

Recent advances in the metric theory of a new continued fraction expansion

Gabriela Ileana Sebe

Abstract. A survey of the metric theory of a new continued fraction expansion discussed in Chan [1], Davison [2] and Sebe [6]-[9] is given. Using the optimal solution to the Gauss-Kuzmin-Lévy problem obtained by Iosifescu and Sebe [5] we derive some spectral properties of the Perron-Frobenius operator of the continued fraction transformation under its invariant measure.

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1 Introduction

One particular expansion in Davison [2], which attracted the attention of Chan [1], has raised to a new type of continued fractions which is quite different from the regular continued fractions.

Define for any $x \in I = [0, 1]$ the transformation

$$\tau(x) = 2^{\{(\log x^{-1})/\log 2\}} - 1, \quad x \neq 0; \quad \tau(0) = 0,$$

where $\{u\}$ denotes the fractionary part of a real u . For any $x \in [0, 1)$ put $a_n(x) = a_1(\tau^{n-1}(x))$, $n \in \mathbf{N}_+ = \{1, 2, \dots\}$, with $\tau^0(x) = x$ the identity map and $a_1(x) = [(\log x^{-1})/\log 2]$, $x \neq 0$; $a_1(0) = \infty$, where $[u]$ denotes the integer part of a real u . Then every irrational $x \in \Omega = I \setminus \mathbf{Q}$ has a unique infinite expansion

$$(1.1) \quad x = \frac{2^{-a_1(x)}}{1 + \frac{2^{-a_2(x)}}{1 + \frac{\dots}{\dots}}}} = [a_1(x), a_2(x), \dots].$$

The incomplete quotients or digits of $x \in \Omega$, $a_n(x)$, $n \in \mathbf{N}_+$ are natural numbers. We usually suppress the dependence on x in the notation of the digits a_1, a_2, \dots . Define

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$[a_1, \dots, a_n]$, $n \in \mathbf{N}_+$, by $[a_1] = 2^{-a_1}$, $[a_1, \dots, a_n] = \frac{2^{-a_1}}{1 + [a_2, \dots, a_n]}$, $n \geq 2$. It then appears that

$$(1.2) \quad x = \lim_{n \rightarrow \infty} [a_1, \dots, a_n]$$

which is the precise meaning of (1.1).

Let \mathcal{B} denote the σ -algebra of Borel subsets of I . The digits a_n , $n \in \mathbf{N}_+$, are non-negative integer-valued random variables which are defined almost surely on (I, \mathcal{B}) with respect to any probability measure on \mathcal{B} that assigns probability 0 to the set $I \setminus \Omega$ of rationals in I . Such a measure is Lebesgue measure λ . But a more important one is the invariant probability measure ν of the transformation τ . It was proved in Sebe ([6], Proposition 4) that

$$\nu(A) = \frac{1}{\log(4/3)} \int_A \frac{dx}{(x+1)(x+2)}, \quad A \in \mathcal{B}.$$

Hence, $\nu(A) = \nu(\tau^{-1}(A))$ for any $A \in \mathcal{B}$ and the sequence $(a_n)_{n \in \mathbf{N}_+}$ is strictly stationary on (I, \mathcal{B}, ν) .

2 Open questions and recent results

It is only recently (see Sebe [6]) that the ergodic properties of these expansions have been studied. It should be stressed that the ergodic theorem does not yield rates of convergence for mixing properties; for this a Gauss-Kuzmin theorem is needed.

It was proved in Sebe ([8], Proposition 3) that the random system with complete connections associated with this new continued fraction expansion is uniformly ergodic and its transition operator is regular with respect to the Banach space of Lipschitz functions. This leads further in Theorem 1 in [8] to a solution of a version of Gauss-Kuzmin problem for these continued fractions.

The study of optimality of the convergence rate remains an open question.

It was investigated in Sebe [7] the Perron-Frobenius operator of the continued fraction transformation τ under different probability measures on \mathcal{B} . The study focussed on the Perron-Frobenius operator of τ under the invariant measure ν induced by the limit distribution function. Using well-known general results (see Iosifescu and Grigorescu ([3], pp. 202 and 262-266), Iosifescu and Kraaikamp ([4], Subsection 2.1.2)), it was derived the asymptotic behaviour of this operator. It should be said that the asymptotic properties of the Perron-Frobenius operator $U : L^1_\nu \rightarrow L^1_\nu$, where $L^1_\nu = \left\{ f : I \rightarrow \mathbf{C} \mid \int_I |f| d\nu < \infty \right\}$, are not strong enough for to lead to a satisfactory solution to Gauss-Kuzmin problem, while when restricting U to $BEV(I) = \{f \mid f \in L^\infty(I, \mathcal{B}, \lambda), f \text{ has a version of bounded variation}\}$ they are substantially better. In the sequel the domain of U will be successively restricted to various Banach spaces.

Recall that the $\text{var}_A f$ over $A \subset I$ of a complex-valued function f on I is defined as $\sup \sum_{i=1}^{k-1} |f(t_i) - f(t_{i-1})|$ the supremum being taken over all $t_1 < \dots < t_k \in A$ and $k \geq 2$. If $\text{var} f = \text{var}_I f < \infty$, then f is called a function of bounded variation. A

variation $\nu(f)$ for $L^\infty(I, \mathcal{B}, \lambda)$, the collection of all classes of λ -essentially bounded measurable complex-valued λ -indistinguishable function on I , is defined as $\nu(f) = \inf \nu \tilde{f}$, the infimum being taken over all versions of f (i.e. all measurable $\tilde{f} : I \rightarrow \mathbf{C}$ such that $\tilde{f} = f$, λ -a.s.). The set $BEV(I)$ is a Banach space under the norm $\|f\|_\nu = \nu(f) + \|f\|_1$, where $\|\cdot\|_1$ is the usual L^1_λ norm $\|f\|_1 = \int_I |f| d\lambda$.

Chan [1] proved a Gauss-Kuzmin-Lévy theorem for the transformation τ . He showed that the convergence rate of the distribution function $\lambda(\tau^n < x)$ of τ^n to its limit $F(x) = \nu([0, x]) = \frac{1}{\log(4/3)} \log \frac{2(x+1)}{x+2}$, $x \in I$ is $O(q^n)$ as $n \rightarrow \infty$ with $q \leq 0.880555$ uniformly in x .

Using a Wirsing-type approach (see [10]), in Sebe [9] it was obtained a better estimate of the convergence rate involved. The strategy in both papers ([1] and [9]) was to restrict the domain of the Perron-Frobenius operator of τ under its invariant measure ν to the Banach space of functions which have a continuous derivative on I . It was noticed in [7] that the Perron-Frobenius operator U of τ under ν is given a.s. on I by $Uf(x) = \sum_{i \in \mathbf{N}} P_i(x) f(u_i(x))$, $x \in I$, for $f \in L^1_\nu$, where

$P_i(x) = \frac{2^{-(i+1)}(x+1)(x+2)}{(2^{-i} + x + 1)(2^{-(i+1)} + x + 1)}$ and $u_i(x) = \frac{2^{-i}}{x+1}$, $x \in I$, $i \in \mathbf{N}$. This means that U takes L^1_ν into itself while

$$\int_A Uf d\nu = \int_{\tau^{-1}(A)} f d\nu, \quad A \in \mathcal{B},$$

for any $f \in L^1_\nu$. The Perron-Frobenius operator U is a bounded linear operator of norm 1 when restricted to $B(I)$, the Banach space of all bounded measurable functions $f : I \rightarrow \mathbf{C}$ under the supremum norm $\|f\| = \sup_{x \in I} |f(x)|$.

Let us consider the sequence $(s_n^a)_{n \in \mathbf{N}}$, $a \geq 0$, where $s_n^a = 2^{-an}/(1 + s_{n-1}^a)$, $n \in \mathbf{N}_+$, with $s_0^a = a$. Also, consider the probability measures ν_a on \mathcal{B} defined by their distribution functions $\nu_a([0, x]) = \frac{(a+2)x}{x+a+1}$, $x \in I$, $a \geq 0$.

The motivation for considering the ν_a , $a \in I$, will appear later on. It is known (see [5]) that $(s_n^a)_{n \in \mathbf{N}}$ is an $I \cup \{a\}$ -valued Markov chain on (I, \mathcal{B}, ν_a) which starts from $s_0^a = a \geq 0$ and has the following transition mechanism: from state $s \in I \cup \{a\}$ the possible transitions are to any state $2^{-i}/(s+1)$ with corresponding transition probability $P_i(s)$, $i \in \mathbf{N}$. Reasoning as in Iosifescu and Kraaikamp ([4], Section 2.1) we get that for any $a \geq 0$ the operator $U : B(I) \rightarrow B(I)$ is the transition operator of the Markov chain $(s_n^a)_{n \in \mathbf{N}_+}$ on (I, \mathcal{B}, ν_a) .

Let $C(I)$ denote the collection of all complex-valued continuous functions on I and $C^1(I)$ the collection of all functions from $C(I)$ that have a continuous derivative. The strategy in Sebe [9] was to restrict the domain of the operator U to the Banach space $C^1(I)$. Define a linear operator $V : C(I) \rightarrow C(I)$ by $Vg = -(Uf)'$, $g \in C(I)$, where $f' = g$. Since U is a Markov operator, V is well defined. It is easy to check that $(U^n f)' = (-1)^n V^n f'$, $n \in \mathbf{N}_+$, $f \in C^1(I)$. It was proved in Sebe ([9], Proposition 4.1 and Corollary 4.2) that there are positive constants $v > 0.206968896$ and $w < 0.209364308$ and a real-valued function $\varphi \in C(I)$ de-

fined by $\varphi(x) = \frac{3bx^2 + 4(b+1)x + 6}{(bx+2)^2(bx+1)^2}$, $x \in I$, where $b = 0.794741181\dots$, such that $v\varphi \leq V\varphi \leq w\varphi$. Next, putting

$$\alpha(f) = \alpha = \min_{x \in I} \frac{\varphi(x)}{f'(x)}, \quad \beta(f) = \beta = \max_{x \in I} \frac{\varphi(x)}{f'(x)}$$

for any $f \in C^1(I)$ for which $f' > 0$, we get

$$\frac{\alpha}{\beta} v^n f' \leq V^n f' \leq \frac{\alpha}{\beta} w^n f', \quad n \in \mathbf{N}_+.$$

Since in this case we cannot use a functional theoretic approach (see Iosifescu and Kraaikamp ([4], Subsection 2.2.2)) which provides the means for computing an eigenvalue λ_0 , $v \leq \lambda_0 \leq w$, of the positive operator V , it follows that we cannot obtain the exact convergence rate by this method.

Actually, in Theorem 4.3 in Sebe [9] there are obtained upper and lower bounds of the convergence rate, respectively $O(w^n)$ and $O(v^n)$ as $n \rightarrow \infty$, which provide a *near-optimal* solution to the Gauss-Kuzmin-Lévy problem. Let us recall this theorem.

Theorem 1. *Let μ be a probability measure on \mathcal{B} such that $\mu \ll \lambda$ and $f_0 \in C^1(I)$ such that $f'_0 > 0$. For any $n \in \mathbf{N}_+$ and $x \in I$,*

$$\begin{aligned} \frac{(\log(4/3))^2 \alpha \min_{x \in I} f'_0(x)}{2\beta} v^n F(x)(1 - F(x)) &\leq |\mu(\tau^n < x) - F(x)| \leq \\ &\leq \frac{(\log(4/3))^2 \beta \max_{x \in I} f'_0(x)}{\alpha} w^n F(x)(1 - F(x)). \end{aligned}$$

In particular, for any $n \in \mathbf{N}_+$ and $x \in I$,

$$0.01023923v^n F(x)(1 - F(x)) \leq |\lambda(\tau^n < x) - F(x)| \leq 0.334467468w^n F(x)(1 - F(x)).$$

Let $BV(I)$ denote the linear space of all functions $f : I \rightarrow \mathbf{C}$ of bounded variation. Note that under the norm $\|f\|_v = |f| + \text{var}f$, $f \in BV(I)$, the linear space $BV(I)$ is a commutative Banach algebra with unit.

By restricting the Perron-Frobenius operator to $BV(I)$ it was proved in Iosifescu and Sebe ([5], Theorem 3) that the *exact (optimal)* convergence rate of ν_a ($s_n^a \leq x$) to $\nu([0, x])$ is $O(g^{2n})$ as $n \rightarrow \infty$ with $g^2 = (3 - \sqrt{5})/2 = 0.38196\dots$ Let us recall this theorem.

Theorem 2. *For any $a \geq 0$ and $n \in \mathbf{N}_+$ we have*

$$(2.1) \quad \frac{(a+1)(a+2)}{2(aF_n + F_{n+1})(aF_{n+1} + F_{n+2})} \leq \sup_{x \in I} |\nu_a(s_n^a \leq x) - \nu([0, x])| \leq k_0 \sigma'(n),$$

where $k_0 \leq 291.665$ and $\sigma'(n) = \sigma(n) = 1/F_n F_{n+1}$ or $\sigma'(n) = \sigma(n-1) = 1/F_{n-1} F_n$ according as $a \leq 1$ or $a > 1$. Here the F_n are the Fibonacci numbers defined recursively by $F_0 = F_1 = 1$, $F_{n+1} = F_n + F_{n-1}$, $n \in \mathbf{N}_+$.

Remark. Since

$$(2.2) \quad \frac{1}{F_{n+1} F_{n+2}} \leq \frac{(a+1)(a+2)}{2(aF_n + F_{n+1})(aF_{n+1} + F_{n+2})}$$

for any $n \in \mathbf{N}_+$ and $a \geq 0$, it follows that both lower and upper bounds in Theorem 2 are $O(g^{2n})$ as $n \rightarrow \infty$.

3 Properties of the Perron-Frobenius operator on $BV(I)$

Let us recall some results obtained in Iosifescu and Sebe [5].

Proposition 1. *For any $n \in \mathbf{N}_+$ we have*

$$v_n = \sup_{f \in BV(I)} \frac{\text{var} U^n f}{\text{var} f} = \sup_{f \in B(I), f \uparrow} \frac{\text{var} U^n f}{\text{var} f} = \sup_{f \in B(I), f \downarrow} \frac{\text{var} U^n f}{\text{var} f} = \sup_{a \in [0,1]} \text{var} U^n I_{(a,1]},$$

where the first two upper bounds are taken over non-constant real-valued functions f . Here \uparrow (\downarrow) means that f is non-decreasing (non-increasing), and $I_{(a,1]}$ stands for the indicator function of the interval $(a, 1]$.

Theorem 3. *We have $v_n \leq k_0 \sigma(n)$, $n \in \mathbf{N}$ where k_0 is a constant not exceeding 291.665 and $\sigma(n) = (F_n F_{n+1})^{-1}$. Here the F_n are the Fibonacci numbers.*

We are now able to derive some results concerning the Perron-Frobenius operator U defined on $BV(I)$.

Proposition 2. *Let $f \in BV(I)$ be real-valued. For any $n \in \mathbf{N}$ we have $|U^n f - U^\infty f| \leq k_0 \sigma(n) \text{var} f$, where $U^\infty f \equiv \int_I f d\nu$.*

Proof. Since $|U^n f - U^\infty f| \leq \text{var} U^n f$, $n \in \mathbf{N}$, the result is implied by Proposition 1 and Theorem 3. The case $n = 0$ can be checked directly. \square

Proposition 3. *The spectral radius of the operator $U - U^\infty$ in $BV(I)$ equals $g^2 = (3 - \sqrt{5})/2 = 0.38196\dots$*

Proof. We should show that

$$\lim_{n \rightarrow \infty} \|U^n - U^\infty\|_v^{1/n} = \lim_{n \rightarrow \infty} \left(\sup_{0 \neq f \in BV(I)} \frac{\|U^n f - U^\infty f\|_v}{\|f\|_v} \right)^{1/n} = g^2.$$

The arguments used in the proof of Proposition 2 and Theorem 3 yields

$$\begin{aligned} \|U^n f - U^\infty f\|_v &= |U^n f - U^\infty f| + \text{var} U^n f \leq 2 \text{var} U^n f \leq \\ &\leq 2k_0 \sigma(n) \text{var} f \leq 2k_0 \sigma(n) \|f\|_v \end{aligned}$$

for any $n \in \mathbf{N}$ and $f \in BV(I)$. Hence $\lim_{n \rightarrow \infty} \|U^n - U^\infty\|_v^{1/n} \leq g^2$.

We have already used in Section 2 the fact that U is the transition operator of the Markov chain $(s_n^a)_{n \in \mathbf{N}_+}$ on $(I, \mathcal{B}_I, \nu_a)$ for any $a \geq 0$. Therefore, in particular, we have

$$(3.1) \quad U^n I_{[0,x]}(a) = E_a(I_{[0,x]}(s_n^a)) = \nu_a(s_n^a \leq x)$$

or

$$(3.2) \quad U^{n-1} I_{[0,x]}(s_1^a) = E_a(I_{[0,x]}(s_n^a) | s_1^a) = \nu_a(s_n^a \leq x | s_1^a)$$

for any $x \in I$ and $n \in \mathbf{N}_+$, according as $a \leq 1$ or $a > 1$. Here E_a stands for the mean value operator with respect to the probability measure ν_a . We have

$$(3.3) \quad U^\infty I_{[0,x]} = \int_I I_{[0,x]} d\nu = \nu([0, x]), \quad x \in I.$$

By (2.1), (2.2), (3.1), (3.2) and (3.3) we obtain

$$(3.4) \quad \frac{1}{F_{n+1}F_{n+2}} \leq \sup_{x \in I} |U^n I_{[0,x]} - U^\infty I_{[0,x]}| \leq \sup_{x \in I} \text{var } U^n I_{[0,x]}$$

for any $n \in \mathbf{N}$. The converse inequality $\lim_{n \rightarrow \infty} \|U^n - U^\infty\|_v^{1/n} \geq g^2$ follows by taking $f = I_{[0,x]}$ and using (3.4). \square

4 Final remarks

(i) Theorem 2 allow a quick derivation of the asymptotic behaviour of $\nu_a(\tau^n \leq x, s_n^a \leq y)$ as $n \rightarrow \infty$ for any $a, x, y \in I$, and of the optimal convergence rate, the same as above.

(ii) The results presented in Section 2 allows us to derive all the results in Iosifescu and Kraaikamp ([4], Sections 2.5.3 and 2.5.4) that hold for the regular continued fraction expansion.

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Author's address:

Gabriela Ileana Sebe
University Politehnica of Bucharest, Department of Mathematics I,
313 Splaiul Independenței, RO - 060042, Bucharest, Romania.
e-mail: gisebe@mathem.pub.ro