Black hole geometric thermodynamics

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Abstract. Section 1 reviews the history of Differential Geometry derived from a fundamental tensor of type $(0, 2)$ and points out which Hessian is more convenient to be fundamental tensor. Section 2 starts with the fundamental tensor $h = \mathrm{Hess}_g f$, gives the Christoffel symbols and the system of geodesics, establishes the relation between the components of the curvature tensors field of h and $(\mathbf{R}^n, g)$, and determines the PDEs representing the coincidence between the Christoffel symbols of h and the Christoffel symbols of g. Section 3 analyses the Reissner-Nordstrom black hole from Pidokrajt point of view and from our point of view, underlying the physical characteristics of the geometrical models via fundamental tensor, Christoffel symbols, geodesics and curvature. The physical geometric models are total different, the most important differences being the degeneration curves, the null length curves and the sign of sectional curvature.

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1 Weinhold and Ruppeiner metrics

The fundamental tensor Geometry is utilized in Mathematical Optimization, Thermodynamics or in Statistics as an important tool for recent research.

Suppose the entropy S and other extensive variables N^a of the system (electric charge, angular momentum, etc) are coordinates of a point in the Euclidean space $R^n, \delta = (\delta_{ij})$). The Euclidean space $(R^n = \{(S, N^a)\}, \delta = (\delta_{ij}))$ is the most simple Riemannian manifold used to analyse a thermodynamic system. But, as it is well-known this space is flat. Accepting the mass (energy) M like a function $M : R^n \to R_+$, in 1975, F. Weinhold [14] introduced in thermodynamics a piecewise pseudo-Riemannian fundamental tensor as the Hessian of the mass (energy) $h^W = \mathrm{Hess}_\delta M$. Taking as starting structure a pseudo-Riemannian manifold $(U \subset R^n, h^W)$, Weinhold developed a new point of view on thermodynamics. The papers of N. Pidokrajt [8]

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and T. Sarkar, G. Sengupta, B. N. Tiwari [9] underline that Weinhold geometry is useful for geometric approach to black hole thermodynamics.

Now suppose that the entropy S is function of mass M and other extensive variables N^a of the system. Geometrically it appear the flat manifold $(R^n = \{(M, N^a)\}, \delta = (\delta_{ij}))$. In 1979, G. Ruppeiner [7] proposed to study thermodynamics using the minus Hessian of the entropy $h^R = -Hess_\delta S$ as a piecewise pseudo Riemannian fundamental tensor, i.e., he changed the initial geometrical structure (R^n, δ) with $(U \subset R^n, h^R)$. Of course Ruppeiner accepted that when the entropy is expressed in terms of other extensive parameters, it basically tells us the thermal physics of the system. The Ruppeiner-Riemannian geometrical model allows the inclusion of the theory of fluctuations into the axioms of equilibrium thermodynamics in the sense that there exist equilibrium states and the distance between them is related to the fluctuation phenomena. In other words, the Ruppeiner-Riemannian geometry is connected to the underlying statistical mechanical model; for example, the flatness is equivalent to non-interacting statistics. The Weinhold geometry is a conformal counterpart of Ruppeiner geometry on a constant level set relating the variables S, M, N^a. More precisely, the arc elements are related by $ds_R^2 = \frac{1}{T} ds_W^2$, where T is the temperature.

J. Donato [2] describes the connection between entropy and curvature.

Since black holes are self-gravitating systems, they do exhibit negative specific heats. Also, the black holes should be treated in microcanonical ensemble. Globally, the Weinhold and Ruppeiner fundamental tensors have variable signature. Consequently the Ruppeiner theory of black holes cannot always be associated to the probability distribution in thermodynamic fluctuation theory.

The previous theory shows that for some applications is more convenient to replace the Euclidean space $(\mathbf{R}^n, \delta)$ with a pseudo-Riemannian manifold $(U \subset \mathbf{R}^n, g)$, where the fundamental tensor g has a piecewise constant signature on R^n. Starting with a function $f : \mathbf{R}^n \to \mathbf{R}$, whose Hessian with respect to the initial piecewise pseudo-Riemannian fundamental tensor g is non-degenerate, we can introduce a pseudo-Riemannian-Hessian manifold $(U \subset \mathbf{R}^n, h = \mathrm{Hess}_g f)$ which gives information about the properties of the system described by the function f and the fundamental tensor g. Also, exploiting the relation between the Christoffel symbols of Levi-Civita connection produced by the fundamental tensor h and the Christoffel symbols of Levi-Civita connection produced by the fundamental tensor g, or the relation between the components of the corresponding curvature tensors, we can produce pseudo-Riemannian fundamental tensor with special properties [13]: metrics with negative curvature, metrics with zero curvature, complete or non-complete metrics, metrics capable to produce convexity, exact sequences of pseudo-Riemannian metrics, metrics representing fluctuation phenomena in thermodynamics, etc.

In our opinion, the pseudo-Riemannian Hessian manifold $(V \subset \mathbf{R}^n,\, k = \mathrm{Hess}_h f)$ is the most suitable to describe the fundamental properties of thermodynamic systems based on function f and the metric k since it re-inforces the information obtained from $(U \subset \mathbf{R}^n, h = \mathrm{Hess}_\delta f)$. Indeed, Weinhold and Ruppeiner started with constant state potentials δ_{ij} (i.e, forcing for a flat space) and built the first order nonconstant state potentials h_{ij} as components of a nonconstant metric. This new metric determines a space $(U \subset \mathbf{R}^n, h = \mathrm{Hess}_\delta f)$ which is more appropriate for thermodynamics theories. Starting with Weinhold-Ruppeiner geometrical structure $(U \subset \mathbf{R}^n, h = \mathrm{Hess}_\delta f)$, and preserving the thermodynamic function f, we build a new fundamental tensor

k. Its components are the second order nonconstant state potentials k_{ij}. Unlike the fundamental tensor h, obtained using an apriori flat space, the fundamental tensor k emerges from the natural Weinhold-Ruppeiner space. So, if this conjecture should prove correct, we must keep one eye on the local tilt of the function f and one to the associated curvature produced by the fundamental tensor $k = \mathrm{Hess}_h f$. The state potentials $\delta_{ij},\ h_{ij},\ k_{ij}$ are different from the usual thermodynamic potentials.

In differential geometry, Hessian metrics are samples of Riemannian or pseudo-Riemannian metrics. For example, Kähler metrics $\dfrac{\partial^2 f}{\partial z^i \partial \bar{z}^j}$ on complex manifolds are Riemannian metrics. The geometry of Hessian manifolds was studied by S. Amari [1], J. Duistermaat [3], N. Hitchin [4], H. Kito [5], Y. Nesterov -M. J. Tood [6], H. Shima [10], H. Shima - K. Yagi [11], B. Totaro [12], C. Udrişte - G. Bercu [13] etc.

2 Geometry via fundamental tensor field

In this section we recall known formulas in the geometry derived from a fundamental tensor field [13].

A *fundamental tensor field of type* $(0,2)$ on the smooth manifold $\mathbf{R}^n$ is a smooth symmetric differential 2-form g on $\mathbf{R}^n$ such that $g_x, x \in \mathbf{R}^n$ is non-degenerate on $T_x\mathbf{R}^n$. We call $(\mathbf{R}^n, g)$ an *Einsenhart manifold.*

Let g_{ij} be the components of the fundamental tensor g. The curve $\gamma(t) = (x^i(t))_{i=1,\dots,n}$ in R^n is called *null length curve* if $g_{ij}(\gamma(t))\dot{x}^i(t)\dot{x}^j(t) = 0$. This definition is proper only in the regions where the fundamental tensor g is not (positive or negative) definite.

The Christoffel symbols G^k_{ij} of the Levi-Civita connection determined by the components $g_{ij},\ g^{ij}$ are

$$G^k_{ij} = \frac{1}{2} g^{kl}\left(\frac{\partial g_{li}}{\partial x^j} + \frac{\partial g_{jl}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^l}\right).$$

They determine the second order differential equations

$$\ddot{x}^i + G^i_{jk}\dot{x}^j\dot{x}^k = 0, i = 1, \dots, n,$$

which describe the autoparallel lines (geodesics) $\gamma(t) = (x^i(t))_{i=1,\dots,n}$. Also they determine the components

$$G^l{}_{ijk} = \frac{\partial G^l_{ki}}{\partial x^j} - \frac{\partial G^l_{ji}}{\partial x^k} + G^r_{ki}G^l_{jr} - G^r_{ji}G^l_{kr}$$

of the curvature tensor field G.

If $f : \mathbf{R}^n \to \mathbf{R}$ is a smooth function, then the second covariant derivative

$$\mathrm{Hess}_g f = \left(\frac{\partial^2 f}{\partial x^i \partial x^j} - G^k_{ij}\frac{\partial f}{\partial x^k}\right) dx^i \otimes dx^j$$

is called the Hessian of f. Notations:

$$f_{,i} = \frac{\partial f}{\partial x^i};\ f_{,ij} = \frac{\partial^2 f}{\partial x^i \partial x^j} - G^m_{ij} f_{,m};\ f_{,ijk} = \frac{\partial f_{,ij}}{\partial x^k} - G^l_{ki} f_{,lj} - G^l_{kj} f_{,li}.$$

Suppose that the Hessian $h = \mathrm{Hess}_g f$ is non-degenerate. Then h is a piecewise pseudo-Riemannian fundamental tensor that produces the Christoffel symbols H^k_{ij}.

2.1. Proposition *Let $f,^{pk}$ be the contravariant components of the fundamental tensor $h_{pk} = f_{,pk}$ and $G^m{}_{ijk}$ be the components of the curvature tensor field produced by the fundamental tensor g_{ij}. Then the Christoffel symbols H^k_{ij} are given by the following formulas*

$$H^p_{ij} = G^p_{ij} + \frac{1}{2} f,^{kp}\, [f_{,ijk} + (G^m{}_{ikj} + G^m{}_{jki}) f_{,m}].$$

2.2. Corollary. *The equality $H^p_{ij} = G^p_{ij}$ is equivalent to the PDEs*

$$f_{,ijk} + (G^l{}_{ikj} + G^l{}_{jki}) f_{,l} = 0,$$

for all $i, j, k = 1, \dots, n$.

2.4. Proposition. *Let h_{ij} be the components of the fundamental tensor $h = Hess_g f$, and k_{ij} be the components of the fundamental tensor $k = Hess_h f$. Then*

$$k_{ij} = h_{ij} - \frac{1}{2} f,^{pk}\, [f_{,ijk} + (G^l{}_{ikj} + G^l{}_{jki}) f_{,l}] f_{,p}.$$

2.5. Proposition. *The curvature tensor field H of the fundamental tensor $h = Hess_g f)$ has the components*:

$$\begin{gathered}
H^h{}_{ijk} = G^h{}_{ijk} + \frac{1}{2}\frac{\partial f,^{hp}}{\partial x^j}[f_{,kip} + (G^m{}_{kpi} + G^m{}_{ipk}) f_{,m}] \\
-\frac{1}{2}\frac{\partial f,^{hp}}{\partial x^k}[f_{,jip} + (G^m{}_{jpi} + G^m{}_{ipj}) f_{,m}] + \frac{1}{2} f,^{hp} \{(G^r{}_{ki} f_{,jrp} - G^r{}_{ji} f_{,krp}) \\
+ \bigg[\frac{\partial}{\partial x^j}(G^s{}_{kpi} - G^r_{pi} G^s_{kr}) - \frac{\partial}{\partial x^k}(G^s{}_{jpi} - G^r_{pi} G^s_{jr}) \\
+ \bigg(G^m{}_{ipk} + G^m{}_{kpi} + \frac{\partial G^m_{ki}}{\partial x^p} + G^r_{ki} G^m_{pr}\bigg) G^s_{jm} - (G^m{}_{ipj} + G^m{}_{jpi} \\
+ \frac{\partial G^m_{ji}}{\partial x^p} + G^r_{ji} G^m_{pr}\bigg) G^s_{km} + G^r_{ki}(G^s{}_{jpr} + G^s{}_{rpj}) - G^r_{ji}(G^s{}_{kpr} + G^s{}_{rpk})\bigg] f_{,s} \\
+ \bigg(G^s{}_{ipk} + G^s{}_{kpi} + \frac{\partial G^s_{ki}}{\partial x^p} + G^r_{ki} G^s_{pr} + \frac{\partial G^s_{pi}}{\partial x^k}\bigg) f_{,js} \\
- \bigg(G^s{}_{ipj} + G^s{}_{jpi} + \frac{\partial G^s_{ji}}{\partial x^p} + G^r_{ji} G^s_{pr} + \frac{\partial G^s_{pi}}{\partial x^j}\bigg) f_{,ks} + \bigg(\frac{\partial G^s_{ji}}{\partial x^k} \\
- \frac{\partial G^s_{ki}}{\partial x^j}\bigg) f_{,sp} + \bigg(\frac{\partial G^s_{pj}}{\partial x^k} - \frac{\partial G^s_{pk}}{\partial x^j}\bigg) f_{,si} + G^s_{ji}\frac{\partial f_{,sp}}{\partial x^k} + G^s_{pj}\frac{\partial f_{,si}}{\partial x^k} \\
+ G^s_{pi}\frac{\partial f_{,sj}}{\partial x^k} - G^s_{ki}\frac{\partial f_{,sp}}{\partial x^j} - G^s_{pk}\frac{\partial f_{,si}}{\partial x^j} - G^s_{pi}\frac{\partial f_{,sk}}{\partial x^j}\bigg\} + \frac{1}{2} f,^{rp} \{[f_{,kip} \\
+ (G^m{}_{kpi} + G^m{}_{ipk}) f_{,m}] G^h_{jr} - [f_{,jip} + (G^m{}_{jpi} \\
+ G^m{}_{ipj}) f_{,m}] G^h_{kr}\} + \frac{1}{4} f,^{rp} f,^{hl} \{[f_{,kip} \\
+ (G^m{}_{kpi} + G^m{}_{ipk}) f_{,m}][f_{,jrl} + (G^m{}_{jlr} + G^m{}_{rlj}) f_{,m}] \\
- [f_{,jip} + (G^m_{jpi} + G^m_{ipj}) f_{,m}][f_{,krl} + (G^m{}_{klr} + G^m{}_{rlk}) f_{,m}]\} .
\end{gathered}$$

3 Reissner-Nordstrom black hole

A Reissner-Nordstrom black hole is formed from non-rotating but electrically-charged matter.

Let M be the mass (energy), Q be the electric charge and S be the entropy of the black hole. The Reissner-Nordstrom black hole is characterized by

$$S = 2M^2 - Q^2 + 2M^2\sqrt{1 - \frac{Q^2}{M^2}}.$$

Solving with respect to the mass M, we find

$$M = \frac{1}{2}\sqrt{S} + \frac{Q^2}{2\sqrt{S}}.$$

Now we take the Euclidean manifold $(\{(S,Q)|S>0\} \subset R^2, \delta)$.

Weinhold point of view [8], [9], [14]. The piecewise pseudo-Riemann-Weinhold fundamental tensor is given by the Hessian matrix $h = Hess_\delta M$, i.e.,

$$h = \frac{1}{\sqrt{S}} \begin{pmatrix} \frac{3Q^2 - S}{8S^2} & -\frac{Q}{2S} \\ -\frac{Q}{2S} & 1 \end{pmatrix}.$$

The degeneration curve of the matrix h is an arc of parabola, $S > 0, S - Q^2 = 0$. The signature of the Hessian h is

$$\text{signature}\,(h) = \begin{cases} (+,+) & \text{for}\, S>0, S-Q^2<0 \\ (+,-) & \text{for}\, S>0, S-Q^2>0, S-3Q^2<0 \\ (-,+) & \text{for}\, S>0, S-3Q^2>0. \end{cases}$$

The Euclidean model $(\{(S,Q)|S>0\} \subset R^2, \delta)$ is changed into a pseudo-Riemannian manifold $(U \subset \{(S,Q)|S>0\} \subset R^2, h = Hess_\delta M)$. The null length curves are described by the differential equations

$$\frac{3Q^2 - S}{8S^2}dS^2 - \frac{Q}{S}dSdQ + dQ^2 = 0$$

or explicitely,

$$-2\sqrt{2}\text{arctan}\frac{Q}{\sqrt{-Q^2+S}} + ln(S) = C_1,\; 2\sqrt{2}\text{arctan}\frac{Q}{\sqrt{-Q^2+S}} + ln(S) = C_2.$$

Now let us compute the Christoffel symbols of the fundamental tensor h. We find the rational functions

$$H^1_{11} = -\frac{3(S-3Q^2)}{4S(S-Q^2)},\quad H^1_{12} = -\frac{2Q}{S-Q^2},\quad H^1_{22} = \frac{2S}{S-Q^2},$$
$$H^2_{11} = \frac{3Q^2}{4S^2(S-Q^2)},\quad H^2_{12} = -\frac{3Q^2+S}{4S(S-Q^2)},\quad H^2_{22} = \frac{Q}{S-Q^2}.$$

These determine the geodesic equations

$$\frac{d^2S}{dt^2} = \frac{3(S-3Q^2)}{4S(S-Q^2)}\left(\frac{dS}{dt}\right)^2 + \frac{4Q}{S-Q^2}\left(\frac{dS}{dt}\right)\left(\frac{dQ}{dt}\right) - \frac{2S}{S-Q^2}\left(\frac{dQ}{dt}\right)^2$$

$$\frac{d^2Q}{dt^2} = -\frac{3Q^2}{4S^2(S-Q^2)}\left(\frac{dS}{dt}\right)^2 + \frac{(3Q^2+S)}{2S(S-Q^2)}\left(\frac{dS}{dt}\right)\left(\frac{dQ}{dt}\right) - \frac{Q}{S-Q^2}\left(\frac{dQ}{dt}\right)^2.$$

The non-zero components of the Riemann curvature tensor are

$$H_{1212} = -\frac{1}{8\sqrt{S^3}(S-Q^2)}, H_{1221} = H_{2112} = -H_{1212}, H_{2121} = H_{1212}.$$

Physical characteristics of Weinhold-Pidokrajt geometrical model: *the fundamental tensor* $h = Hess_\delta M$ *is piecewise pseudo-Riemannian; the components* h_{ij} *of the fundamental tensor* h *may be assimilated to state potentials; the Christoffel symbols and the curvature tensor diverges in the extremal limit and along the curve where the fundamental tensor change the signature; the manifold* $(U \subset \{(S,Q)|S > 0\} \subset R^2, h = Hess_\delta M)$ *is non-flat; the sign of sectional curvature depends only of the sign of* $det\,(h)$ *since* $(det\,(h))H_{1212}$ *is positive throughout.*

Our point of view. We start with the pseudo-Riemannian manifold $(U \subset \{(S,Q)|S > 0\} \subset R^2, h = \mathrm{Hess}_\delta M)$. First, let us compute the covariant derivative of the differential $a = dM$ of the function M (a covariant vector) with respect to h, i.e., we compute the Hessian $k = \mathrm{Hess}_h M$. In this way we obtain a new pseudo-Riemannian manifold $(V \subset \{(S,Q)|S > 0\} \subset R^2, k = \mathrm{Hess}_h M)$.

Fig. 1. Signature of fundamental tensor field k

Since $a = dM = (a_1, a_2)$, where $a_1 = \dfrac{S-Q^2}{4\sqrt{S^3}}, a_2 = \dfrac{Q}{\sqrt{S}}$, we find

$$k = \frac{1}{\sqrt{S}}\begin{pmatrix} \dfrac{-S^2+9Q^4+4Q^2S}{16S^2(Q^2-S)} & -\dfrac{(3Q^2+S)Q}{4S(Q^2-S)} \\ -\dfrac{(3Q^2+S)Q}{4S(Q^2-S)} & \dfrac{3Q^2-S}{2(Q^2-S)} \end{pmatrix}.$$

The degeneration curve of the matrix k is an arc of the sextic curve, $S > 0$, $-9S^2Q^2 + S^3 + 9Q^6 - 9Q^4S = 0$, having the algebraic singularities

$$\{[[1,0,0],3,6,3],[[0,0,1],3,6,3]\}.$$

The signature of the Hessian k is not constant (Fig. 1). For example, the signature is (+,+) in the region

$$S > 0,\ -9S^2Q^2 + S^3 + 9Q^6 - 9Q^4S > 0.$$

That is why k represents a piecewise pseudo-Riemannian fundamental tensor. The null length curves are described by the differential equations

$$\frac{-S^2 + 9Q^4 + 4Q^2S}{8S^2}dS^2 - \frac{(3Q^2+S)Q}{S}dSdQ + (3Q^2 - S)dQ^2 = 0.$$

Now let us compute the Christoffel symbols of the metric k. We find the rational functions

$$K^1_{11} = -1/2(536Q^6S^2 + 972Q^{10} - 1740Q^8S + 264Q^4S^3 - 36S^4Q^2 + 4S^5 + 33Q^6S$$
$$-27Q^8 + 23S^2Q^4 + 3S^3Q^2)/S/(Q^2 - S)/(-80S^2Q^4 + 64S^3Q^2 - 8S^4 + 216Q^8$$
$$-192Q^6S - 9Q^6 - 6Q^4S - S^2Q^2)$$
$$K^1_{12} = 1/2Q(864Q^8 - 1824Q^6S + 992S^2Q^4 + 32S^3Q^2 - 64S^4 - 9Q^6 + 21Q^4S$$
$$+5S^2Q^2 - S^3)/(Q^2 - S)/(-80S^2Q^4 + 64S^3Q^2 - 8S^4 + 216Q^8 - 192Q^6S$$
$$-9Q^6 - 6Q^4S - S^2Q^2)$$
$$K^1_{22} = -S(9Q^6 - 27Q^4S + 7S^2Q^2 + 3S^3)/(Q^2 - S)/(-80S^2Q^4 + 64S^3Q^2 - 8S^4$$
$$+216Q^8 - 192Q^6S - 9Q^6 - 6Q^4S - S^2Q^2)$$
$$K^2_{11} = -Q(112S^2Q^4 - 27Q^8 + 126Q^6S - 14S^3Q^2 - 5S^4 - 896Q^6S^2$$
$$+1728Q^4S^3 + 96S^4Q^2 - 64S^5 + 2592Q^10 - 3456Q^8S)/S^2/(-80S^2Q^4$$
$$+64S^3Q^2 - 8S^4 + 216Q^8 - 192Q^6S - 9Q^6 - 6Q^4S - S^2Q^2)$$
$$K^2_{12} = 2(81Q^8 + 12Q^6S - 26S^2Q^4 - 20S^3Q^2 + S^4)/S/(-80S^2Q^4$$
$$+64S^3Q^2 - 8S^4 + 216Q^8 - 192Q^6S - 9Q^6 - 6Q^4S - S^2Q^2)$$
$$K^2_{22} = -1/2Q(9Q^6 - 33Q^4S - 21S^2Q^2 - 3S^3 - 160S^3Q^2 + 32S^4$$
$$+288Q^6S - 160S^2Q^4)/(Q^2 - S)/(-80S^2Q^4 + 64S^3Q^2 - 8S^4 + 216Q^8$$
$$-192Q^6S - 9Q^6 - 6Q^4S - S^2Q^2).$$

The geodesic equations are of the form

$$\frac{d^2S}{dt^2} + K^1_{11}\left(\frac{dS}{dt}\right)^2 + 2K^1_{12}\frac{dS}{dt}\frac{dQ}{dt} + K^1_{22}\left(\frac{dQ}{dt}\right)^2 = 0$$

$$\frac{d^2Q}{dt^2} + K^2_{11}\left(\frac{dS}{dt}\right)^2 + 2K^2_{12}\frac{dS}{dt}\frac{dQ}{dt} + K^2_{22}\left(\frac{dQ}{dt}\right)^2 = 0.$$

The non-zero components of the Riemannian curvature tensor are

$$K_{1221} = K_{2112} = -K_{1212}, K_{2121} = K_{1212},$$

where

$$\begin{gathered} K_{1212} = -\big(14S^7 + 128S^8 + 378Q^8S^3 + 710Q^6S^4 - 866Q^4S^5 + \\ +90Q^{12}S + 738Q^{10}S^2 + 45Q^{10}S + 33Q^8S^2 + 15Q^6S^3 \\ +70Q^2S^6 + 3Q^4S^4 - 1134Q^{14} + 15552Q^{16} \\ +15040Q^{10}S^3 - 51776Q^8S^4 - 46656Q^{14}S \\ +40768Q^6S^5 - 13888Q^4S^6 + 1088Q^2S^7 + 39744Q^{12}S^2\big) / \\ \big[4S^{2.5}(80Q^4S^2 - 64Q^2S^3 + 8S^4 - 216Q^8 + 192Q^6S + \\ +9Q^6 + 6Q^4S + Q^2S^2)(-Q^2 + S)^3\big] . \end{gathered}$$

Physical characteristics of our geometrical model: *the fundamental tensor* $k = Hess_h M$ *is piecewise pseudo-Riemannian; the components* k_{ij} *of the fundamental tensor* k *are like state potentials; the Christoffel symbols and the curvature tensor diverges in the extremal limit and along the curve where the fundamental tensor change the signature; the manifold* $(V \subset \{(S,Q)|S > 0\} \subset R^2, k = Hess_h M)$ *is non-flat; the sectional curvature has piecewise constant sign depending on the sign of det* (k) *and the sign of* K_{1212}.

Of course, there are essential differences between the previous physical geometric models, the most important being the degeneration curves, the null length curves and the sign of sectional curvature. When convexity of the mass fails (we pass from signature $(+,+)$ to signature $(+,-)$ or $(-,+)$) altogether we have instability, which is mathematically a bifurcation.

We underline that the covariant derivative can be computed using cov_diff in MAPLE.

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