:

Theorem 1.1. *If in matrix equation (1.2) the functions f, a_0, \dots, a_{n-1} are continuous on the definition interval $J \subset R$, then equation (1.2) has a unique solution $W(x)$, a column vector, so that $W(x_0) = W_0$.*

Are defined any n linear independent solutions of the homogenous system $W' = AW$ as the fundamental system of solutions. With the linear independent vectors, placed one after the other, one forms a square matrix W called the fundamental matrix of homogenous system which verifies the matrix equation $W' = AW$.

Theorem 1.2. *If W is any fundamental matrix of system $W' = AW$ then any solution of this system can be written as $w = Wc$ where c is a constant vector; if the initial condition is $w(x_0) = w_0$ then the solution is $w(x) = W(x)W^{-1}(x_0)w_0$. Any fundamental matrix of system can be deduced from another multiplying at right with a constant matrix.*

Theorem 1.3. *W is any fundamental matrix of the system $w' = Aw$, and $w(x_0) = w_0$, then any solution of the inhomogeneous system $w' = Aw + g$, $A, g \in C^0(J)$, is $w(x) = W(x)W^{-1}(x_0)w_0 + W(x) \int (W^{-1}g)(t) dt$.*

2 Matrix functions

We present in the following paragraphs some details about the matrix functions. For the beginning we consider the polynomial function. If $A \in M_n$ with proper values $\lambda_1, \dots, \lambda_n$ then $A^k \in M_n, k \in N$ and $p(A)$ is:

$$(2.1) \quad p(A) = A^m + b_1 A^{m-1} + \dots + A + b_m, b_j \in C, j = 1, n$$

For a matrix which admits a diagonal form, that means that there is a matrix D with non zero values only on its diagonal, and an invertible matrix S with $A = SD^kS^{-1}$ then $A^k = SD^kS^{-1}$; $p(A) = Sp(D)S^{-1}$, where:

$$p(D) = \begin{pmatrix} p(\lambda_1) & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & p(\lambda_n) \end{pmatrix}$$

We extend the matrix function definition for differentiable functions in a domain in C which contains the proper values of A . For a closed rectifiable curve γ which includes inside a point ζ , where g is differentiable, but which does not include a singularity of g , is known that $g(\zeta) = \frac{1}{2\pi i} \int_{\gamma} (\zeta - z)^{-1} g(z) dz$. We define $g(A)$ as $g(A) = \frac{1}{2\pi i} \int_{\gamma} (A - zI)^{-1} g(z) dz$, where γ is a closed rectifiable curve which includes the spectrum of A , but does not include any singularity of g . For

the exponential function $g(z) = e^{xz}$, $x, z \in C$, one defines $g(A) = e^{xA}$ as $g(A) = \frac{1}{2\pi i} \int_{\gamma} (A - zI)^{-1} e^{xz} dz$, where γ is a closed rectifiable curve which includes the proper values of A . We differentiate:

$$(2.2) \quad \begin{aligned} \frac{d}{dx} (e^{xA}) &= \frac{1}{2\pi i} \frac{d}{dx} \left(\int_{\gamma} (A - zI)^{-1} e^{xz} dz \right) = \\ &= \frac{1}{2\pi i} \int_{\gamma} (A - zI)^{-1} z e^{xz} dz = Ae^{xA} \end{aligned}$$

because the last integral is the matrix function for $g(z) = ze^{xz}$. We obtain that e^{xA} verifies the matrix equation $W' = AW$ and for $x = 0$ we have $e^{0A} = \frac{1}{2\pi i} \int_{\gamma} (A - zI)^{-1} 1 dz = I$, where is I the unit matrix.

The matrix e^{xA} is a fundamental matrix for the differential system: $W' = AW + g$, $A, g \in C^0(J)$ and the general solution, with the initial conditions $w(x_0) = w_0$, is: $w(x) = e^{(x-x_0)A} w_0 + e^{xA} \int e^{-tA} g dt$.

3 Differential systems with periodical coefficients

Consider the linear homogenous differential system $W' = AW$, $A \in M_n$, $A \in C^0(J)$, $J \subset R$.

We suppose that there is $p \in R_+$ so that $A(x+p) = A(x)$ for any $x \in J$. Then the system is periodic with the period p . We mention the following theorem (Floquet):

Theorem 3.1. *If the system $W' = AW$ is periodic, with the period $p \in R_+$ then any W , fundamental matrix of the system, can be expressed as $W(x) = W_1(x) e^{xR}$, where $W_1(x) \in M_n$ is a periodical matrix with the period p , and $R \in M_n$ is a constant matrix $R = \frac{1}{p} \ln(C)$, with constant matrix C defined by $W(x+p) = W(x)C$, $C \in M_n$*

4 Stability theory aspects

Consider the differential system $y' = Ay$, $A \in M_n$, with components defined and continuous on $I \subset R$. Consider also $t_0 \in I$ and $\tilde{y}_0 \in R^n$. From theorem 1, the solution $\tilde{y} : I \rightarrow R^n$, exists, it is unique, so that $\tilde{y}(t_0) = \tilde{y}_0$. Another solution $y : I \rightarrow R^n$ of the system with the initial condition $y(t_0) = y_0$, and y_0 different from \tilde{y}_0 , is called a perturbed solution of system, reported to \tilde{y} . This solution is called Lyapunov stable if for any ε positive exists δ so that, for $|y_0 - \tilde{y}_0| < \delta$ then $|y - \tilde{y}| < \varepsilon$ for any $t > t_0$, where $|y| = \max\{|y_1(t)|, \dots, |y_n(t)|; t \geq t_0\}$. If, supplementary $\lim_{t \rightarrow \infty} |y_j(t) - \tilde{y}_j| = 0$, for any $j = 1, \dots, n$ and $t \rightarrow \infty$, then the solution is called asymptotic stable.

Theorem 4.1. (Floquet) *If the system $W' = AW$ is periodic, with the period $p \in R_+$ and W , any fundamental matrix of the system, expressed as: $W(x) = W_1(x) e^{xR}$, where $W_1(x) \in M_n$ is a periodical matrix with the period p , and $R \in M_n$ is a constant matrix, then, if the proper values of R have negative real part, the solution of the periodical system is asymptotically stable, and if at least a proper value of the matrix R is strictly positive, the solution of the periodical system is unstable. If the proper values of the matrix R have zero real part, then the solution of the periodical system is undecided (stable, unstable or periodical).*

Theorem 4.2. *If $y : I \rightarrow R^n$ is a stable solution of the system $y' = Ay$, with periodical matrix and continuous components, defined by parameters, for fixed parameters, there is a neighbourhood of fixed parameters where the solution y is also stable. For an unstable solution of the system we can formulate an analogue property.*

This theorem is used for separation of the stable and unstable zones in the plane of principal parameters by curves of periodical solutions of the system.

5 Application

The dimensionless system of equations [6] that specifies the stated problem for the dynamical system described by pantograph and contact wire is:

$$\begin{aligned} (1 - \mu) \tilde{y}_3'' + 2\zeta_s \tilde{\omega}_{ns} (\tilde{y}_3' - \tilde{y}_2') + \tilde{\omega}_{ns}^2 (\tilde{y}_3 - \tilde{y}_2) + \omega_{nL}^2 \tilde{y}_3 + 2\zeta_L \tilde{\omega}_{nL} \tilde{y}_3' &= 0 \\ \mu \tilde{y}_2'' + (1 - \mu) \tilde{y}_3'' + & \\ + \tilde{\Omega}_n^2 \left(\tilde{y}_2 - \sum [T_j(\tau) + w_j] \frac{\sin j\Delta}{j\Delta} \sin j\tau \right) + \tilde{\omega}_{nL}^2 \tilde{y}_3 + 2\zeta_L \tilde{\omega}_{nL} \tilde{y}_3' &= 0 \\ \frac{d^2 T_j}{d\tau^2} + \frac{1}{\tilde{\nu}_\beta} \frac{dT_j}{d\tau} + \left(\frac{j^4}{\tilde{\nu}_{EI}^2} + \frac{j^2}{\tilde{\nu}_T^2} \right) T_j = & \\ = -2\tilde{M} (\mu \tilde{y}_2'' + (1 - \mu) \tilde{y}_3'' + \tilde{\omega}_{nL}^2 \tilde{y}_3 + 2\zeta_L \tilde{\omega}_{nL} \tilde{y}_3') \sin j\tau; \quad j = 1, \dots, 5 & \end{aligned}$$

with $\tilde{\nu}_{EI}^2 = \frac{mL^2}{EI\pi^2} \nu^2$, $\tilde{\nu}_T^2 = \frac{m}{T} v^2$, $\tilde{\nu}_\beta = \frac{m\pi}{\beta L} v$, where T is the tension in the wire and β is the viscous damping of the wire and where we consider known the initial conditions for the problem:

$$\tilde{y}_3(0) = \tilde{y}_{03}, \quad \tilde{y}_3'(0) = \tilde{y}'_{03}, \quad \tilde{y}_2(0) = \tilde{y}_{02}, \quad \tilde{y}_2'(0) = \tilde{y}'_{02}, \quad T_i(0) = T_{0i}, \quad T_i'(0) = T'_{0i}$$

Now we consider the participation of the external forces by additional values in the coefficients of the series development of the contact force between pantograph and contact wire, in the right hand of the third equation from the system.

A_j , $j \in N$ are the additional term of the coefficient of $\sin jt$ that appear in the third equation of motion. We perform an analysis with the following values of the parameters:

$$\begin{aligned} \tilde{\Omega}_n &= 4.77, \quad \zeta_s = 0.3, \quad \tilde{M} = 0.58, \quad \tilde{\nu}_\beta = 6.4, \quad \mu = 0.1, \quad \tilde{\omega}_{nL} = 0.72, \\ \zeta_L &= 0.45, \quad \tilde{\nu}_{EI} = 85.6 \end{aligned}$$

The dimensionless parameters in the plane of parameters are chosen, in this case $\tilde{\lambda}$, and $\tilde{\nu}_T$, where $\tilde{\lambda} = \frac{\tilde{\omega}_{nL}}{\tilde{\omega}_{ns}}$. We analyse the stability of motion for the concentrated mass with the displacement \tilde{y}_2 . In fig.1 is plotted with a continuous line the domain of the periodic solutions of \tilde{y}_2 for $\tilde{A}_j = 0$ and with a discontinuous line the domain for periodic solutions when $\tilde{A}_1 = 0.003$ and the others are zero.

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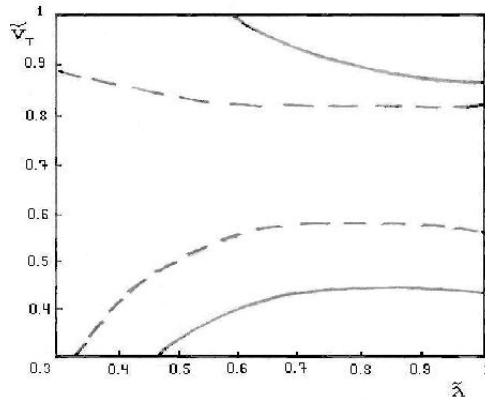


Fig.1 Stable and unstable zones in the plane of parameters

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