

# The dynamical rigid body with memory

Ion Doru Albu, Mihaela Neamțu and Dumitru Oprea

**Abstract.** In the present paper we describe the dynamics of the revised rigid body, the dynamics of the rigid body with distributed delays and the dynamics of the fractional rigid body. We analyze the stationary states for given values of the rigid body's parameters.

**M.S.C. 2000:** 26A33, 53C63, 58A05, 58A40.

**Key words:** rigid body, revised rigid body, distributed delay, fractional derivative.

## 1 Introduction

In mechanical problems, the dynamics of the rigid body has an important role. In many papers M. Puta analyzed the dynamics of the rigid body with control and he obtained important results.

Recently, the dynamics of revised rigid body, the dynamics of the rigid body with distributed delay and the dynamics of the rigid body with fractional derivative have been studied. The last two aspects represent the dynamics of the rigid body with memory.

Our paper studies the dynamics of the revised rigid body obtained by a metriplectic structure which is canonically associated. We define the dynamics of the rigid body with distributed delays. For the Euler-Poincare dynamics of the rigid body we analyze the linearized system in an equilibrium point. We obtain the existence conditions for the Hopf bifurcation with respect to the parameter of the repartition density which defines the distributed delay. Also, we define the fractional rigid body dynamics using the Caputo fractional derivative.

## 2 The revised differential equations for the rigid body

The differential equations for the rigid body in  $\mathbb{R}^3$  are described by a 2-antisymmetric tensor field  $P$  and the Hamiltonian function  $h$  given by:

---

BSG Proceedings 15. The International Conference "Differential Geometry - Dynamical Systems" DGDS-2007, October 5-7, 2007, Bucharest-Romania, pp. 1-10.  
© Balkan Society of Geometers, Geometry Balkan Press 2008.

(2.1)

$$P(x) = (P^{ij}(x)) = \begin{pmatrix} 0 & x^3 & -x^2 \\ -x^3 & 0 & x^1 \\ x^2 & -x^1 & 0 \end{pmatrix}, \quad h = \frac{1}{2}(a_1(x^1)^2 + a_2(x^2)^2 + a_3(x^3)^2),$$

where  $(x^1, x^2, x^3)^T \in \mathbb{R}^3$ ,  $a_i \in \mathbb{R}_+$ ,  $i = 1, 2, 3$ ,  $a_1 > a_2 > a_3$ . These differential equations are:

(2.2)

$$\dot{x}(t) = P(x(t))\nabla_x h(x(t)),$$

where  $\dot{x}(t) = (\dot{x}^1(t), \dot{x}^2(t), \dot{x}^3(t))^T$  and  $\nabla_x h$  is the gradient of  $h$  with respect to the canonical metric on  $\mathbb{R}^3$ . The differential equations (2.2) have been studied by M. Puta in [8].

Let  $\mathbb{R}^3$  be the space interpreted as the space of body angular velocities  $\Omega$  equipped with the cross product as the Lie bracket. On this space, we consider the standard Lagrangian kinetic energy  $L(\Omega) = \frac{1}{2}I \cdot \Omega$ , where  $I = \text{diag}(I_1, I_2, I_3)$  is the moment of inertial tensor, so that the general Euler-Poincare equations become the standard rigid body equations for a freely spinning rigid body:

(2.3)

$$I\dot{\Omega} = (I \cdot \Omega) \times \Omega.$$

If  $M = I \cdot \Omega$  is the angular momentum, then, (2.3) is:

(2.4)

$$\dot{M} = M \times \Omega.$$

If  $M = (I_1x(t), I_2y(t), I_3z(t))^T$ ,  $\Omega = (x(t), y(t), z(t))^T$  from (2.4) results:

(2.5)

$$\dot{x}(t) = \frac{I_2 - I_3}{I_1}y(t)z(t), \quad \dot{y}(t) = \frac{I_3 - I_1}{I_2}x(t)z(t), \quad \dot{z}(t) = \frac{I_1 - I_2}{I_3}x(t)y(t),$$

with  $I_1 > I_2 > I_3$ . The differential equations (2.5) have been studied by M. Puta in [8]. The revised differential equations for the rigid body given by (2.2) have been studied in [5]. They are described by  $P$ ,  $h$  and the tensor fields  $g = (g^{ij}(x))$ , where  $P$  and  $h$  are given by (2.1) and  $g$  is defined by  $g(x) = (g^{ij}(x))$ ,

$$g^{ij}(x) = \frac{\partial h(x)}{\partial x^i} \frac{\partial h(x)}{\partial x^j}, i \neq j, g^{ij}(x) = - \sum_{k=1, k \neq i}^3 \left( \frac{\partial h(x)}{\partial x^k} \right)^2, i = 1, 2, 3$$

and the Casimir function of the Poisson structure  $P$  is:

$$c(x) = \frac{1}{2}((x^1)^2 + (x^2)^2 + (x^3)^2).$$

The structure  $(\mathbb{R}^3, P, g, h, c)$  is called a metriplectic manifold of second kind. The revised differential system associated to (2.2) is given by:

(2.6)

$$\dot{x}(t) = P(x)\nabla_x h(x) + g(x)\nabla_x c(x).$$

**Proposition 2.1.** (i) The differential equations (2.2) are given by:

$$(2.7) \quad \dot{x}^1(t) = (a_2 - a_3)x^2(t)x^3(t), \dot{x}^2(t) = (a_3 - a_1)x^1(t)x^3(t), \dot{x}^3(t) = (a_1 - a_2)x^1(t)x^2(t);$$

(ii) The differential equations (2.6) are given by:

$$(2.8) \quad \begin{aligned} \dot{x}^1(t) &= (a_2 - a_3)x^2(t)x^3(t) + a_2(a_1 - a_2)x^1(t)(x^2(t))^2 + a_3(a_1 - a_3)x^1(t)(x^3(t))^2 \\ \dot{x}^2(t) &= (a_3 - a_1)x^1(t)x^3(t) + a_3(a_2 - a_3)x^2(t)(x^3(t))^2 + a_1(a_2 - a_1)x^2(t)(x^1(t))^2 \\ \dot{x}^3(t) &= (a_1 - a_2)x^1(t)x^2(t) + a_1(a_3 - a_1)x^3(t)(x^1(t))^2 + a_2(a_3 - a_2)x^3(t)(x^2(t))^2. \end{aligned}$$

The equilibrium points of the system (2.7) are studied in [8] and the equilibrium points of the system (2.8) in [5].

### 3 The differential system with distributed delay for the rigid body

We consider the space  $\mathbb{R}^3$ , the product  $\mathbb{R}^3 \times \mathbb{R}^3 = \{(\tilde{x}, x), \tilde{x} \in \mathbb{R}^3, x \in \mathbb{R}^3\}$  and the canonical projections  $\pi_i : \mathbb{R}^3 \times \mathbb{R}^3 \rightarrow \mathbb{R}^3$ ,  $i = 1, 2$ . A vector field  $X \in \mathcal{X}(\mathbb{R}^3 \times \mathbb{R}^3)$  satisfying the condition  $X(\pi_1^* f) = 0$ , for any  $f \in C^\infty(\mathbb{R}^3)$  is given by:

$$X(\tilde{x}, x) = \sum_{i=1}^3 X_i(\tilde{x}, x) \frac{\partial}{\partial x_i}.$$

The differential system associated to  $X$  is given by:

$$\dot{x}^i(t) = X^i(\tilde{x}(t), x(t)), i = 1, 2, 3.$$

A differential system with distributed delay is a differential system associates to a vector field  $X \in \mathcal{X}(\mathbb{R}^3 \times \mathbb{R}^3)$  for which  $X(\pi_1^* f) = 0$ , for any  $f \in C^\infty(\mathbb{R}^3)$  and it is given by (1), where  $\tilde{x}(t)$  is:

$$\tilde{x}(t) = \int_0^k k(s)x(t-s)ds,$$

where  $k(s)$  is a density of repartition. In what follows, we will consider the case of the following densities of repartition:

(i) the uniform density with:

$$k_\tau(s) = \begin{cases} 0, & 0 \leq s \leq a \\ \frac{1}{\tau}, & a \leq s \leq a + \tau \\ 0, & s > a + \tau \end{cases},$$

where  $a > 0$ ,  $\tau > 0$  are given numbers;

(ii) the exponential density, with  $k_\alpha(s) = \alpha e^{-\alpha s}$ ,  $\alpha > 0$ ;

(iii) the Erlang density, with  $k_\alpha(s) = \alpha^2 s e^{-\alpha s}$ ,  $\alpha > 0$ ;

(iv) the Dirac density, with  $k_\tau(s) = \delta(s - \tau)$ ,  $\tau > 0$ .

The initial condition is:  $x(s) = \varphi(s)$ ,  $s \in (-\infty, 0]$ , with  $\varphi : (-\infty, 0] \rightarrow \mathbb{R}^3$  a smooth map. Some systems of differential equations with distributed delay in  $\mathbb{R}^3$  were studied in [1], [2]. For such a system, we consider relevant the geometric properties of the vector field which defines the system, for example first integrals (constants of the motion), Morse functions, almost metriplectic structure, etc. The differential equations with distributed delay for rigid body are generated by a 2-antisymmetric tensor field  $P$  on  $\mathbb{R}^3 \times \mathbb{R}^3$  that satisfies the following relations:  $P(\pi_1^* f_1, \pi_2^* f_2) = 0$ ,  $P(\pi_2^* f_1, \pi_2^* f_2) = 0$ , for all  $f_1, f_2 \in C^\infty(\mathbb{R}^3)$  and  $h \in C^\infty(\mathbb{R}^3 \times \mathbb{R}^3)$ . The differential equation is given by:

$$(3.1) \quad \dot{x} = P(\tilde{x}, x) \nabla_x h(\tilde{x}, x).$$

Let  $P(\tilde{x}, x)$  be the tensor field with the components given by:

$$(3.2) \quad (P^{ij}(x, \tilde{x})) = \begin{pmatrix} 0 & x^3 & -\tilde{x}^2 \\ -x^3 & 0 & x^1 \\ \tilde{x}^2 & -x^1 & 0 \end{pmatrix}$$

and

$$(3.3) \quad h(\tilde{x}, x) = a_1 \tilde{x}^1 x^1 + a_2 \tilde{x}^2 x^2 + a_3 \tilde{x}^3 x^3.$$

The differential equations (3.1) are:

$$\begin{aligned} \dot{x}^1(t) &= a_2 \tilde{x}^2(t) x^3(t) - a_3 \tilde{x}^2(t) \tilde{x}^3(t), \\ \dot{x}^2(t) &= a_3 x^1(t) \tilde{x}^3(t) - a_1 \tilde{x}^1(t) x^3(t), \\ \dot{x}^3(t) &= a_1 \tilde{x}^1(t) \tilde{x}^2(t) - a_2 x^1(t) \tilde{x}^2(t). \end{aligned}$$

The differential system:

$$(3.4) \quad \dot{x} = P(\tilde{x}, x) \nabla_x h(\tilde{x}, x) + g(\tilde{x}, x) \nabla_x c(\tilde{x}, x),$$

where the components of  $g(\tilde{x}, x)$  are:

$$(3.5) \quad g^{ij}(x) = \frac{\partial h(x, \tilde{x})}{\partial x^i} \frac{\partial c(x, \tilde{x})}{\partial x^j}, i \neq j, g^{ii}(x) = - \sum_{k=1, k \neq i}^3 \frac{\partial h(x, \tilde{x})}{\partial x^k} \frac{\partial c(x, \tilde{x})}{\partial x^k}, i = 1, 2, 3$$

is called the revised differential system with distributed delay associated to the differential system (3.1).

From (3.2), (3.3), (3.5) and  $c(\tilde{x}, x) = \frac{1}{2}(x^1)^2 + x^2 \tilde{x}^2 + \frac{1}{2}(x^3)^2$  results:

$$(g^{ij}) = \begin{pmatrix} -a_2^2 x^2 \tilde{x}^2 - a_3 x^3 \tilde{x}^3 & a_1 a_2 \tilde{x}^1 x^2 & a_1 a_3 \tilde{x}^1 x^3 \\ a_1 a_2 \tilde{x}^1 x^2 & -a_1^2 x^1 \tilde{x}^1 - a_3 x^3 \tilde{x}^3 & a_2 a_3 \tilde{x}^2 x^3 \\ a_1 a_3 \tilde{x}^1 x^2 & a_2 a_3 \tilde{x}^2 x^3 & -a_1^2 x^1 \tilde{x}^1 - a_2^2 x^2 \tilde{x}^2 \end{pmatrix}.$$

The differential system (3.4) is given by:

$$(3.6) \quad \begin{aligned} \dot{x}^1(t) &= a_2 \tilde{x}^2(t) x^3(t) - a_3 \tilde{x}^2(t) \tilde{x}^3(t) + a_1 a_2 \tilde{x}^1(t) (x^2(t))^2, \\ \dot{x}^2(t) &= a_3 x^1(t) \tilde{x}^3(t) - a_1 \tilde{x}^1(t) x^3(t) - a_1^2 x^1(t) \tilde{x}^1(t) x^2(t) - a_3^2 x^2(t) x^3(t) \tilde{x}^3(t), \\ \dot{x}^3(t) &= a_1 \tilde{x}^1(t) x^2(t) - a_2 x^1(t) \tilde{x}^2(t) + a_2 a_3 \tilde{x}^2(t) x^2 x^3. \end{aligned}$$

For the Dirac distribution, system (3.6) was analyzed in [6]. The other types of densities will be analyzed in our future papers.

The Euler-Poincare equation for the free rigid body with distributed delay is defined by:

$$\dot{M} = M \times \Omega + \alpha M \times (\tilde{M} \times \tilde{\Omega})$$

where  $M = (I_1 x(t), I_2 y(t), I_3 z(t))^T$ ,  $\Omega = (x(t), y(t), z(t))^T$ ,  $\tilde{\Omega} = (\tilde{x}(t), \tilde{y}(t), \tilde{z}(t))^T$ ,  $\tilde{M} = I\tilde{\Omega}$ ,  $I_1 > 0$ ,  $I_2 > 0$ ,  $I_3 > 0$  and  $\alpha \in \mathbb{R}$ .

The equilibrium points of our system are  $\Omega_1 = (\frac{m}{I_1}, 0, 0)^T$ ,  $\Omega_2 = (0, \frac{m}{I_2}, 0)^T$ ,  $\Omega_3 = (0, 0, \frac{m}{I_3})^T$ ,  $m \in \mathbb{R}^*$ .

**Proposition 3.1.** *The equilibrium point  $\Omega_1$  has the following behavior:*

(i) *The corresponding linear system is given by:*

$$\dot{U}(t) = AU(t) + \alpha B\tilde{U}(t)$$

where  $U(t) = (u^1(t), u^2(t), u^3(t))^T$  and

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{I_3 - I_1}{I_1 I_2} m \\ 0 & \frac{I_1 - I_2}{I_1 I_3} m & 0 \end{pmatrix}, B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{I_2 - I_1}{I_1 I_2} m^2 & 0 \\ 0 & 0 & \frac{I_3 - I_1}{I_1 I_3} m^2 \end{pmatrix};$$

(ii) *The characteristic equation is:*

$$\lambda[\lambda^2 - \frac{\alpha m^2}{I_1} (\frac{I_2 - I_1}{I_2} + \frac{I_3 - I_1}{I_3}) \lambda k^{(1)}(\lambda) + \frac{\alpha^2 m^4}{I_1^2 I_2 I_3} (I_2 - I_1)(I_3 - I_1) k^{(1)}(\lambda)^2 - \frac{(I_1 - I_2)(I_3 - I_1)}{I_1^2 I_2 I_3} m^2] = 0;$$

(iii) *On the tangent space at  $\Omega_1$  to the sphere of radius  $m$  the linear operator given by the linearized vector field has the characteristic equation:*

$$\lambda^2 - \frac{\alpha m^2}{I_1} (\frac{I_2 - I_1}{I_2} + \frac{I_3 - I_1}{I_3}) \lambda k^{(1)}(\lambda) + \frac{\alpha^2 m^4}{I_1^2 I_2 I_3} (I_2 - I_1)(I_3 - I_1) k^{(1)}(\lambda)^2 - \frac{(I_1 - I_2)(I_3 - I_1)}{I_1^2 I_2 I_3} m^2] = 0;$$

(iv) *If  $I_1 > I_2$ ,  $I_1 > I_3$  and  $k^{(1)}(\lambda) = e^{-\tau\lambda}$ ,  $\tau > 0$ , for  $0 \leq \tau < \tau_c$ , where*

$$\tau_c = \frac{I_1(I_3(I_1 - I_2) + I_2(I_1 - I_3))}{3|\alpha|m^2(I_1 - I_2)(I_1 - I_3)},$$

*then the equilibrium point  $\Omega_1$  is asymptotically stable.*

The analysis of the equilibrium point  $\Omega_1$  for the Dirac density is given in [1].

## 4 Fractional differential systems for the rigid body

Generally speaking, the fractional derivative, Riemann-Liouville fractional derivative and Caputo's fractional derivative are mostly used. In the present paper we discuss the Caputo derivative:

$$D_t^\alpha x(t) = I^{m-\alpha} \left( \frac{d}{dt} \right)^m x(t), \quad \alpha > 0,$$

where  $m - 1 < \alpha \leq m$ ,  $\left( \frac{d}{dt} \right)^m = \frac{d}{dt} \circ \dots \circ \frac{d}{dt}$ ,  $I^\beta$  is the  $\beta$ th order Riemann-Liouville integral operator, which is expressed as follows:

$$I^\beta x(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} x(s) ds, \quad \beta > 0.$$

In this paper, we suppose that  $\alpha \in (0, 1)$ .

Examples of the fractional differential systems are: the fractional order of Chua's system, the fractional order of Rossler's system and the fractional Duffing oscillator. The geometrical and mechanical interpretation of the fractional derivative is given in [7]. The geometry of fractional oscillator bundle of higher order was made in [3] using the fractional differential forms defined in [4].

A fractional system of differential equations with distributed delay in  $\mathbb{R}^3$  is given by:

$$(4.1) \quad D_t^\alpha x(t) = X(x(t), \tilde{x}(t)), \quad \alpha \in (0, 1),$$

where  $x(t) = (x^1(t), x^2(t), x^3(t))^T \in \mathbb{R}^3$ . The linearized of (4.1) in the equilibrium point  $x_0$ ,  $(X(x_0, x_0) = 0)$  is given by the following linear fractional differential system:

$$(4.2) \quad D_t^\alpha u(t) = Au(t) + B\tilde{u}(t),$$

where  $A = \left( \frac{\partial X}{\partial x} \right) |_{x=x_0}$ ,  $B = \left( \frac{\partial X}{\partial \tilde{x}} \right) |_{x=x_0}$ .

The characteristic equation of (4.2) is:

$$(4.3) \quad \Delta(\lambda) = \det(\lambda^\alpha I - A - k^{(1)}(\lambda)B) = 0,$$

where  $k^{(1)}(\lambda) = \int_0^\infty k(s)e^{-\lambda s} ds$ .

From (4.2) we have:

**Proposition 4.1.** ([4]) (i) If all the roots of characteristic equation  $\Delta(\lambda) = 0$  have negative real parts, then the equilibrium point  $x_0$  of (4.1) is asymptotically stable;

(ii) If  $k(s)$  is the Dirac distribution, the characteristic equation (4.3) is given by:

$$\Delta(\lambda) = \det(\lambda^\alpha I - A - e^{-\lambda\tau} B) = 0.$$

If  $\tau = 0$ ,  $\alpha \in (0, 1)$  and all the roots of the equation  $\det(\lambda I - A - B) = 0$  satisfy  $|\arg(\lambda)| > \frac{\alpha\pi}{2}$ , then the equilibrium point  $x_0$  is asymptotically stable;

(iii) If  $\alpha \in (0.5, 1)$  and the equation  $\det(\lambda I - A - Be^{-\lambda\tau}) = 0$  has no purely imaginary roots for any  $\tau > 0$ , then the equilibrium point  $x_0$  is asymptotically stable.

For the following delayed fractional equation (see [4])

$$(4.4) \quad D_t^\alpha x(t) = ax(t - \tau)$$

where  $\alpha \in (0, 1)$ ,  $a \in \mathbb{R}$  and  $\tau > 0$  the stability condition is:

If  $a < 0$ ,  $(-a)^{\frac{1}{\alpha}} \neq \frac{1}{\tau}((2k+1)\pi - \frac{\alpha}{2}\pi)$  and  $(-a)^{\frac{1}{\alpha}} \neq -\frac{1}{\tau}((2k+1)\pi - \frac{\alpha}{2}\pi)$ ,  $k \in \mathbb{Z}$ , then the zero solution of (4.4) is asymptotically stable.

For the following delayed fractional equation (see [4])

$$(4.5) \quad \begin{aligned} D_t^\alpha x(t) &= y(t) - k_1 x(t) \\ D_t^\alpha y(t) &= -(k_1 + k_2)y(t) + x(t - \tau), \end{aligned}$$

where  $\alpha \in (0, 1)$ ,  $k_1 \geq 0$ ,  $k_2 > 0$ ,  $\tau > 0$ , the stability condition is:

If  $k_1 > 0$ ,  $k_2 > \frac{1}{k_1} - k$ , then the zero solution of system (4.5) is asymptotically stable.

For  $f \in C^\infty(\mathbb{R}^3)$ , by  $D_{x^1}^\alpha f$ ,  $D_{x^2}^\alpha f$ ,  $D_{x^3}^\alpha f$  we denote the Caputo partial derivatives defined by:

$$(4.6) \quad D_{x^i}^\alpha f(x) = \frac{1}{\Gamma(1-\alpha)} \int_0^{x^i} \frac{\partial f(x^1, \dots, x^{i-1}, s, x^{i+1}, \dots, x^n)}{\partial x^i} \frac{1}{(x^i - s)^\alpha} ds, \quad i = 1, 2, 3$$

where  $(x^i)$  are the coordinate functions on  $\mathbb{R}$ , and  $(\frac{\partial}{\partial x^i})$ ,  $i = 1, 2, 3$  is the canonical base of the vector field on  $\mathbb{R}^n$ .

From (4.6) it results:

$$D_{x^i}^\alpha (x^i)^\gamma = \frac{(x^i)^{\gamma-\alpha} \Gamma(1+\gamma)}{\Gamma(1+\gamma-\alpha)}, \quad D_{x^i}^\alpha (x^j) = 0, \quad i \neq j.$$

Let  $\mathcal{X}^\alpha(\mathbb{R}^3)$  be the module of the fractional vector fields generated by the operators  $\{D_{x^i}^\alpha, i = 1, 2, 3\}$  and the module  $\mathcal{D}(\mathbb{R}^3)$  generated by 1-forms  $\{d(x^i)^\alpha, i = 1, 2, 3\}$ . The fractional exterior derivative  $d^\alpha : C^\infty(\mathbb{R}^3) \rightarrow \mathcal{D}(\mathbb{R}^3)$  is defined by:

$$d^\alpha(f) = d(x^i)^\alpha D_{x^i}^\alpha(f).$$

Let  $\overset{\alpha}{P} \in \mathcal{X}^\alpha(\mathbb{R}^3) \times \mathcal{X}^\alpha(\mathbb{R}^3)$  be a fractional 2-skew-symmetric tensor field and  $d^\alpha f, d^\alpha g \in \mathcal{D}(\mathbb{R}^3)$ . The bilinear map  $[\cdot, \cdot]^\alpha : C^\infty(\mathbb{R}^3) \times C^\infty(\mathbb{R}^3) \rightarrow C^\infty(\mathbb{R}^3)$  defined by:

$$(4.7) \quad [f, g]^\alpha = \overset{\alpha}{P}(d^\alpha f, d^\alpha g), \quad \forall f, g \in C^\infty(\mathbb{R}^3)$$

is called the fractional Leibniz bracket.

If  $\overset{\alpha}{P} = \overset{\alpha}{P}^{ij} D_{x^i}^\alpha \otimes D_{x^j}^\alpha$  then from (4.7) it follows that:

$$[f, g]^\alpha = \overset{\alpha}{P}^{ij} D_{x^i}^\alpha f D_{x^j}^\alpha g.$$

From the properties of the fractional Caputo, results:

$$\begin{aligned} [fh, g]^\alpha &= \sum_{k=0}^{\infty} \binom{\alpha}{k} \overset{\alpha}{P}^{ij} (D_{x^i}^{\alpha-k} f)(D_{x^j}^\alpha g) \left(\frac{\partial}{\partial x^i}\right)^k h \\ [f, gh]^\alpha &= \sum_{k=0}^{\infty} \binom{\alpha}{k} \overset{\alpha}{P}^{ij} (D_{x^i}^\alpha f)(D_{x^j}^{\alpha-k} g) \left(\frac{\partial}{\partial x^i}\right)^k h. \end{aligned}$$

If  $\overset{\alpha}{P}$  is skew-symmetric we say that  $(\mathbb{R}^3, [\cdot, \cdot]^\alpha)$  is a fractional almost Poisson manifold. If  $\alpha \rightarrow 1$  then we obtain the concepts from [8].

For  $h \in C^\infty(\mathbb{R}^3)$ , the fractional almost Poisson dynamic system is given by:

$$(4.8) \quad D_t^\alpha x^i(t) = [x^i(t), h(t)]^\alpha, \text{ where } [x^i, h]^\alpha = \overset{\alpha}{P}^{ij} D_{x^j}^\alpha h.$$

Let  $\overset{\alpha}{P}$  be a 2-skew-symmetric fractional tensor field and  $\overset{\alpha}{g}$  a 2-symmetric fractional tensor field on  $\mathbb{R}^3$ . We define the bracket  $[\cdot, \cdot]^\alpha : C^\infty(\mathbb{R}^3) \times C^\infty(\mathbb{R}^3) \rightarrow C^\infty(\mathbb{R}^3)$  by:

$$[f, h]^\alpha = \overset{\alpha}{P}(d^\alpha f, d^\alpha g) + \overset{\alpha}{g}(d^\alpha f, d^\alpha h), \quad f, h \in C^\infty(\mathbb{R}^3).$$

The structure  $(M, \overset{\alpha}{P}, \overset{\alpha}{g}, [\cdot, \cdot]^\alpha)$  is called fractional almost metriplectic manifold. The fractional dynamic system associated to  $h \in C^\infty(\mathbb{R}^3)$  is

$$(4.9) \quad D_t^\alpha x^i(t) = [x^i(t), h(t)]^\alpha, \text{ where } [x^i, h]^\alpha = \overset{\alpha}{P}^{ij} D_{x^j}^\alpha h + \overset{\alpha}{g}^{ij} D_{x^j}^\alpha h.$$

If we define the bracket  $[\cdot, (\cdot, \cdot)]^\alpha : C^\infty(\mathbb{R}^3) \times C^\infty(\mathbb{R}^3) \times C^\infty(\mathbb{R}^3) \rightarrow C^\infty(\mathbb{R}^3)$  by:

$$[f, (h_1, h_2)]^\alpha = \overset{\alpha}{P}(d^\alpha f, d^\alpha h_1) + \overset{\alpha}{g}(d^\alpha f, d^\alpha h_2), \quad \forall f, h_1, h_2 \in C^\infty(\mathbb{R}^3),$$

then, the fractional vector field  $\overset{\alpha}{X}_{h_1 h_2}$  defined by:

$$\overset{\alpha}{X}_{h_1 h_2}(f) = [f, (h_1, h_2)]^\alpha, \quad \forall f \in C^\infty(\mathbb{R}^3)$$

is called the fractional almost Leibniz vector field associated to the functions  $h_1, h_2 \in C^\infty(\mathbb{R}^3)$ .

The fractional almost Leibniz dynamical system is given by:

$$D_t^\alpha x^i(t) = \overset{\alpha}{P}^{ij} D_{x^j}^\alpha h_1 + \overset{\alpha}{g}^{ij} D_{x^j}^\alpha h_2.$$

Let  $\overset{\alpha}{P} = (P^{ij})$ ,  $\overset{\alpha}{g} = (g^{ij})$  be the fractional 2-tensor fields on  $\mathbb{R}^3$  and  $h \in C^\infty(\mathbb{R}^3)$  given by:

$$h = \frac{1}{\Gamma(\alpha + 1)} [a_1(x^1)^{\alpha+1} + a_2(x^2)^{\alpha+1} + a_3(x^3)^{\alpha+1}].$$

**Proposition 4.2.** (i) The fractional dynamic system (4.8) is:

$$(4.10) \quad D_t^\alpha x^1 = (a_2 - a_3)x^2x^3, D_t^\alpha x^2 = (a_3 - a_1)x^1x^3, D_t^\alpha x^3 = (a_1 - a_2)x^1x^2;$$

(ii) The fractional dynamic system (4.9) is:

$$(4.11) \quad \begin{aligned} D_t^\alpha x^1 &= (a_2 - a_3)x^2x^3 + a_2(a_1 - a_2)x^1(x^2)^2 + a_3(a_1 - a_3)x^1(x^3)^2 \\ D_t^\alpha x^2 &= (a_3 - a_1)x^1x^3 + a_3(a_2 - a_3)x^2(x^3)^2 + a_1(a_2 - a_1)x^2(x^1)^2 \\ D_t^\alpha x^3 &= (a_1 - a_2)x^1x^2 + a_1(a_3 - a_1)x^3(x^1)^2 + a_2(a_3 - a_2)x^3(x^2)^2; \end{aligned}$$

(iii) The fractional dynamic systems (4.10) and (4.11) have the equilibrium points  $M_1(m, 0, 0)$ ,  $M_2(0, m, 0)$ ,  $M_3(0, 0, m)$ ,  $m \in \mathbb{R}^*$ ;

(iv) The characteristic equations for (4.10) are:

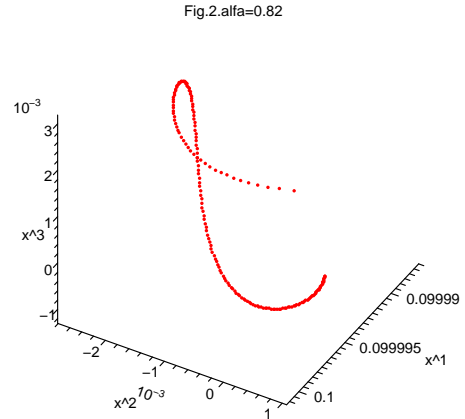
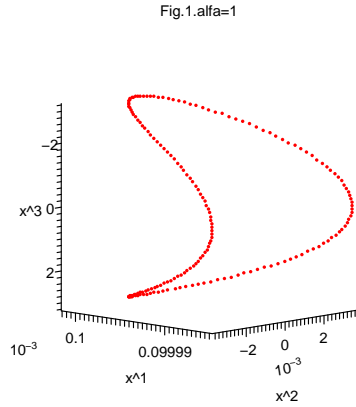
in  $M_1(m, 0, 0)$ :  $\lambda^\alpha(\lambda^{2\alpha} + (a_1 - a_3)(a_1 - a_2)m^2) = 0$ ,  
 in  $M_2(0, m, 0)$ :  $\lambda^\alpha(\lambda^{2\alpha} - (a_1 - a_2)(a_2 - a_3)m^2) = 0$ ,  
 in  $M_3(0, 0, m)$ :  $\lambda^\alpha(\lambda^{2\alpha} + (a_1 - a_3)(a_2 - a_3)m^2) = 0$ ; (v) The characteristic equations in  $M_1(m, 0, 0)$ ,  $M_2(0, m, 0)$ ,  $M_3(0, 0, m)$  for (4.11) are:

$$\begin{aligned} \lambda^\alpha(\lambda^{2\alpha} - a_1(a_2 + a_3 - 2a_1)m^2\lambda^\alpha + (a_1 - a_3)(a_1 - a_2)m^2(a_1^2m^2 + 1)) &= 0, \\ \lambda^\alpha(\lambda^{2\alpha} - a_2(a_1 + a_3 - 2a_2)m^2\lambda^\alpha - (a_1 - a_2)(a_2 - a_3)m^2(a_2^2m^2 + 1)) &= 0, \\ \lambda^\alpha(\lambda^{2\alpha} - a_3(a_1 + a_2 - 2a_3)m^2\lambda^\alpha + (a_1 - a_3)(a_2 - a_3)m^2(a_3^2m^2 + 1)) &= 0; \end{aligned}$$

The above findings allow the analysis of the equilibrium points with respect to the parameters of the characteristic equations.

Equations (4.10) are called the fractional equations of the rigid body and equations (4.11) are called revised fractional equations of the rigid body. If  $\alpha \rightarrow 1$ , the results from Proposition 4.2 lead to results from [8].

For  $a_1 := 3$ ,  $a_2 := 2$ ,  $a_3 := 1$ ,  $\alpha = 1$  the dynamics  $(x^1(t), x^2(t), x^3(t))$  of (4.10) is given in figure Fig.1 and for  $\alpha = 0.82$  in figure Fig.2. The numerical algorithm used is Adam-Moulton-Bashford.



## 5 Conclusions

In the present paper we present the dynamics of the rigid body with memory. The memory was described by the variables with distributed delay and by the Caputo fractional derivative.

## References

- [1] I.D. Albu, M. Neamtu and D. Opris, *Dissipative mechanical systems with delay*, Tensor N.S. 67 (2006), 1-27.
- [2] I.D. Albu, D.Opris, *Leibnitz dynamics with time delay*, arXiv.math/0508225.

- [3] I.D. Albu, M. Neamtu and D. Opris, *The geometry of fractional osculator bundle of higher order and applications*, International Conference on Differential Geometry Lagrange and Hamilton Spaces, Iasi, September, 3-8, 2007, Scientific Annals of "Al. I. Cuza" University, Mathematics, LIII (2007):21-32, arXiv:0709.2000v1, math.DG, 2007.
- [4] W. Deng, C. Li and J. Lü, *Stability analysis of linear differential systems with multiple time delay*, Nonlinear Syn. 48 (2007), 409-416.
- [5] D. Fish, *Dissipative perturbations of 3D Hamiltonian systems*, arXiv:math/0506047v1, 2005.
- [6] M. Neamtu, D. Opris and M. Popescu, *Hopf Bifurcation for the Rigid Body with Time Delay*, Proceedings of the 5th conference of Balkan Society of Geometers, Geometry Balkan Press, Bucharest 2006, 83-92.
- [7] I. Podlubny, *Geometrical and physical interpretation of fractional integration and fractional differentiation*, Fractional Calculus and Applied Analysis 5, 4 (2002), 367-386.
- [8] M. Puta, *Hamiltonian systems and geometric quantisation*, Mathematics and Applications, Vol. 260, Kluwer Academic Publishers 2004.

*Authors' addresses:*

Ion Doru Albu and Dumitru Opris  
Department of Mathematics, Faculty of Mathematics,  
West University of Timișoara,  
4 Bd. V. Parvan, 300223, Timișoara, Romania.  
E-mail: albud@math.uvt.ro, opris@math.uvt.ro

Mihaela Neamtu  
Department of Economic Informatics and Statistics,  
Faculty of Economics, West University of Timișoara,  
16A Pestalozzi St., 300115, Timișoara, Romania.  
E-mail:mihaela.neamtu@fse.uvt.ro