Kinetic PDEs on the first order jet bundle

Constantin Udrişte, Ariana Pitea and Janina Mihăilă

Abstract. This paper reformulates a problem of Sharafutdinov [4] regarding the kinetic equation on a Riemannian manifold and extends the new variant from the single-time context to the multi-time context. Section 1 is dedicated to the single time case. It begins by pointing out that a symmetric tensor field of type $(0, m)$ and a geodesic determine a function which is homogeneous of degree $m - 1$. Also, it is proved that the vector field H associated to the geodesic spray G^t is tangent to the spherical bundle SM. The kinetic equation on $J^1(\mathbf{R}, M)$ associated to a symmetric tensor field is deduced. Then the paper presents the connection between the geodesic vector field (differential operator) H and the inner differentiation operator d. Section 2 extends the single-time case to the multi-time case. It is proved that p symmetric d-tensor fields and a minimal submanifold determine a function which is homogeneous of degree $m - 1$. Also,it points out that the affine vector fields H_α generate a completely integrable distribution tangent to the spherical bundle. The kinetic PDEs system is written explicitly and the vector fields H_α are connected with the inner differential operator d.

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1 The Kinetic PDE on $J^1(\mathbf{R}, M)$

Definition 1.1. *Let (M, g) be a compact Riemannian manifold and ∂M be the boundary of M. The Riemannian metric g is called simple if any two points $p, q \in \partial M$ can be joined by a unique geodesic*

$$x_{pq} \colon [0,1] \to \overline{M}, \quad \overline{M} = M \cup \partial M,$$
$$x(0) = p, \quad x(1) = q, \quad x(0,1) \subset M.$$

Let (M, g) be a Riemannian manifold, simple, compact, with the dimension n and $\tau_M = (TM, \pi, M)$, the tangent bundle, respectively $\tau'_M = (T'M, \pi', M)$ the cotangent bundle.

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Let us denote by $S^m\tau'_M$ the set of the symmetric tensor fields of type $(0,m)$ on τ'_M and by $C^\infty(S^m\tau'_M)$ the space of the smooth sections of this bundle.

Let

$$TM = \{(x,\xi)\,|\, x\in M, \xi \in T_xM\}$$

be the tangent space to M and

$$T^\circ M = \{(x,\xi)\in TM\,|\,\xi\neq 0\}$$

be the manifold of the nonzero tangent vectors. In a neighborhood $\mathcal{U}\subset M$ of the point $x\in M$ we consider a local coordinate system $(x^1,\dots,x^n)$. Then $\xi = \xi^i(x)\frac{}{\partial}\partial x^i(x)$ and $(x^1,\dots,x^n;\xi^1,\dots,\xi^n)$ is a local coordinate system on $\pi^{-1}(\mathcal{U})\subset TM$, where $\pi\colon TM\to M$ is the projection of the tangent bundle.

Let us consider the product bundle $(\mathbf{R}\times M, P, \mathbf{R})$, where P is the projection of $\mathbf{R}\times M$ onto $\mathbf{R}$. A map $\phi\colon I\subset\mathbf{R}\to\mathbf{R}\times M$ is called local section of $(\mathbf{R}\times M, P, \mathbf{R})$ if it satisfies the condition $P\circ\phi = id_I$. The set of all local sections of $(\mathbf{R}\times M, P, \mathbf{R})$ whose domains contain the point t will be denoted $\Gamma_t(P)$.

If $\phi\in\Gamma_t(P)$ and (t^1,x^i) are coordinate functions around $\phi(t)\in\mathbf{R}\times M$, then $\phi^i = x^i\circ\phi$.

Definition 1.2. *Two local sections $\phi,\psi\in\Gamma_t(P)$ are called* 1- *equivalent at the point t if*

$$\phi(t)=\psi(t),\quad \frac{d\phi^i}{dt}(t) = \frac{d\psi^i}{dt}(t).$$

The equivalence class containing ϕ is called the 1- jet of ϕ at t and is denoted by $j^1_t\phi$.

Definition 1.3. *The set*

$$J^1(\mathbf{R},M) = \{j^1_t\phi | t\in T, \phi\in\Gamma_t(P)\},$$

is called the first order jet manifold.

For $(x,\xi)\in T^\circ M$, we denote $\gamma_{x,\xi}\colon [\tau_-(x,\xi),\tau_+(x,\xi)]\to M$ the geodesic with maximal domain of definition determined by the initial conditions $\gamma_{x,\xi}(0)=x$ and $\dot\gamma_{x,\xi}(0)=\xi$.

Proposition 1.1. *Consider the tensor field $f\in C^\infty(S^m\tau'_M)$. The function*

$$u\colon T^\circ M\to\mathbf{R},\quad u(x,\xi) = \int_{\tau_-(x,\xi)}^0 f_{i_1\dots i_m}(\gamma_{x,\xi}(t))\dot\gamma^{i_1}_{x,\xi}(t)\cdots\dot\gamma^{i_m}_{x,\xi}(t)dt, \tag{1.1}$$

is positively homogeneous of degree $m-1$ in its second argument, that is

$$u(x,\lambda\xi) = \lambda^{m-1}u(x,\xi),\quad \lambda>0.$$

Proof. Let λ be a strictly positive real number. Because $\tau_-(x,\lambda\xi) = \lambda^{-1}\tau_-(x,\xi)$, the definition of the function u leads to

$$u(x,\lambda\xi) = \int_{\lambda^{-1}\tau_-(x,\xi)}^0 f_{i_1\dots i_m}(\gamma_{x,\lambda\xi}(t))\dot\gamma^{i_1}_{x,\lambda\xi}(t)\cdots\dot\gamma^{i_m}_{x,\lambda\xi}(t)dt.$$

Taking $t = \lambda^{-1}s$, $dt = \lambda^{-1}ds$, it follows

$$\begin{aligned}
u(x,\lambda\xi) &= \int_{\tau_-(x,\xi)}^{0} f_{i_1\ldots i_m}(\gamma_{x,\lambda\xi}(\lambda^{-1}s))\dot{\gamma}^{i_1}_{x,\lambda\xi}(\lambda^{-1}s)\cdots\dot{\gamma}^{i_m}_{x,\lambda\xi}(\lambda^{-1}s)\lambda^{-1}ds \\
&= \int_{\tau_-(x,\xi)}^{0} f_{i_1\ldots i_m}(\gamma_{x,\xi}(s))\lambda\dot{\gamma}^{i_1}_{x,\xi}(s)\cdots\lambda\dot{\gamma}^{i_m}_{x,\xi}(s)\lambda^{-1}ds \\
&= \lambda^{m-1}\int_{\tau_-(x,\xi)}^{0} f_{i_1\ldots i_m}(\gamma_{x,\xi}(s))\dot{\gamma}^{i_1}_{x,\xi}(s)\cdots\dot{\gamma}^{i_m}_{x,\xi}(s)ds \\
&= \lambda^{m-1}u(x,\xi).
\end{aligned}$$

Corollary 1.1. *The function u defined by the formula* (1.1) *is uniquely determined by its restriction to the spherical bundle (compact manifold of the unit tangent vectors), $SM = \{(x,\xi)\in TM \mid \|\xi\|^2 = g_{ij}\xi^i\xi^j = 1\}$.*

The border ∂SM of SM splits in two submanifolds

$$\partial_+SM = \{(x,\xi)\in SM \mid x\in\partial M, <\xi,\nu(x)>\geq 0\}$$

and

$$\partial_-SM = \{(x,\xi)\in SM \mid x\in\partial M, <\xi,\nu(x)>\leq 0\},$$

where $\nu(x)$ is the unit vector of the outer normal to the border ∂M.

Let γ_{pq} be the geodesic joining the points p and q.Comparing the single-ray transform [8]

$$I_f(\gamma_{pq}) = \int_{\gamma_{pq}} f_{i_1\ldots i_m}(x)\dot{x}^{i_1}\cdots\dot{x}^{i_m}dt$$

with the function u, it follows that u fulfills the following boundary conditions:

$$u(x,\xi)\Big|_{x\in\partial M} = \begin{cases} 0, & \text{if } (x,\xi)\in\partial_-SM \\ I_f(\gamma_{x,\xi}), & \text{if } (x,\xi)\in\partial_+SM. \end{cases} \tag{1.2}$$

Let $\gamma = \gamma_{x,\xi}$. The derivative of the function $u\circ\gamma$ with respect to t at $t=0$, is

$$\frac{d(u\circ\gamma)}{dt} = \frac{\partial u}{\partial x^i}\dot{\gamma}^i + \frac{\partial u}{\partial\xi^i}\ddot{\gamma}^i.$$

Using the equations of the geodesics, $\ddot{\gamma}^i + \Gamma^i_{jk}\dot{\gamma}^j\dot{\gamma}^k = 0$ (Γ^i_{jk} are the Christoffel symbols of (M,g)), it follows

$$\frac{d(u\circ\gamma)}{dt} = \frac{\partial u}{\partial x^i}\xi^i - \Gamma^i_{jk}\xi^j\xi^k\frac{\partial u}{\partial\xi^i}.$$

Thus, it appears the vector field $H = \xi^i\frac{\partial}{\partial x^i} - \Gamma^i_{jk}\xi^j\xi^k\frac{\partial}{\partial\xi^i}$ defined on TM, which is called geodesic vector field.

G^t, the 1-parameter group generated by H, is called the geodesic spray (geodesic pulverization).

Theorem 1.1. *The vector field H is tangent to the spherical bundle*

$$SM = \left\{(x,\xi) \in TM \mid \|\xi\|^2 = g_{ij}\xi^i\xi^j = 1\right\}.$$

Proof. The normal vector field to SM is $N_{SM} = \dfrac{\partial g_{jk}}{\partial x^i}\xi^j\xi^k dx^i + 2g_{iq}\xi^q d\xi^i$.
It is to notice that

$$\begin{aligned} < N_{SM}, H > &= \frac{\partial g_{jk}}{\partial x^i}\xi^i\xi^j\xi^k - 2\Gamma^i_{jk} g_{iq}\xi^j\xi^k\xi^q \\ &= \frac{\partial g_{jk}}{\partial x^i}\xi^i\xi^j\xi^k - \left(\frac{\partial g_{jq}}{\partial x^k} + \frac{\partial g_{kq}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^q}\right)\xi^j\xi^k\xi^q \\ &= 0. \end{aligned}$$

Proposition 1.2. *The vector field H and the integrand from* (1.1) *are connected by the relation (partial derivative equation)*

$$H(u) = f_{i_1\ldots i_m}(x)\xi^{i_1}\cdots\xi^{i_m}, \tag{1.3}$$

on the manifold $T^\circ M$.

The partial derivative equation (1.3) *is called the kinetic equation on $J^1(\mathbf{R}, M)$ associated to f.*

Proof. Consider the point $(x,\xi) \in T^\circ M$ and the geodesic $\gamma = \gamma_{x,\xi}$.Using the equality $\gamma_{\gamma(t_0),\dot\gamma(t_0)}(t) = \gamma(t+t_0)$ in (1.1), we obtain

$$u(\gamma(t_0), \dot\gamma(t_0)) = \int_{\tau_-(x,\xi)}^{t_0} f_{i_1\ldots i_m}(\gamma(t))\dot\gamma^{i_1}(t)\cdots\dot\gamma^{i_m}(t)dt.$$

Differentiating this equality with respect to t_0 and then considering $t_0 = 0$, it follows

$$\frac{\partial u}{\partial x^i}\frac{d\gamma^i}{dt} + \frac{\partial u}{\partial \xi^i}\frac{d\dot\gamma^i}{dt} = f_{i_1\ldots i_m}\xi^{i_1}\cdots\xi^{i_m}$$

or

$$\frac{\partial u}{\partial x^i}\dot\gamma^i + \frac{\partial u}{\partial \xi^i}\ddot\gamma^i = f_{i_1\ldots i_m}\xi^{i_1}\cdots\xi^{i_m},$$

where $\dot\gamma(0) = \xi$.

From the equations of the geodesics, $\ddot\gamma^i + \Gamma^i_{jk}\gamma^j\gamma^k = 0$, the last equality becomes

$$\frac{\partial u}{\partial x^i}\dot\gamma^i - \Gamma^i_{jk}\frac{\partial u}{\partial \xi^i}\dot\gamma^j\dot\gamma^k = f_{i_1\ldots i_m}\xi^{i_1}\cdots\xi^{i_m}.$$

Because $\dot\gamma(0) = \xi$, we obtained the required relation.

Remark 1.1. *The geodesic flow has the following geometric meaning: G^t is the translation along the geodesics at the moment t, i.e., $G^t(x,\xi) = (\gamma_{x,\xi}(t), \dot\gamma_{x,\xi}(t))$.*

In particular, the geodesic vector field H is tangent to the manifold SM (Theorem 1.1). By consequence, the equation (1.3) can be considered on SM.

Proposition 1.3. *The differential operator H and the operator of inner differentiation (the composition of the covariant derivative and the symmetrization) are related by the equality*

$$H(v_{i_1\dots i_{m-1}}(x)\xi^{i_1}\cdots\xi^{i_{m-1}}) = (dv)_{i_1\dots i_m}(x)\xi^{i_1}\cdots\xi^{i_m}.$$

Proof. The following relations hold

$$\begin{aligned}
&H(v_{i_1\dots i_{m-1}}(x)\xi^{i_1}\cdots\xi^{i_{m-1}})\\
&= \left(\xi^{i_m}\frac{\partial}{\partial x^{i_m}} - \Gamma^{i_m}_{jk}\xi^j\xi^k\frac{\partial}{\partial \xi^{i_m}}\right)(v_{i_1\dots i_{m-1}}(x)\xi^{i_1}\cdots\xi^{i_{m-1}})\\
&= \frac{\partial v_{i_1\dots i_{m-1}}}{\partial x^{i_m}}(x)\xi^{i_1}\cdots\xi^{i_{m-1}}\xi^{i_m} - \Gamma^{i_m}_{jk}v_{i_1\dots i_{m-1}}(x)\frac{\partial}{\partial\xi^{i_m}}(\xi^{i_1}\cdots\xi^{i_{m-1}})\xi^j\xi^k\\
&= \frac{\partial v_{i_1\dots i_{m-1}}}{\partial x^{i_m}}(x)\xi^{i_1}\cdots\xi^{i_{m-1}}\xi^{i_m}\\
&\quad -\Gamma^q_{jk}v_{i_1\dots i_{s-1}qi_{s+1}\dots i_{m-1}}(x)\xi^{i_1}\cdots\xi^{i_{s-1}}\xi^{i_{s+1}}\cdots\xi^{i_{m-1}}\xi^j\xi^k;\\
&(dv)_{i_1\dots i_m}(x)\xi^{i_1}\cdots\xi^{i_m} = \frac{1}{m!}\sum_{\sigma\in\Sigma_m} v_{\sigma(i_1)\dots\sigma(i_{m-1});\sigma(i_m)}(x)\xi^{i_1}\cdots\xi^{i_m},
\end{aligned}$$

where Σ_m is the set of the permutations of the elements $i_1,\dots,i_m$.

Because of the commutativity of the multiplication, $\xi^{a_1}\cdots\xi^{a_m}$ can be obtained in $m!$ ways. It follows

$$\begin{aligned}
(dv)_{i_1\dots i_m}(x)\xi^{i_1}\cdots\xi^{i_m} &= v_{a_1\dots a_{m-1};a_m}\xi^{a_1}\cdots\xi^{a_m}\\
&= \frac{\partial v_{a_1\dots a_{m-1}}}{\partial x^{a_m}}(x)\xi^{a_1}\cdots\xi^{a_m}\\
&\quad -\Gamma^q_{a_sa_m}v_{a_1\dots a_{s-1}qa_{s+1}\dots a_{m-1}}(x)\xi^{a_1}\cdots\xi^{a_m}.
\end{aligned}$$

The equality from the proposition was proved.

Thus, we arrive at the following problem [8]:

Problem 1. Let (M,g) be a simple compact differentiable manifold. Is the right-hand part of the equation (1.3), considered on SM, determined by the values $u\Big|_{\partial SM}$ of its solution on the boundary? In particular, does the equality $u\Big|_{\partial SM} = 0$ imply the homogeneity of order $m-1$ in ξ of the polynomial $u(x,\xi)$?

Definition 1.4. *The equation $H(u) = F(x,\xi)$, considered on SM, where F is a smooth function on $M\times T^\circ M$ and the right-hand side depends arbitrarily on ξ, is called the kinetic equation of the metric g and of the function F.*

Further, we shall present a physical meaning of this equation. Let us imagine a stationary distribution of particles moving on M. Any particle moves on a geodesic of the metric g with unit speed. The particles do not influence one another and do not influence the medium. Let us suppose that there are sources of the particles in M. The functions u and F are the densities of the particles and of the sources with respect to the volume element

$$dV^{2n} = \det(g_{ij})d\xi^1\wedge\dots\wedge d\xi^n\wedge dx^1\dots\wedge dx^n.$$

Then the equality $H(u) = F(x, \xi)$ is accomplished.

If the source F is known, in order to obtain a unique solution u of the PDE $H(u) = F(x, \xi)$ we must fix the value $u\Big|_{\partial_- SM}$. In particular, the first of the conditions (1.2) means the absence of the flow. The second one is used for the inverse problem of determining the sourse. For this, the sourse must depend polynomially on ξ; if $v \in C^\infty(S^{m-1}\tau'_M)$ and $v\Big|_{\partial M} = 0$, then the source $F(x,\xi) = (dv)_{i_1 \dots i_m}\xi^{i_1}\cdots\xi^{i_m}$ is invisible from outside. Does this construction exhaust all the sources invisible from outside, which are polynomial in ξ?

2 The Kinetic PDEs system on $J^1(T, M)$

Let (T, h) and (M, g) be Riemannian manifolds of dimensions p and n. Local coordinates on T and M will be written $t = (t^\alpha)$, $\alpha = \overline{1,p}$, and $x = (x^i)$, $i = \overline{1,n}$. The Christoffel symbols will be denoted respectively by $H^\alpha_{\beta\gamma}$, Γ^i_{jk}.

Let us consider the bundle $(T \times M, P, T)$, where P is the projection $P\colon T \times M \to T$.

A map $\phi\colon I \subset T \to T \times M$ is called local section of $(T \times M, P, T)$ if it satisfies the condition $P \circ \phi = id_I$. If $t \in T$, then the set of all local sections of $(T \times M, P, T)$, whose domains contain the point t, will be denoted $\Gamma_t(P)$.

If $\phi \in \Gamma_t(P)$ and (t^α, x^i) are coordinate function around $\phi(t) \in T \times M$, then $x^i(\phi(t)) = \phi^i(t)$.

Definition 2.1. *Two local sections $\phi, \psi \in \Gamma_t(P)$ are called* 1 *- equivalent at the point t if*

$$\phi(t) = \psi(t), \quad \frac{\partial \phi^i}{\partial t^\alpha}(t) = \frac{\partial \psi^i}{\partial t^\alpha}(t).$$

The equivalence class containing ϕ is called the 1- jet of ϕ at t and is denoted by $j^1_t\phi$.

Definition 2.2. *The set*

$$J^1(T, M) = \{j^1_t\phi | t \in T, \phi \in \Gamma_t(P)\}$$

is called first order jet bundle.

Definition 2.3. *The pair of metrics (h, g) is called simple if for any $\sigma \in \partial M$ there is a unique minimal submanifold T represented by $x\colon \Omega \cup \partial\Omega \to M \cup \partial M$, $x\Big|_{\partial\Omega} = \sigma$, Ω parallelipiped, $x(T \setminus \partial T) \subset M$, such that x depends smoothly on σ.*

Let (T, h) be a minimal submanifold with a maximal domain of definition, represented by $\gamma_{x,\xi_\mu}\colon \Omega \to M$, $\Omega = \prod_{\alpha=1}^{p}[\tau^\alpha_-(x, \xi_\mu), \tau^\alpha_+(x, \xi_\mu)]$, determined by the condition $\gamma_{x,\xi_\mu}(0) = x$, $(\gamma_{x,\xi_\mu})^i_\alpha(0) = \xi^i_\alpha(x)$, $\alpha = \overline{1,p}$ and fixed by a closed border σ of dimension $p-1$ included in ∂M. Suppose that ∂M is foliated by submanifolds of type σ.

Let $(x; \xi_1(x), \dots, \xi_p(x))$, $x \in M$, $\xi_\alpha(x) \in T_xM$, $\xi_\alpha(x) \neq 0$, $\alpha = \overline{1,p}$. In a neighborhood $\mathcal{U} \subset M$ of $x \in M$ we consider a local coordinate system $(x^1, \dots, x^n)$. Then $\xi_\alpha(x) = \xi^i_\alpha(x)\dfrac{\partial}{\partial x^i}(x)$.

Let us denote

$$(T^{\circ}T)^p = \{x; \xi_1, \dots, \xi_p\} \,|\, x \in M, \xi_\alpha(x) \in T_x M, \xi_\alpha \neq 0, \alpha = \overline{1,p}\}.$$

Proposition 2.1. *Let* $(F_\beta) = \left(F^{\alpha_1 \dots \alpha_m}_{i_1 \dots i_m \beta}\right)$ *be* p *symmetric d-tensor fields of class* C^∞ *on* $\Omega \times M$ *such that* $\dfrac{\partial F_\alpha}{\partial t^\beta} = \dfrac{\partial F_\beta}{\partial t^\alpha}$, $\alpha \neq \beta$. *The function*

$$u(t^\alpha, x^i, \xi^i_\alpha) = \int_{\Gamma_{\tau_-(x,\xi_\mu),0}} F^{\alpha_1 \dots \alpha_m}_{i_1 \dots i_m \beta}(\gamma_{x,\xi_\mu}(t^\alpha))(\gamma_{x,\xi_\mu})^{i_1}_{\alpha_1}(t^\alpha) \cdots \cdots (\gamma_{x,\xi_\mu})^{i_m}_{\alpha_m}(t^\alpha) dt^\beta, \tag{2.1}$$

where $\Gamma_{\tau_-(x,\xi_\mu),0}$ *is a arbitrary* C^1 *- curve joining the points*

$$\tau_-(x, \xi_\mu) = (\tau^1_-(x, \xi_\mu), \dots, \tau^p_-(x, \xi_\mu))$$

and $0 = (0, \dots, 0)$, *is positively homogeneous of degree* $m-1$, *that is*

$$u(t^\alpha, x^i, \lambda \xi^i_\alpha) = \lambda^{m-1} u(t^\alpha, x^i, \xi^i_\alpha), \quad \lambda > 0.$$

Proof. It is to notice that

$$u(t^\alpha, x^i, \lambda\xi^i_\alpha) = \int_{\Gamma_{\tau_-(x,\xi_\mu),0}} F^{\alpha_1 \dots \alpha_m}_{i_1 \dots i_m \beta}(\gamma_{x,\lambda\xi_\mu}(t^\alpha))(\gamma_{x,\lambda\xi_\mu})^{i_1}_{\alpha_1}(t^\alpha) \cdots \cdots (\gamma_{x,\lambda\xi_\mu})^{i_m}_{\alpha_m}(t^\alpha) dt^\beta, \tag{2.2}$$

must be path independent. Consequently, in normal coordinates, we can use the straight line joining the points $\tau_-(x, \lambda\xi_\mu)$ and 0

$$\frac{t^1}{\tau^1_-(x, \lambda\xi_\mu)} = \frac{t^2}{\tau^2_-(x, \lambda\xi_\mu)} = \dots = \frac{t^p}{\tau^p_-(x, \lambda\xi_\mu)} = s \in [0,1] \Rightarrow$$

$$\Rightarrow \begin{cases} t^1 = \lambda^{-1} s \tau^1_-(x, \xi_\mu); \\ t^2 = \lambda^{-1} s \tau^2_-(x, \xi_\mu); \\ \dots \\ t^p = \lambda^{-1} s \tau^p_-(x, \xi_\mu), \quad s \in [0,1]. \end{cases}$$

It follows

$$\begin{aligned} u(t^\alpha, x^i, \lambda\xi^i_\alpha) &= \int_1^0 F^{\alpha_1 \dots \alpha_m}_{i_1 \dots i_m \beta}(\gamma_{x,\lambda\xi_\mu}(\lambda^{-1} s \tau^\alpha_-(x, \xi_\mu)))(\gamma_{x,\lambda\xi_\mu})^{i_1}_{\alpha_1}(\lambda^{-1} s \tau^\alpha_-(x, \xi_\mu)) \\ &\quad \cdots (\gamma_{x,\lambda\xi_\mu})^{i_m}_{\alpha_m}(\lambda^{-1} s \tau^\alpha_-(x, \xi_\mu)) \lambda^{-1} \tau^\beta_-(x, \xi_\mu) ds \\ &= \int_1^0 F^{\alpha_1 \dots \alpha_m}_{i_1 \dots i_m \beta}(\gamma_{x,\xi_\mu}(s \tau^\alpha_-(x, \xi_\mu)))\lambda \; (\gamma_{x,\xi_\mu})^{i_1}_{\alpha_1}(s \tau^\alpha_-(x, \xi_\mu)) \\ &\quad \cdots \lambda \; (\gamma_{x,\xi_\mu})^{i_m}_{\alpha_m}(s \tau^\alpha_-(x, \xi_\mu)) \lambda^{-1} \tau^\beta_-(x, \xi_\mu) ds \\ &= \lambda^{m-1} \int_1^0 F^{\alpha_1 \dots \alpha_m}_{i_1 \dots i_m \beta}(\gamma_{x,\xi_\mu}(s \tau^\alpha_-(x, \xi_\mu)))(\gamma_{x,\xi_\mu})^{i_1}_{\alpha_1}(s \tau^\alpha_-(x, \xi_\mu)) \\ &\quad \cdots (\gamma_{x,\xi_\mu})^{i_m}_{\alpha_m}(s \tau^\alpha_-(x, \xi_\mu)) \tau^\beta_-(x, \xi_\mu) ds = \lambda^{m-1} u(t^\alpha, x^i, \xi^i_\alpha). \end{aligned}$$

Corollary 2.1. *The above-mentioned function u is uniquely determined by its restriction to the spherical bundle (compact manifold)*

$$SM^p = \{(x;\xi_1,\dots,\xi_p) \,|\, x \in M,\ \xi_\alpha \in T_xM,\ g_{ij}h^{\alpha\beta}\xi^i_\alpha\xi^j_\beta = 1\}.$$

The border ∂SM^p of the manifold SM^p is divided in two submanifolds

$$\Omega_+SM^p = \{(x;\xi_1,\dots,\xi_p) \in SM^p \,|\, \exists\alpha \in \{1,\dots,p\}, < \xi_\alpha(x),\ \nu(x) > \geq 0\}$$

and

$$\Omega_-SM^p = \{(x;\xi_1,\dots,\xi_p) \in SM^p \,|\ < \xi_\alpha,\ \nu(x) > \leq 0,\ \forall\alpha \in \{1,\dots,p\}\},$$

where $\nu(x)$ is the unit vector of the outer normal to ∂SM^p.

The function u fulfills the following boundary conditions:

$$u(t^\alpha, x^i, \xi^i_\alpha)\Big|_{x\in\partial SM^p} = \begin{cases} 0, \quad (x;\xi_1,\dots,\xi_p) \in \Omega_+SM^p \\ \displaystyle\int_{\Gamma_{\tau_-(x,\xi_\mu),0}} F^{\alpha_1\dots\alpha_m}_{i_1\dots i_m\beta}(\gamma_{x,\xi_\mu}(t^\alpha))(\gamma_{x,\xi_\mu})^{i_1}_{\alpha_1}(t^\alpha)\cdots \\ \qquad\cdots(\gamma_{x,\xi_\mu})^{i_m}_{\alpha_m}(t^\alpha)dt^\beta, (x;\xi_1,\dots,\xi_p) \in \Omega_-SM^p. \end{cases}$$

Definition 2.4. *A map $\gamma\colon \Omega \to M$ is called affine if it moves the geodesics of the manifold (Ω, h) into geodesics of the manifold (M, g).*

Proposition 2.2. *If the map $\gamma\colon \Omega \to M$, $\gamma^i = \gamma^i(t^1,\dots,t^p)$, is affine, then*

$$\frac{\partial^2\gamma^i}{\partial t^\alpha\partial t^\beta} - H^\nu_{\alpha\beta}\gamma^i_\nu - \Gamma^i_{jk}\gamma^j_\alpha\gamma^k_\beta = 0, \quad \alpha,\beta,\gamma = \overline{1,p}, \quad i,j,k = \overline{1,n},$$

$(t^1,\dots,t^p)$ and $(x^1,\dots,x^n)$ being local coordinate systems on (T,h) and (M,g) respectively.

Proof. Let $t^\alpha = t^\alpha(s)$, $s \in I \subseteq \mathbf{R}$, $0 \in I$, $t^\alpha(0) = 0$, $\alpha = \overline{1,p}$, be a geodesic of Ω, i.e. a solution of the differential system

$$\frac{d^2t^\alpha}{ds^2} + H^\alpha_{\beta\nu}\frac{dt^\beta}{ds}\frac{dt^\nu}{ds} = 0.$$

From $\gamma^i(s) = \gamma^i(t^\alpha(s))$, it follows $\dfrac{d\gamma^i}{ds} = \dfrac{\partial\gamma^i}{\partial t^\nu}\dfrac{dt^\nu}{ds}$.

$$\begin{aligned} \frac{d^2\gamma^i}{ds^2} &= \frac{\partial^2\gamma^i}{\partial t^\alpha\partial t^\beta}\frac{dt^\alpha}{ds}\frac{dt^\beta}{ds} + \frac{\partial\gamma^i}{\partial t^\alpha}\frac{d^2t^\alpha}{ds^2} \\ &= \frac{\partial^2\gamma^i}{\partial t^\beta\partial t^\nu}\frac{dt^\beta}{ds}\frac{dt^\nu}{ds} - H^\alpha_{\beta\nu}\frac{dt^\beta}{ds}\frac{dt^\nu}{ds}\frac{\partial\gamma^i}{\partial t^\alpha} \\ &= \left(\frac{\partial^2\gamma^i}{\partial t^\beta\partial t^\nu} - H^\alpha_{\beta\nu}\gamma^i_\alpha\right)\frac{dt^\beta}{ds}\frac{dt^\nu}{ds}. \end{aligned}$$

Then

$$\frac{d^2\gamma^i}{ds^2} + \Gamma^i_{jk}\frac{d\gamma^j}{ds}\frac{d\gamma^k}{ds} = \left(\frac{\partial^2\gamma^i}{\partial t^\beta\partial t^\gamma} - H^\alpha_{\beta\gamma}\gamma^i_\alpha + \Gamma^i_{jk}\gamma^j_\beta\gamma^k_\gamma\right)\frac{dt^\beta}{ds}\frac{dt^\gamma}{ds}.$$

$\gamma^i = \gamma^i(s)$ being geodesic of M, $\dfrac{d^2\gamma^i}{ds^2} + \Gamma^i_{jk}\dfrac{d\gamma^i}{ds}\dfrac{d\gamma^k}{ds} = 0$, whence we obtain the equality from the proposition.

Let us consider now the PDEs system

$$\frac{\partial^2\gamma^i}{\partial t^\alpha \partial t^\beta} - H^\nu_{\alpha\beta}\gamma^i_\nu + \Gamma^i_{jk}\gamma^j_\alpha\gamma^k_\beta = 0.$$

If this system has solutions of class C^3, it follows

$$\begin{aligned}
\frac{\partial^3\gamma^i}{\partial t^\lambda\partial t^\alpha\partial t^\beta} &= \frac{\partial H^\nu_{\alpha\beta}}{\partial t^\lambda}\gamma^i_\nu + H^\nu_{\alpha\beta}\frac{\partial^2\gamma^i}{\partial t^\nu\partial t^\lambda} - \frac{\partial\Gamma^i_{jk}}{\partial x^l}\gamma^l_\lambda\gamma^j_\alpha\gamma^k_\beta - \Gamma^i_{jk}\frac{\partial^2\gamma^j}{\partial t^\lambda\partial t^\alpha}\gamma^k_\beta \\
&\quad -\Gamma^i_{jk}\gamma^j_\alpha\frac{\partial^2\gamma^k}{\partial t^\lambda\partial t^\beta} \\
&= \frac{\partial H^\nu_{\alpha\beta}}{\partial t^\lambda}\gamma^i_\nu + H^\nu_{\alpha\beta}\left(H^\epsilon_{\nu\lambda}\gamma^i_\epsilon - \Gamma^i_{jk}\gamma^j_\nu\gamma^k_\lambda\right) - \frac{\partial\Gamma^i_{jk}}{\partial x^l}\gamma^l_\lambda\gamma^j_\alpha\gamma^k_\beta \\
&\quad -\Gamma^i_{jk}\gamma^k_\beta\left(H^\epsilon_{\alpha\lambda}\gamma^j_\epsilon - \Gamma^j_{lq}\gamma^l_\alpha\gamma^q_\lambda\right) - \Gamma^i_{jk}\gamma_a l^j\left(H^\epsilon_{\beta\lambda}\gamma^k_\epsilon - \Gamma^k_{lq}\gamma^l_\beta\gamma^q_\lambda\right) \\
&= \frac{\partial H^\nu_{\alpha\beta}}{\partial t^\lambda}\gamma^i_\nu + H^\nu_{\alpha\beta}H^\epsilon_{\nu\lambda}\gamma^i_\epsilon - \frac{\partial\Gamma^i_{jk}}{\partial x^l}\gamma^l_\lambda\gamma^j_\alpha\gamma^k_\beta + \Gamma^i_{jk}\Gamma^j_{lq}\gamma^l_\alpha\gamma^k_\beta\gamma^q_\lambda \\
&\quad -H^\nu_{\alpha\beta}\Gamma^i_{jk}\gamma^j_\nu\gamma^k_\lambda - H^\epsilon_{\alpha\lambda}\Gamma^i_{jk}\gamma^j_\epsilon\gamma^k_\beta - H^\epsilon_{\beta\lambda}\Gamma^i_{jk}\gamma^j_\alpha\gamma^k_\epsilon + \Gamma^i_{jk}\Gamma^k_{lq}\gamma^j_\alpha\gamma^l_\beta\gamma^q_\lambda.
\end{aligned}$$

Analogously,

$$\begin{aligned}
\frac{\partial^3\gamma^i}{\partial t^\beta\partial t^\alpha\partial t^\lambda} &= \frac{\partial H^\nu_{\alpha\lambda}}{\partial t^\beta}\gamma^i_\nu + H^\nu_{\alpha\lambda}H^\epsilon_{\nu\beta}\gamma^i_\epsilon - \frac{\Gamma^i_{jk}}{\partial x^l}\gamma^l_\beta\gamma^j_\alpha\gamma^k_\lambda + \Gamma^i_{jk}\Gamma^j_{lq}\gamma^l_\alpha\gamma^k_\lambda\gamma^q_\beta \\
&\quad -H^\nu_{\alpha\lambda}\Gamma^i_{jk}\gamma^j_\nu\gamma^k_\beta - H^\epsilon_{\alpha\beta}\Gamma^i_{jk}\gamma^j_\epsilon\gamma^k_\lambda - H^\epsilon_{\lambda\beta}\Gamma^i_{jk}\gamma^j_\alpha\gamma^k_\epsilon + \Gamma^i_{jk}\Gamma^k_{lq}\gamma^j_\alpha\gamma^l_\lambda\gamma^q_\beta.
\end{aligned}$$

The complete integrability conditions, $\dfrac{\partial^3\gamma^i}{\partial t^\lambda\partial t^\alpha\partial t^\beta} = \dfrac{\partial^3\gamma^i}{\partial t^\beta\partial t^\alpha\partial t^\lambda}$, lead to

$$\begin{aligned}
&\frac{\partial H^\nu_{\alpha\beta}}{\partial t^\lambda}\gamma^i_\nu + H^\nu_{\alpha\beta}H^\epsilon_{\nu\lambda}\gamma^i_\epsilon - \frac{\partial\Gamma^i_{jk}}{\partial x^l}\gamma^l_\lambda\gamma^j_\alpha\gamma^k_\beta + \Gamma^i_{jk}\Gamma^j_{lq}\gamma^l_\alpha\gamma^k_\beta\gamma^q_\lambda \\
&= \frac{\partial H^\nu_{\alpha\lambda}}{\partial t^\beta}\gamma^i_\nu + H^\nu_{\alpha\lambda}H^\epsilon_{\nu\beta}\gamma^i_\epsilon - \frac{\partial\Gamma^i_{jk}}{\partial x^l}\gamma^l_\beta\gamma^j_\alpha\gamma^k_\lambda + \Gamma^i_{jk}\Gamma^j_{lq}\gamma^l_\alpha\gamma^k_\lambda\gamma^q_\beta \\
&\Leftrightarrow \left(\frac{\partial H^\nu_{\alpha\beta}}{\partial t^\lambda} - \frac{\partial H^\nu_{\alpha\lambda}}{\partial t^\beta} + H^\epsilon_{\alpha\beta}H^\nu_{\epsilon\lambda} - H^\epsilon_{\alpha\lambda}H^\nu_{\epsilon\beta}\right)\gamma^i_\nu = \Bigg(\frac{\partial\Gamma^i_{jk}}{\partial\gamma^l} - \frac{\partial\Gamma^i_{jl}}{\partial\gamma^k} - \Gamma^i_{kr}\Gamma^r_{jl} \\
&\qquad\qquad + \Gamma^i_{rl}\Gamma^r_{jk}\Bigg)\gamma^l_\lambda\gamma^j_\alpha\gamma^k_\beta \\
&\Leftrightarrow H^\nu_{\alpha;\lambda\beta}\gamma^i_\nu = G^i_{j;lk}\gamma^l_\lambda\gamma^j_\alpha\gamma^k_\beta,
\end{aligned}$$

where $H^\gamma_{\alpha;\lambda\beta}$ and $G^i_{j;kl}$ are the Riemann tensors of the manifolds (Ω, h), (M, g) respectively.

The total derivative of the function $u \circ \gamma$, $\gamma = \gamma_{x,\xi_\mu}$, defined in (2.1), with respect to t^β, at $(t^1, \dots, t^p) = (0, \dots, 0)$ is

$$\begin{aligned}
\frac{d(u\circ\gamma)}{dt^\beta} &= \frac{\partial u}{\partial t^\beta} + \frac{\partial u}{\partial x^i}\xi^i_\beta + \frac{\partial u}{\partial\xi^i_\alpha}\frac{\partial^2\gamma^i}{\partial t^\alpha\partial t^\beta} \\
&= \frac{\partial u}{\partial t^\beta} + \frac{\partial u}{\partial x^i}\xi^i_\beta + \left(H^\gamma_{\alpha\beta}\xi^i_\gamma - \Gamma^i_{jk}\xi^j_\alpha\xi^k_\beta\right)\frac{\partial u}{\partial\xi^i_\alpha}.
\end{aligned}$$

Thus, we obtain the vector fields

$$H_\beta = \frac{\partial}{\partial t^\beta} + \xi^i_\beta \frac{\partial}{\partial x^i} + \left(H^\nu_{\alpha\beta}\xi^i_\nu - \Gamma^i_{jk}\xi^j_\alpha \xi^k_\beta\right)\frac{\partial}{\partial \xi^i_\alpha}, \quad \beta = \overline{1,p},$$

which are called affine vector fields.

Theorem 2.1. *The affine vector fields H_β, $\beta = \overline{1,p}$, are tangent to the manifold*

$$SM^p = \{(x;\xi_1,\dots,\xi_p)\,|\, x \in M,\ \xi_\alpha = T_xM,\ g_{ij}h^{\alpha\beta}\xi^i_\alpha \xi^j_\beta = 1\}.$$

Proof. The field N_{SM^p}, normal to SM^p, is

$$N_{SM^p} = \frac{\partial h^{\alpha\beta}}{\partial t^\nu} g_{ij}\xi^i_\alpha \xi^j_\beta dt^\nu + h^{\alpha\beta}\frac{\partial g_{ij}}{\partial x^k}\xi^i_\alpha\xi^j_\beta dx^k + 2h^{\nu\beta} g_{lj}\xi^j_\beta d\xi^l_\nu.$$

It is to notice that

$$\begin{aligned} < N_{SM^p}, H_\lambda > &= \frac{\partial h^{\alpha\beta}}{\partial t^\lambda} g_{ij}\xi^i_\alpha\xi^j_\beta + h^{\alpha\beta}\frac{\partial g_{ij}}{\partial x^k}\xi^i_\alpha\xi^j_\beta\xi^k_\lambda + 2h^{\nu\beta}g_{kj}\xi^j_\beta\xi^k_\epsilon H^\epsilon_{\nu\lambda} \\ &\quad -2h^{\nu\beta}g_{jk}\xi^j_\beta\xi^l_\nu\xi^q_\lambda\Gamma^k_{lq} = \left(\frac{\partial h^{\alpha\beta}}{\partial t^\lambda} + 2h^{\nu\beta}H^\alpha_{\nu\lambda}\right) g_{ij}\xi^i_\alpha\xi^j_\beta \\ &\quad + h^{\alpha\beta}\left(\frac{\partial g_{ij}}{\partial x^k} - 2g_{lj}\Gamma^l_{ik}\right)\xi^i_\alpha\xi^j_\beta\xi^k_\lambda = 0, \end{aligned}$$

because of the equalities

$$\Gamma^l_{ik} = \frac{1}{2}g^{lj}\left(\frac{\partial g_{kj}}{\partial x^i} + \frac{\partial g_{ij}}{\partial x^k} - \frac{\partial g_{ik}}{\partial x^j}\right); \quad \frac{\partial h^{\alpha\beta}}{\partial t^\lambda} = -h^{\alpha\nu}H^\beta_{\lambda\nu} - h^{\nu\beta}H^\alpha_{\lambda\nu}.$$

Theorem 2.2. *a) The distribution generated by the vector fields H_α, $\alpha = \overline{1,p}$, is completely integrable.*

b) The vector fields H_α, $\alpha = \overline{1,p}$, generate an affine p-flow.

Proof. The following relations hold

$$\begin{aligned} H_\beta(H_\lambda(u)) &= \left[\frac{\partial}{\partial t^\beta} + \xi^i_\beta\frac{\partial}{\partial x^i} + \left(H^\nu_{\alpha\beta}\xi^i_\nu - \Gamma^i_{rk}\xi^r_\alpha\xi^k_\beta\right)\frac{\partial}{\partial \xi^i_\alpha}\right] \\ &\quad \left[\frac{\partial u}{\partial t^\lambda} + \xi^j_\lambda\frac{\partial u}{\partial x^j} + \left(H^\epsilon_{\sigma\lambda}\xi^j_\epsilon - \Gamma^j_{lq}\xi^l_\sigma\xi^q_\lambda\right)\frac{\partial u}{\partial \xi^j_\sigma}\right] \\ &= \frac{\partial^2 u}{\partial t^\beta \partial t^\lambda} + \xi^j_\lambda \frac{\partial^2 u}{\partial t^\beta \partial x^j} + \left(H^\epsilon_{\sigma\lambda}\xi^j_\epsilon - \Gamma^j_{lq}\xi^l_\sigma\xi^q_\lambda\right)\frac{\partial^2 u}{\partial t^\beta \partial \xi^j_\sigma} \\ &\quad + \xi^i_\beta\frac{\partial^2 u}{\partial t^\lambda \partial x^i} + \xi^i_\beta\xi^j_\lambda\frac{\partial^2 u}{\partial x^i \partial x^j} - \frac{\partial \Gamma^j_{lq}}{\partial x^i}\xi^i_\beta\xi^l_\sigma\xi^q_\lambda\frac{\partial u}{\partial \xi^j_\sigma} \end{aligned}$$

$$
\begin{aligned}
&+\left(H^{\epsilon}_{\sigma\lambda}\xi^j_\epsilon-\Gamma^j_{lq}\xi^l_\sigma\xi^q_\lambda\right)\xi^i_\beta\frac{\partial^2 u}{\partial x^i\partial\xi^j_\sigma}+\left(H^{\nu}_{\alpha\beta}\xi^i_\nu-\Gamma^i_{rk}\xi^r_\alpha\xi^k_\beta\right)\frac{\partial^2 u}{\partial t^\lambda\partial\xi^i_\alpha}\\
&+\left(H^{\nu}_{\alpha\beta}\xi^i_\nu-\Gamma^i_{rk}\xi^r_\alpha\xi^k_\beta\right)\delta^j_i\delta^\lambda_\alpha\frac{\partial u}{\partial x^j}+\left(H^{\nu}_{\alpha\beta}\xi^i_\nu-\Gamma^i_{rk}\xi^r_\alpha\xi^k_\beta\right)\xi^j_\lambda\frac{\partial^2 u}{\partial x^j\partial\xi^i_\alpha}\\
&+\left(H^{\nu}_{\alpha\beta}\xi^i_\nu-\Gamma^i_{rk}\xi^r_\alpha\xi^k_\beta\right)\left(H^{\epsilon}_{\sigma\lambda}\delta^j_i\delta^\alpha_\epsilon-\Gamma^j_{lq}\delta^l_i\delta^\sigma_\alpha\xi^q_\lambda-\Gamma^j_{lq}\xi^l_\sigma\delta^q_i\delta^\lambda_\alpha\right)\frac{\partial u}{\partial\xi^j_\sigma}\\
&+\left(H^{\nu}_{\alpha\beta}\xi^i_\nu-\Gamma^i_{rk}\xi^r_\alpha\xi^k_\beta\right)\left(H^{\epsilon}_{\sigma\lambda}\xi^j_\epsilon-\Gamma^j_{lq}\xi^l_\sigma\xi^q_\lambda\right)\frac{\partial^2 u}{\partial\xi^i_\alpha\partial\xi^j_\sigma}=\frac{\partial^2 u}{\partial t^\beta\partial t^\lambda}\\
&+\xi^j_\lambda\frac{\partial^2 u}{\partial t^\beta\partial x^j}+\left(H^{\epsilon}_{\sigma\lambda}\xi^j_\epsilon-\Gamma^j_{lq}\xi^l_\sigma\xi^q_\lambda\right)\frac{\partial^2 u}{\partial t^\beta\partial\xi^j_\sigma}+\xi^i_\beta\frac{\partial^2 u}{\partial t^\lambda\partial x^i}\\
&+\xi^i_\beta\xi^j_\lambda\frac{\partial^2 u}{\partial x^i\partial x^j}-\frac{\partial\Gamma^j_{lq}}{\partial x^i}\xi^i_\beta\xi^l_\sigma\xi^q_\lambda\frac{\partial u}{\partial\xi^j_\sigma}\\
&+\left(H^{\epsilon}_{\sigma\lambda}\xi^i_\beta\xi^j_\epsilon-\Gamma^j_{lq}\xi^i_\beta\xi^l_\sigma\xi^q_\lambda\right)\frac{\partial^2 u}{\partial x^i\partial\xi^j_\sigma}+\left(H^{\nu}_{\alpha\beta}\xi^i_\nu-\Gamma^i_{rk}\xi^r_\alpha\xi^k_\beta\right)\frac{\partial^2 u}{\partial t^\lambda\partial\xi^i_\alpha}\\
&+\left(H^{\nu}_{\lambda\beta}\xi^j_\nu-\Gamma^j_{rk}\xi^r_\lambda\xi^k_\beta\right)\frac{\partial u}{\partial x^j}+\left(H^{\nu}_{\alpha\beta}\xi^i_\nu\xi^j_\lambda-\Gamma^i_{rk}\xi^r_\alpha\xi^k_\beta\xi^j_\lambda\right)\frac{\partial^2 u}{\partial x^j\partial\xi^i_\alpha}\\
&+\Big(H^{\nu}_{\alpha\beta}H^{\alpha}_{\sigma\lambda}\xi^j_\nu-H^{\alpha}_{\sigma\lambda}\Gamma^j_{rk}\xi^r_\alpha\xi^k_\beta-H^{\nu}_{\sigma\beta}\Gamma^j_{lq}\xi^l_\nu\xi^q_\lambda+\Gamma^l_{rk}\Gamma^j_{lq}\xi^r_\sigma\xi^k_\beta\xi^q_\lambda\\
&-H^{\nu}_{\lambda\beta}\Gamma^j_{lq}\xi^q_\nu\xi^l_\sigma+\Gamma^q_{rk}\Gamma^j_{lq}\xi^r_\lambda\xi^k_\beta\xi^l_\sigma\Big)\frac{\partial u}{\partial\xi^j_\sigma}\\
&+\left(H^{\nu}_{\alpha\beta}\xi^i_\nu-\Gamma^i_{rk}\xi^r_\alpha\xi^k_\beta\right)\left(H^{\epsilon}_{\sigma\lambda}\xi^j_\epsilon-\Gamma^j_{lq}\xi^l_\sigma\xi^q_\lambda\right)\frac{\partial^2 u}{\partial\xi^i_\alpha\xi^j_\sigma}.
\end{aligned}
$$

It follows

$$
\begin{aligned}
[H_\beta,H_\lambda](u)&=H_\beta(H_\lambda(u))-H_\lambda(H_\beta(u))\\
&=\Bigg(-\frac{\partial\Gamma^j_{lq}}{\partial x^i}\xi^i_\beta\xi^l_\sigma\xi^q_\lambda+\frac{\partial\Gamma^j_{lq}}{\partial x^i}\xi^i_\lambda\xi^l_\sigma\xi^q_\beta+H^{\nu}_{\alpha\beta}H^{\alpha}_{\sigma\lambda}\xi^j_\nu-H^{\nu}_{\alpha\lambda}H^{\alpha}_{\sigma\beta}\xi^j_\nu\\
&\qquad+\Gamma^l_{rk}\Gamma^j_{lq}\xi^r_\sigma\xi^k_\beta\xi^q_\lambda-\Gamma^l_{rk}\Gamma^j_{lq}\xi^r_\sigma\xi^k_\lambda\xi^q_\beta\Bigg)\frac{\partial u}{\partial\xi^j_\sigma}\\
&=\Bigg[\left(H^{\nu}_{\alpha\beta}H^{\alpha}_{\sigma\lambda}-H^{\nu}_{\alpha\lambda}H^{\alpha}_{\sigma\beta}\right)\xi^j_\nu+\left(\frac{\partial\Gamma^j_{lq}}{\partial x^i}-\frac{\partial\Gamma^j_{li}}{\partial x^q}+\Gamma^r_{lq}\Gamma^j_{ri}-\Gamma^r_{li}\Gamma^j_{rq}\right)\\
&\qquad\cdot\xi^i_\lambda\xi^l_\sigma\xi^q_\beta\Bigg]\frac{\partial u}{\partial\xi^j_\sigma}=\left[\left(H^{\nu}_{\alpha\beta}H^{\alpha}_{\sigma\lambda}-H^{\nu}_{\alpha\lambda}H^{\alpha}_{\sigma\beta}\right)\xi^i_\nu-G^j_{l,iq}\xi^i_\lambda\xi^l_\sigma\xi^q_\beta\right]\frac{\partial u}{\partial\xi^j_\sigma}.
\end{aligned}
$$

In this way $[H_\beta,H_\lambda]=C^{\alpha}_{\beta\lambda}H_\alpha$.

Proposition 2.3. *The vector fields H_β, $\beta=\overline{1,p}$, and the d-tensor fields from the definition* 2.4 *are related by*

$$
H_\beta(u)=F^{\alpha_1\ldots\alpha_m}_{i_1\ldots i_m\beta}(x)\xi^{i_1}_{\alpha_1}\cdots\xi^{i_m}_{\alpha_m},\quad \beta=\overline{1,p}. \tag{2.3}
$$

The system (2.3) *is called the kinetic PDEs system.*

Proof. Let P be the potential of the vector field

$$X = (X_\beta) = \left(F^{\alpha_1\dots\alpha_m}_{i_1\dots i_m\beta}\xi^{i_1}_{\alpha_1}\cdots\xi^{i_m}_{\alpha_m}\right).$$

Let $t = (t^1, \dots, t^p)$. It follows that

$$P(t) = \int_0^{t^1} X_1(s,0,\dots,0)ds + \int_0^{t^2} X_2(t^1,s,0,\dots,0)ds + \cdots + \int_0^{t^p} X_p(t^1,t^2,\dots,t^{p-1},s)ds + \mathcal{C}.$$

Let us denote $t_0 = (t_0^1, \dots, t_0^p)$. The equality $\gamma_{\gamma(t_0),\gamma^i_\alpha(t_0)} = \gamma(t+t_0)$ implies

$$u(t_0, x(t_0), x^i_\alpha(t_0)) = \int_{\tau_-(x,\xi_\mu)}^{t_0} X_\beta dt^\beta \Rightarrow$$
$$\Rightarrow u(t_0, x(t_0), x^i_\alpha(t_0)) = P(t_0) - P(\tau_-(x,\xi_\mu)).$$

Differentiating the last relation with respect to t^β and then considering $t_0 = 0$, it follows

$$H_\beta(u) = F^{\alpha_1\dots\alpha_m}_{i_1\dots i_m\beta}(x)\xi^{i_1}_{\alpha_1}\cdots\xi^{i_m}_{\alpha_m}.$$

Proposition 2.4. *The differential operators H_β, $\beta = \overline{1,p}$, and the inner differentiation operator d are connected by the equality*

$$H_\beta\left(v^{\alpha_1\dots\alpha_{m-1}}_{i_1\dots i_{m-1}}(t,x)\xi^{i_1}_{\alpha_1}\cdots\xi^{i_{m-1}}_{\alpha_{m-1}}\right) = (dv)^{\alpha_1\dots\alpha_{m-1}}_{i_1\dots i_{m-1}\beta}(t,x)\xi^{i_1}_{\alpha_1}\cdots\xi^{i_{m-1}}_{\alpha_{m-1}}, \beta = \overline{1,p}.$$

Proof. The following relations hold

$$\begin{aligned}
&H_\beta\left(v^{\alpha_1\dots\alpha_{m-1}}_{i_1\dots i_{m-1}}(t,x)\xi^{i_1}_{\alpha_1}\cdots\xi^{i_{m-1}}_{\alpha_{m-1}}\right)\\
&= \left[\frac{\partial}{\partial t^\beta} + \xi^k_\beta\frac{\partial}{\partial x^k} + \left(H^\nu_{\epsilon\beta}\xi^k_\nu - \Gamma^k_{jl}\xi^j_\epsilon\xi^l_\beta\right)\frac{\partial}{\partial \xi^k_\epsilon}\right]\left(v^{\alpha_1\dots\alpha_{m-1}}_{i_1\dots i_{m-1}}(t,x)\xi^{i_1}_{\alpha_1}\cdots\xi^{i_{m-1}}_{\alpha_{m-1}}\right)\\
&= \frac{\partial v^{\alpha_1\dots\alpha_{m-1}}_{i_1\dots i_{m-1}}}{\partial t^\beta}(t,x)\xi^{i_1}_{\alpha_1}\cdots\xi^{i_{m-1}}_{\alpha_{m-1}} + \frac{\partial v^{\alpha_1\dots\alpha_{m-1}}_{i_1\dots i_{m-1}}}{\partial x^k}\xi^{i_1}_{\alpha_1}\cdots\xi^{i_{m-1}}_{\alpha_{m-1}}\xi^k_\beta\\
&+v^{\alpha_1\dots\alpha_{m-1}}_{i_1\dots i_{m-1}}(t,x)\left(H^\nu_{\epsilon\beta}\xi^k_\nu - \Gamma^k_{jl}\xi^j_\epsilon\xi^l_\beta\right)\frac{\partial}{\partial \xi^k_\epsilon}\left(\xi^{i_1}_{\alpha_1}\cdots\xi^{i_{m-1}}_{\alpha_{m-1}}\right)\\
&= \frac{\partial v^{\alpha_1\dots\alpha_{m-1}}_{i_1\dots i_{m-1}}}{\partial t^\beta}(t,x)\xi^{i_1}_{\alpha_1}\cdots\xi^{i_{m-1}}_{\alpha_{m-1}} + \frac{\partial v^{\alpha_1\dots\alpha_{m-1}}_{i_1\dots i_{m-1}}}{\partial x^k}(t,x)\xi^{i_1}_{\alpha_1}\cdots\xi^{i_{m-1}}_{\alpha_{m-1}}\xi^k_\beta\\
&+\left(H^\nu_{\epsilon\beta}\xi^k_\nu - \Gamma^k_{jl}\xi^j_\epsilon\xi^l_\beta\right)v^{\alpha_1\dots\alpha_{s-1}\epsilon\alpha_{s+1}\dots\alpha_{m-1}}_{i_1\dots i_{s-1}k i_{s+1}\dots i_{m-1}}\cdot\xi^{i_1}_{\alpha_1}\cdots\widehat{\xi^{i_s}_{\alpha_s}}\cdots\xi^{i_{m-1}}_{\alpha_{m-1}}
\end{aligned}$$

$$= \frac{\partial v^{\alpha_1\ldots\alpha_{m-1}}_{i_1\ldots i_{m-1}}}{\partial t^\beta}(t,x)\xi^{i_1}_{\alpha_1}\cdots\xi^{i_{m-1}}_{\alpha_{m-1}} + H^\nu_{\epsilon\beta}\xi^k_\nu\xi^{i_1}_{\alpha_1}\cdots\widehat{\xi^{i_s}_{\alpha_s}}\cdots\xi^{i_{m-1}}_{\alpha_{m-1}}$$

$$\cdot v^{\alpha_1\ldots\alpha_{s-1}\epsilon\alpha_{s+1}\ldots\alpha_{m-1}}_{i_1\ldots i_{s-1}ki_{s+1}\ldots i_{m-1}} + \frac{\partial v^{\alpha_1\ldots\alpha_{m-1}}_{i_1\ldots i_{m-1}}}{\partial x^{i_m}}(t,x)\xi^{i_1}_{\alpha_1}\cdots\xi^{i_{m-1}}_{\alpha_{m-1}}\xi^{i_m}_\gamma$$

$$-\Gamma^{i_m}_{jl}\xi^j_\epsilon\xi^l_\beta v^{\alpha_1\ldots\alpha_{s-1}\epsilon\alpha_{s+1}\ldots\alpha_{m-1}}_{i_1\ldots i_{s-1}i_m i_{s+1}\ldots i_{m-1}}\xi^{i_1}_{\alpha_1}\ldots\hat{\xi}^{i_s}_{\alpha_s}\ldots\xi^{i_{m-1}}_{\alpha_{m-1}}$$

$$= v^{\alpha_1\ldots\alpha_{m-1}}_{i_1\ldots i_{m-1};\beta}(t,x)\xi^{i_1}_{\alpha_1}\cdots\xi^{i_{m-1}}_{\alpha_{m-1}} + v^{\alpha_1\ldots\alpha_{m-1}}_{i_1\ldots i_{m-1};i_m}(t,x)\xi^{i_1}_{\alpha_1}\cdots\xi^{i_{m-1}}_{\alpha_{m-1}}\xi^{i_m}_\beta$$

$$= \frac{1}{[(m-1)!]^2}\sum_{\substack{\sigma\in\Sigma_{m-1}\\ \tau\in\Sigma'_{m-1}}} v^{\sigma(\alpha_1)\ldots\sigma(\alpha_{m-1})}_{\tau(i_1)\ldots\tau(i_{m-1})\beta}\xi^{\tau(i_1)}_{\sigma(\alpha_1)}\cdots\xi^{\tau(i_{m-1})}_{\sigma(\alpha_{m-1})}$$

$$+\frac{1}{(m-1)!m!}\cdot\sum_{\substack{\sigma\in\Sigma_{m-1}\\ \mu\in\Sigma'_m}} v^{\sigma(\alpha_1)\ldots\sigma(\alpha_{m-1})}_{\mu(i_1)\ldots\mu(i_{m-1});\mu(i_m)}\xi^{\mu(i_1)}_{\sigma(\alpha_1)}\cdots\xi^{\mu(i_{m-1})}_{\sigma(\alpha_{m-1})}\xi^{\mu(i_m)}_\beta$$

$$= (dv)^{\alpha_1\ldots\alpha_{m-1}}_{i_1\ldots i_{m-1};\beta}(t,x)\xi^{i_1}_{\alpha_1}\cdots\xi^{i_{m-1}}_{\alpha_{m-1}},$$

where Σ_{m-1}, Σ'_{m-1} and Σ'_m are the sets of permutations of $\{\alpha_1,\ldots,\alpha_{m-1}\}$, $\{i_1,\ldots,i_{m-1}\}$ and $\{i_1,\ldots,i_m\}$ respectively.

Thus, we arrive at the following problem [8]:

Problem 2. Let (M,g) be a compact differentiable manifold. To what extent the right-hand side of (2.3), considered on SM^p, is determined by the values $u\Big|_{\partial SM^p}$ of the solution on the boundary? In particular, does the equality $u\Big|_{\partial SM^p} = 0$ imply the homogeneity of order $m-1$ in $(\xi^1,\ldots,\xi^p)$ of the polynomial u?

Definition 2.5. *The system formed by the equations $H_\beta(u) = F_\beta(x;\xi^1,\ldots,\xi^p)$, considered on SM^p, is called the kinetic partial equations system of (h,g) and of the functions (F_β) on $J^1(T,M)$. The function u is called the density of field and the set of functions F_β is called the density of sources with respect to the volume form*

$$dV^{p+n+np} = \det(g_{ij})dt^1\wedge\ldots\wedge dt^p\wedge dx^1\wedge\ldots\wedge dx^n\Big(\bigwedge_{\substack{\alpha=\overline{1,p}\\ i=\overline{1,n}}} d\xi^i_\alpha\Big).$$

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Authors' addresses:

Constantin Udrişte and Ariana Pitea
University Politehnica of Bucharest,
Faculty of Applied Sciences, Department Mathematics I,
Splaiul Independenţei 313, RO-060042, Bucharest, Romania.
e-mail: udriste@mathem.pub.ro and apitea@mathem.pub.ro

Janina Mihăilă
Ecological University of Bucharest,
Vasile Milea 1G,Ro-061341, Bucharest, Romania.
e-mail: janinamihaelamihaila@yahoo.it