Investigations of the T system with time delay

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Abstract. A three-dimensional nonlinear system with time delay is analyzed in this paper. The method used is based on the normal form theory and the center manifold theorem in order to identify the conditions such that the system to display periodic oscillations. Partially numerical computations are performed.

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1 Introduction

This work deals with the nonlinear three-dimensional T system with time delay given by:

(1.1)
$$\dot{x}(t) = a(y(t-\tau) - x(t)), \ \dot{y} = (c-a)x(t) - ax(t-\tau)z(t), \ \dot{z} = -bz(t) + x(t)y(t),$$

where a, b, c, $a \neq 0$ are the real parameters of the system. The system (1.1) without delay, i.e. $\tau = 0$ has been previously investigated in some works [8],[9],[3]. Recently similar studied systems are the Rössler system [4], the Chen system [2], the Lü system [5] and the Bloch system [7]. All these systems possess the so called property of dependence to initial conditions which is the first idea to chaos. Such systems can be used in secure communications [1].
Straightforward computations lead us to the the equilibrium points, more exactly, if $\frac{b}{a}(c-a) > 0$, the system T has three equilibrium isolated points: $O(0,0,0)$, $E_1(x_0, x_0, y_0)$, $E_2(-x_0, -x_0, y_0)$ with $x_0 = \sqrt{\frac{b}{a}(c-a)}$, $y_0 = \frac{c-a}{a}$ and for $b \neq 0$, $\frac{b}{a}(c-a) \leq 0$ it has only one isolated equilibrium point, $O(0,0,0)$. As reported in [8], we have the following results for the system T without delay:

THEOREM 1.1. *For $b \neq 0$ the following statements are true:*

a) If $(a > 0, b > 0, c \leq a)$, then $O(0,0,0)$ is asymptotically stable ,
b) If $(b < 0)$ or $(a < 0)$ or $(a > 0, c > a)$, then $O(0,0,0)$ is unstable.

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THEOREM 1.2. *If $(a + b > 0, ab(c - a) > 0, b(2a^2 + bc - ac) > 0)$, the equilibrium points $E_{1,2}(\pm x_0, \pm x_0, y_0)$ are asymptotically stable.*

2 Analyzes of the system near the equilibrium point $O(0,0,0)$

We investigate in the following the behavior of the system (1.1) in the neighborhood of the equilibrium point $O(0,0,0)$. Denoting by $X(t) = (x(t), y(t), z(t))^T$, the linear system associated to (1.1) is given by

$$\dot{X}(t) = JX(t) + KX(t - \tau) \tag{2.1}$$

with

$$J = \begin{pmatrix} -a & 0 & 0 \\ c - a & 0 & 0 \\ 0 & 0 & -b \end{pmatrix}, K = \begin{pmatrix} 0 & a & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{2.2}$$

The characteristic polynomial corresponding to system (2.1) is given by

$$det\left(\lambda I - J - Ke^{-\lambda\tau}\right) = 0, \tag{2.3}$$

which leads to:

$$(b + \lambda)\left(a\lambda + \lambda^2 - ace^{-\lambda\tau} + a^2 e^{-\lambda\tau}\right) = 0. \tag{2.4}$$

The first negative real eigenvalue is $\lambda_1 = -b$, (assume $b > 0$) and the others can be obtain from

$$e^{\lambda\tau} = F(\lambda) \text{ where } F(\lambda) = \frac{a(c - a)}{\lambda(a + \lambda)}. \tag{2.5}$$

In what follows, we investigate the existence of the Hopf bifurcation for system (1.1), using time delay as the bifurcation parameter. We are looking for the value τ_0 such that the equilibrium point $(0,0,0)$ changes from local asymptotic instability to stability or vice versa. This corresponds to pure imaginary roots for the characteristic polynomial. Let $\lambda_{2,3} = \pm i\omega$, $\omega > 0$, be these solutions. It is sufficient to look for $\lambda = i\omega$, root of (2.4). Replacing $\lambda = i\omega$ in (2.4) one gets $\sin\tau\omega = -\frac{\omega}{c-a}$ and $\cos\tau\omega = -\frac{\omega^2}{ac-a^2}$ which leads to

$$a^4 - \omega^4 - 2a^3c + a^2c^2 - a^2\omega^2 = 0. \tag{2.6}$$

Since $\omega > 0$ we get the positive root $\omega_0 = \sqrt{\frac{-a^2+\sqrt{d}}{2}}$ where $d = 5a^4 - 8a^3c + 4a^2c^2$.

It is clear that $\left|e^{i\omega_0\tau}\right| = 1$. Then from (2.5) we have that the value of τ_0 is the smallest positive value of τ satisfying:

$$\tau_0 = \frac{1}{\omega_0}\left(\arg F\left(i\omega_0\right) + 2k\pi\right), k = 0, 1, ... \tag{2.7}$$

Differentiating now (2.4) implicitly with respect to τ, we get

$$\frac{\partial\lambda}{\partial\tau} = \frac{-\lambda a\left(c-a\right)\left(b+\lambda\right)}{e^{\lambda\tau}\left(ab+2a\lambda+2b\lambda+3\lambda^2\right)+a\left(c-a\right)\left(b\tau+\lambda\tau-1\right)}, \tag{2.8}$$

which leads to:

$$Re\left(\frac{\partial\lambda}{\partial\tau}\right)\Big|_{(\lambda=i\omega_0,\tau=\tau_0)} = \left(A^2+B^2\right)^{-1}\left(B\omega_0 - bA\right)\left(c-a\right)a\omega_0, \tag{2.9}$$

with

$$A = \omega_0\left(a^4\tau_0 - a^2b + a\omega_0^2 - 2b\omega_0^2 - 2a^3c\tau_0 + a^2c^2\tau_0\right)/\left(ac-a^2\right),$$

$$B = \left(2a^3c - a^4 + 3\omega_0^4 + a^2b\tau_0\left(a-c\right)^2 + ab\omega_0^2 - a^2c^2 + 2a^2\omega_0^2\right)/\left(ac-a^2\right).$$

Assume $Re\left(\frac{\partial\lambda}{\partial\tau}\right)\big|_{(\lambda=i\omega_0,\tau=\tau_0)} \neq 0$.

Let us in the following compute the first Lyapunov coefficient. The method employed is based on the normal form theory and the center manifold theorem presented in [6].

From the above study we have that for any root $\lambda\left(\tau\right) = \beta\left(\tau\right) \pm i\omega\left(\tau\right)$ of Eq.(2.4) we have $\beta\left(\tau_0\right) = 0, \omega\left(\tau_0\right) = \omega_0 > 0$ and $Re\left(\frac{\partial\lambda}{\partial\tau}\right)\big|_{(\lambda=i\omega_0,\tau=\tau_0)} \neq 0$.

Then the system (1.1) reads:

$$\dot{X}\left(t\right) = JX\left(t\right) + KX\left(t-\tau\right) + F\left(X\left(t\right), X\left(t-\tau\right)\right) \tag{2.10}$$

where $F\left(X\left(t\right), X\left(t-\tau\right)\right) = \left(0, F^2\left(x\left(t-\tau\right), z\left(t\right)\right), F^3\left(x\left(t\right), y\left(t\right)\right)\right)^T$ with $F^2\left(x\left(t-\tau\right), z\left(t\right)\right) = -ax\left(t-\tau\right)z\left(t\right), F^3\left(x\left(t\right), y\left(t\right)\right) = x\left(t\right)y\left(t\right)$.

Denote in the following the space of continuous real-valued functions as $C = C\left(\left[-\tau_0, 0\right], \mathbb{R}^3\right)$ and $\tau = \tau_0 + \alpha$, α real number. Assume α as the bifurcation parameter.

Define the linear operator $L\left(\alpha, \varphi\right) = J\varphi\left(0\right) + K\varphi\left(-\tau\right)$ for any $\varphi \in C$.

For $\varphi \in C^1\left(\left[-\tau, 0\right], \mathbb{C}^3\right)$ define also the operator

$$\Lambda\left(\alpha\right)\varphi\left(\theta\right) = \begin{cases} \frac{\partial\varphi(\theta)}{\partial\theta}, \theta \in \left[-\tau, 0\right) \\ J\varphi\left(0\right) + K\varphi\left(-\tau\right), \theta = 0, \end{cases}$$

and for $\varphi^* \in C^1\left(\left[0, \tau\right], \mathbb{C}^3\right)$ we define the adjoint operator Λ^* of Λ by

$$\Lambda^*\varphi^*\left(s\right) = \begin{cases} -\frac{\partial\varphi^*(s)}{\partial s}, s \in \left(0, \tau\right] \\ \varphi^*\left(0\right)J + \varphi^*\left(\tau\right)K, s = 0. \end{cases}$$

Denote by $\varphi_1\left(\theta\right) = \varphi_1\left(0\right)e^{\lambda_1\theta}, \varphi_2\left(\theta\right) = \varphi_2\left(0\right)e^{\lambda_2\theta}$, $\varphi_3\left(\theta\right) = \bar{\varphi}_2\left(\theta\right)$, respectively $\varphi_1^*\left(s\right) = \varphi_1^*\left(0\right)e^{-\lambda_1 s}, \varphi_2^*\left(s\right) = \varphi_2^*\left(0\right)e^{-\lambda_2 s}, \varphi_3^*\left(s\right) = \bar{\varphi}_2^*\left(s\right)$, the eigenvectors of the operator $\Lambda\left(0\right)$, respectively of the operator Λ^* corresponding to the eigenvalues $\lambda_1 = -b, \lambda_2 = i\omega_0, \lambda_3 = -i\omega_0$.

For $\varphi \in C^1\left(\left[-\tau, 0\right], \mathbb{C}^3\right)$ and $\varphi^* \in C^1\left(\left[0, \tau\right], \mathbb{C}^3\right)$ we define the bilinear form

$$\langle \varphi^*(s), \varphi(\theta) \rangle = \bar{\varphi}^*(0)\varphi(0) - \bar{\varphi}^*(\tau) K H(\varphi)(-\tau) \tag{2.11}$$

where $H(\varphi)(\theta) = \sum_{i=1}^{3} a_i \int_0^{\theta} e^{-\lambda_i \xi} \varphi(\xi)\, d\xi, \quad \varphi^*(s) = \sum_{i=1}^{3} a_i \varphi_i^*(s)$.

It can be checked that $\Lambda(0)$ and Λ^* are adjoint operators with respect to this bilinear form.

As $\varphi_1(\theta) = \varphi_1(0)e^{\lambda_1\theta}$ and $\varphi_2(\theta) = \varphi_2(0)e^{\lambda_2\theta}$ are the eigenvectors of the operator $\Lambda(0)$ corresponding to the eigenvalues $-b, i\omega_0$, then $\varphi_1(0)$ is a solution of $\left(-bI - J - Ke^{b\tau_0}\right)u = 0$ and $\varphi_2(0)$ is a solution of $\left(i\omega_0 I - J - Ke^{-i\omega_0\tau_0}\right)u = 0$ which leads to $\varphi_1(0) = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}^T$ and $\varphi_2(0) = \begin{pmatrix} -i\omega_0 & a-c & 0 \end{pmatrix}^T$.

Similarly, $\varphi_1^*(0)$ is a solution of $v\left(-bI - J - Ke^{b\tau_0}\right) = 0$ and $\varphi_2^*(0)$ is a solution of $v\left(i\omega_0 I - J - Ke^{-i\omega_0\tau_0}\right) = 0$, which gives $\varphi_1^*(0) = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}$ and $\varphi_2^*(0) = \begin{pmatrix} c-a & a+i\omega_0 & 0 \end{pmatrix}$. With these notations it is not difficult to observe that

$$\langle \varphi_i^*(s), \varphi_j(\theta) \rangle = \bar{\varphi}_i^*(0)\varphi_j(0) - \frac{e^{-\lambda_j\tau_0} - e^{-\lambda_i\tau_0}}{\lambda_j - \lambda_i} \bar{\varphi}_i^*(0) K \varphi_j(0), i \neq j \tag{2.12}$$

and

$$\langle \varphi_i^*(s), \varphi_i(\theta) \rangle = \bar{\varphi}_i^*(0)\varphi_i(0) + \tau_0 e^{\lambda_i\tau_0} \bar{\varphi}_i^*(0) K \varphi_i(0), i = j, \tag{2.13}$$

for $\theta \in [-\tau, 0], s \in [0, \tau]$ which leads to:
$\langle \varphi_1^*(\theta), \varphi_1(\theta) \rangle = 1 := e_{11}$, $\langle \varphi_1^*(\theta), \varphi_2(\theta) \rangle := e_{12} = \langle \varphi_1^*(\theta), \varphi_3(\theta) \rangle := e_{13}$,
$\langle \varphi_2^*(\theta), \varphi_1(\theta) \rangle = 0 := e_{21}$, $\langle \varphi_2^*(\theta), \varphi_2(\theta) \rangle = (c-a)\left(ia\tau_0\omega_0 - a + \tau_0\omega_0^2\right) := e_{22}$,
$\langle \varphi_2^*(\theta), \varphi_3(\theta) \rangle = 2\omega_0 i (c-a) := e_{23}$, $\langle \varphi_3^*(\theta), \varphi_1(\theta) \rangle = 0 := e_{31}$,
$\langle \varphi_3^*(\theta), \varphi_2(\theta) \rangle := e_{32} = \bar{e}_{23}$, $\langle \varphi_3^*(\theta), \varphi_3(\theta) \rangle := e_{33} = \bar{e}_{22}$.

Denote $F = (f_{ij})_{i,j=1,2,3}$ the inverse matrix of $E = (e_{ij})_{i,j=1,2,3}$. After the computations we have that
$f_{11} = 1, f_{12} = f_{13} = 0$,
$f_{21} = 0,\ f_{22} = \frac{-a - ia\tau_0\omega_0 + \tau_0\omega_0^2}{\gamma},\ f_{23} = -2i\frac{\omega_0}{\gamma}$,
$f_{31} = 0,\ f_{32} = 2i\frac{\omega_0}{\gamma},\ f_{33} = \frac{-a + ia\tau_0\omega_0 + \tau_0\omega_0^2}{\gamma}$, where
$\gamma = \left(a^2 - 4\omega_0^2 - 2a\tau_0\omega_0^2 + \tau_0^2\omega_0^4 + a^2\tau_0^2\omega_0^2\right)(c-a)$.
Then
$\psi_1(s) = f_{11}\varphi_1^*(s) + f_{12}\varphi_2^*(s) + f_{13}\varphi_3^*(s)$,
$\psi_2(s) = f_{21}\varphi_1^*(s) + f_{22}\varphi_2^*(s) + f_{23}\varphi_3^*(s)$,
$\psi_3(s) = f_{31}\varphi_1^*(s) + f_{32}\varphi_2^*(s) + f_{33}\varphi_3^*(s)$,
are generalized eigenvectors of Λ^* satisfying $\langle \psi_i(s), \varphi_j(\theta) \rangle = \delta_{ij}$, $i, j = 1, 2, 3$ where $\delta_{ij} = \begin{cases} 1, i = j \\ 0, i \neq j. \end{cases}$

It is known from [6] that the partially locally stable manifold at $O(0,0,0)$ contains elements of the form $\psi = z\varphi_2 + \overline{z\varphi_2} + v\underline{\varphi_1 + w}(z, \overline{z}, v)$, where $w : [-\tau, 0] \times \mathbb{C}^2 \times \mathbb{R} \to \mathbb{R}^3$, is given by $w(\theta, z, \overline{z}, v) = w(z\varphi_2(\theta) + \overline{z}\overline{\varphi_2(\theta)} + v\varphi_1(\theta))$. From the Taylor expansion of the function w,

$$w(\theta, z, \bar{z}, v) = w_{200}(\theta)\frac{z^2}{2} + w_{110}(\theta) z\bar{z} + w_{020}(\theta)\frac{\bar{z}^2}{2} + w_{002}(\theta) v^2 + \\ + w_{101}(\theta) zv + w_{011}(\theta)\bar{z}v + ... \tag{2.14}$$

we have that

$$F(v\varphi_1(0) + z\varphi_2(0) + \bar{z}\bar{\varphi}_2(0) + w(0, z, \bar{z}, v), v\varphi_1(-\tau_0) + z\varphi_2(-\tau_0) + \\ + \bar{z}\bar{\varphi}_2(-\tau_0) + w(-\tau, z, \bar{z}, v)) = \\ = F_{200}\frac{z^2}{2} + F_{110}z\bar{z} + F_{020}\frac{\bar{z}^2}{2} + F_{210}\frac{z^2\bar{z}}{2} + F_{002}v^2 + F_{101}zv + F_{011}\bar{z}v + ... \tag{2.15}$$

where $F_{200} = \begin{pmatrix} 0 \\ 0 \\ 2(c-a)i\omega_0 \end{pmatrix}, F_{110} = F_{002} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, F_{020} = \begin{pmatrix} 0 \\ 0 \\ 2(a-c)i\omega_0 \end{pmatrix}$,

$F_{210} = \begin{pmatrix} 0 \\ c^2_{210} \\ c^3_{210} \end{pmatrix}$, $F_{101} = \begin{pmatrix} 0 \\ (c-a)^{-1}(-a-i\omega_0)\omega_0^2 \\ 0 \end{pmatrix}$,

$F_{011} = \begin{pmatrix} 0 \\ -i(c-a)^{-3}a^{-2}(\omega_0 - ia)(i\omega_0 - a)^2\omega_0^4 \\ 0 \end{pmatrix}$,

and

$$c^2_{210} = 2(c-a)^{-3}a^{-2}\left(-iw^3_{110}(0)a^2(a-c)^2 + \tfrac{1}{2}i\omega_0^2 w^3_{200}(0)(ia+\omega_0)^2\right)(\omega_0 - ia)\omega_0^2,$$

$$c^3_{210} = 2aw^1_{110}(0) - 2cw^1_{110}(0) - 2iw^2_{110}(0)\omega_0 - cw^1_{200}(0) + iw^2_{200}(0)\omega_0 + aw^1_{200}(0).$$

Then

$w_{200}(\theta) = -\frac{g_{200}}{i\omega_0}\varphi_2(0)e^{i\omega_0\theta} - \frac{\bar{g}_{020}}{3i\omega_0}\bar{\varphi}_2(0)e^{-i\omega_0\theta} + \frac{1}{-b-2i\omega_0}\left(h_{200} + \bar{h}_{020}\right)\varphi_1(0)e^{-b\theta} + E_{200}e^{2i\omega_0\theta}$,

$w_{110}(\theta) = \frac{g_{110}}{i\omega_0}\varphi_2(0)e^{i\omega_0\theta} - \frac{\bar{g}_{110}}{i\omega_0}\bar{\varphi}_2(0)e^{-i\omega_0\theta} + \frac{h_{110}+\bar{h}_{110}}{-b}\varphi_1(0)e^{-b\theta} + E_{110}$,

$w_{101}(\theta) = \frac{g_{101}}{b}\varphi_2(0)e^{i\omega_0\theta} + \frac{\bar{g}_{101}}{b-2i\omega_0}\bar{\varphi}_2(0)e^{-i\omega_0\theta} - \frac{2h_{101}}{i\omega_0}\varphi_1(0)e^{-b\theta} + E_{101}e^{(i\omega_0-b)\theta}$,

where

$E_{200} = -\left(J + e^{-2i\omega_0\tau_0}K - 2i\omega_0 I\right)^{-1}F_{200}$,
$E_{110} = -(J+K)^{-1}F_{110}$,
$E_{101} = -\left(J + e^{-(i\omega_0-b)\tau_0}K - (i\omega_0 - b)I\right)^{-1}F_{101}$,

and

$g_{200} = \bar{\varphi}_2^*(0)F_{200}, g_{110} = \bar{\varphi}_2^*(0)F_{110}, g_{020} = \bar{\varphi}_2^*(0)F_{020}$,
$g_{002} = \bar{\varphi}_2^*(0)F_{002}, g_{101} = \bar{\varphi}_2^*(0)F_{101}, g_{011} = \bar{\varphi}_2^*(0)F_{011}, g_{210} = \bar{\varphi}_2^*(0)F_{210}$,
$h_{200} = \varphi_1^*(0)F_{200}, h_{110} = \varphi_1^*(0)F_{110}, h_{020} = \varphi_1^*(0)F_{020}$,
$h_{101} = \varphi_1^*(0)F_{101}, h_{011} = \varphi_1^*(0)F_{011}$.

Denoting in the following $\mu_2 = -\frac{Re(C_1)}{Re(M)}$, $T_2 = -\frac{Im(C_1)+\mu_2 Im(M)}{\omega_0}$, $\beta_2 = 2Re(C_1)$ where $C_1 = \frac{i}{2\omega_0}\left(g_{200}g_{110} - 2|g_{110}|^2 - \frac{1}{3}|g_{020}|^2\right) + \frac{g_{210}}{2} \neq 0$, and $M = \left(\frac{\partial\lambda}{\partial\tau}\right)|_{(\lambda=i\omega_0, \tau=\tau_0)}$ we have the following result:

Theorem 2.1. *The following statements are true:*

a) If $\mu_2 > 0, (< 0)$, *the Hopf bifurcation is supercritical (subcritical) and the periodic solutions of the bifurcation exist for* $\tau > \tau_0, (< \tau_0)$.

b) If $\beta_2 < 0, (> 0)$, *the periodic solutions are stable.*

c) If $T_2 > 0, (< 0)$, *the period of the periodic solutions are increasing (decreasing).*

Remark 2.1. *In some particular cases it is possible that* $C_1 = 0$, *so there could be Bautin bifurcations.*

An example of this type is given by $a = 1, b = 2, c = 5$. In this case we have:
$\omega_0 = 1.8791,\ f_{22} = 4.4169 \times 10^{-2} - 2.7542 \times 10^{-2} i\ , f_{23} = -0.02851 i,$

$\psi_2(0) = \begin{pmatrix} 0.17668 - 0.22421i & 0.04235 + 2.6946 \times 10^{-2} i & 0 \end{pmatrix},$

$F_{200} = \begin{pmatrix} 0 & 0 & 15.033i \end{pmatrix}^T, F_{110} = F_{002} = F_{210} = E_{110} = \begin{pmatrix} 0 & 0 & 0 \end{pmatrix}^T,$
$F_{020} = \begin{pmatrix} 0 & 0 & -15.033i \end{pmatrix}^T,\ F_{101} = \begin{pmatrix} 0.0 & -0.88275 - 1.6588i & 0 \end{pmatrix}^T,$
$E_{200} = \begin{pmatrix} 0 & 0 & 3.1172 + 1.6589i \end{pmatrix}^T,$
$E_{101} = \begin{pmatrix} 0.20679 + 0.41803i & 1.8088 \times 10^{-2} + 1.0325 \times 10^{-2} i & 0.0 \end{pmatrix}^T.$

We have also
$g_{200} = g_{110} = g_{020} = g_{002} = g_{210} = 0,$
$g_{101} = -8.2082 \times 10^{-2} - 4.6464 \times 10^{-2} i,$
$g_{011} = 7.3130 \times 10^{-3} + 9.4031 \times 10^{-2} i,$
$h_{200} = 15.033i,\ h_{020} = -15.033i, h_{110} = h_{101} = h_{011} = 0,$
$w_{200}(0) = \begin{pmatrix} 0 & 0 & -3.1173 - 1.6589i \end{pmatrix}^T, w_{110}(\theta) = \begin{pmatrix} 0 & 0 & 0 \end{pmatrix}^T,$
$w_{101}(0) = \begin{pmatrix} 0.18548 + 0.46003i & 0.25702 + 0.15083i & 0 \end{pmatrix}^T,$

which lead to $C_1 = 0$.

3 Conclusions

In this paper we investigate a three dimensional nonlinear system with time delay. The system without delay, for some values of the system's parameters, presents the property of dependence to initial conditions. It has three equilibrium points, the origin $O(0, 0, 0)$ and another two points $E_{1,2}$. Using notions from the normal form theory and the center manifold theorem we point out the conditions on which the system with delay displays periodic solutions. In doing that we studied Hopf bifurcations of the system assuming the delay parameter τ as bifurcation parameter. Finally, we pointed out a numerical case to illustrate that the system could present Bautin bifurcations. In this case the first Lyapunov coefficient is equal to zero.

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