Adomian decomposition method and two coupled scalar fields

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Abstract. In this paper, we use the Adomian decomposition method (ADM) for non-linear systems. We consider a two coupled scalar fields system and obtain the exact solution by the ADM approach. Also, we consider the static case and draw $\phi(x)$ and $\chi(x)$ with respect to different parameters r. These lead us to have a solitonic and non-solitonic solutions.

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1 Introduction

A wide class of stochastic and deterministic problems in Physics, engineering and the other sciences are modeled mathematically by differential equations as linear and non-linear differential equations, fractional differential equations, stochastic differential and integro-differential equations systems [13, 2, 5, 10]. Some of these problems are solved by Adomian decomposition method (ADM).
This method can solve also the linear problem in colloidal polymer technology, antigen-antibody aggregation, cluster formation in galaxies [2], mechanical systems with several springs attached in series [5, 10, 3] and nonlinear systems of harmonic oscillators exhibiting passage through resonance [3, 4, 15], the nonlinear Klein-Gordon equations and nonlinear Lane-Emden equation [17] and the Cauchy problem arising in one dimensional nonlinear thermo-elasticity [16], the Kaup-Kupershmidt (KK) equation [12], the coupled Drinfeld-Sokolov-Wilson equation [11], coupled Schrodinger-KdV, generalized KdV and shallow water equations [1]. We note that the Adomian decomposition method provides a solution in terms of a rapidly convergent power series. The modified form of the Adomian decomposition method was introduced by Wazwaz, which is direct, without any need of transformation formulas [2].
The series terms approach zero as $\frac{1}{(mn)!}$ where n is the order of the highest linear

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differential operator and m is the number of terms in the approximation φ_n. ADM method works for both initial-value and boundary-value problems.
In here, we are going to discuss the notable example of the completely non-linear Schrödinger equation(NLS) with two coupled scalar fields. The applications of two coupled scalar fields for hexagonal network defect were presented by [14, 8, 6, 7], and in domain walls were discussed by [7]. Also two fields solutions in the Einstein equations for describing black holes with the cosmic strings were discussed by [9], which give us motivation to study two scalar fields. Now, we use ADM method for solving this problem exactly. At first, we review ADM method, then in Section 3, we apply ADM to a coupled scalar fields with an individual initial condition and investigate it for different values of parameter r. We draw the $\phi(x)$ and $\chi(x)$ in terms of x and see that in the plotted figures [13, 2, 5]. In Section 4 we add up the results and to offer suggestions for obtaining more accurate results by using this method.

2 A brief review of Adomian decomposition method (ADM)

At first, we consider the general form of equation,

$$Fu = g(t), \tag{2.1}$$

where F is a nonlinear operator, and expand it into the following equation,

$$Lu + Ru + Nu = g(t), \tag{2.2}$$

where the linear term is represented by Lu, and L is a linear operator and easily invertible. We choose L as the highest-order derivative and R as the reminder of the linear operator - a term consisting only of u as coefficient (constant or variable). The nonlinear term is represented by Nu, L^{-1} is defined as n-fold integration for $L = \frac{d^n}{dt^n}$. As an example, for $L = \frac{d^2}{dt^2}$, we will have $L^{-1} = \int_0^t \int_0^t [.]\, dtdt$,
and

$$L^{-1}L = u - u(0) - tú(0). \tag{2.3}$$

From (2.2) we infer,

$$\begin{aligned} Lu &= g(t) - Ru - Nu \\ L^{-1}Lu &= L^{-1}g(t) - L^{-1}Ru - L^{-1}Nu, \end{aligned} \tag{2.4}$$

and in this case one can obtain u as follows,

$$u = u(0) + tú(0) + L^{-1}g(t) - L^{-1}Ru - L^{-1}Nu. \tag{2.5}$$

The first three terms are identified as $u = \sum_{n=0}^{\infty} u_n$,

$$\begin{aligned} u_0 &= u(0) + tú(0) + L^{-1}g(t) \\ u_1 &= -L^{-1}Ru_0 - L^{-1}Nu_0 \\ u_2 &= -L^{-1}Ru_1 - L^{-1}Nu_1 \\ &\vdots \\ u_n &= -L^{-1}Ru_{n-1} - L^{-1}Nu_{n-1}, \end{aligned} \tag{2.6}$$

and also we have,

$$Nu = \sum_{n=0}^{\infty} A_n(u_0, u_1, ..., u_n),$$

where A_n are called Adomian polynomials and depend only on the u components and make a rapidly convergent series (any nonlinearity is written here in terms of A_n and Nu which need not even be analytic).
The Adomian polynomial is generated by following formula:

$$A_n(u_0, u_1, ..., u_n) = \frac{1}{n!}[\frac{d^n}{d\lambda^n} N(\sum_{i=0}^{n} u_i \lambda^i)]_{\lambda=0}. \tag{2.7}$$

We write here the first five Adomian polynomials for convenience,

$$\begin{aligned}
A_0 &= f(u_0) \\
A_1 &= u_1 f^{(1)}(u_0) \\
A_2 &= u_2 f^{(1)}(u_0) + \left(\frac{1}{2!}\right) u_1^2 f^{(2)}(u_0) \\
A_3 &= u_3 f^{(1)}(u_0) + u_1 u_2 f^{(2)}(u_0) + \left(\frac{1}{3!}\right) u_1^3 f^{(3)}(u_0) \\
A_4 &= u_4 f^{(1)}(u_0) + \left[\left(\frac{1}{2!}\right) u_2^2 + u_1 u_3\right] f^{(2)}(u_0) + \left(\frac{1}{2!}\right) u_1^2 u_2 f^{(3)}(u_0) + \left(\frac{1}{4!}\right) u_1^4 f^{(4)}(u_0). \\
&\vdots \qquad (2.8)
\end{aligned}$$

So, the practical solution for the n-term approximation is,

$$\begin{aligned}
\varphi_n &= \sum_{i=0}^{n-1} u_i \\
u &= \lim_{n\to\infty} \varphi_n = \sum_{i=0}^{\infty} u_i. \qquad (2.9)
\end{aligned}$$

3 Decomposition method applied to a coupled differential equation

A general system of two real scalar fields in a space-time of dimension 1+1 are described by the following Lagrangian density,

$$\ell = \frac{1}{2}\partial_\alpha \phi \partial^\alpha \phi + \frac{1}{2}\partial_\alpha \chi \partial^\alpha \chi + U(\phi, \chi). \tag{3.1}$$

where $U(\phi, \chi)$ is a potential dependent on the fields ϕ and χ.
Now we consider the special example of two scalar fields potential,

$$U(\phi, \chi) = \frac{1}{2}(1 - \phi^2)^2 - r\chi^2 + r(1 + 2r)\phi^2\chi^2 + \frac{1}{2}r^2\chi^4. \tag{3.2}$$

The corresponding Euler-Lagrange equation is

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{\phi}}\right) = \frac{\partial L}{\partial \phi}$$

and

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{\chi}}\right) = \frac{\partial L}{\partial \chi}$$

lead us to have a following equations.

$$\begin{aligned} \frac{d^2\phi}{dx^2} &= -2\,\phi + 2r\,(2r+1)\,\phi\chi^2 + 2\,\phi^3 \\ \frac{d^2\chi}{dx^2} &= -2r\,\chi + 2r\,(2r+1)\,\phi^2\chi + 2r^2\,\chi^3 \end{aligned} \tag{3.3}$$

Applying the ADM formalism and using the equations (2.4), (2.5), and (2.6), we have,

$$\begin{aligned} L\phi &= -2\,\phi + 2r\,(2r+1)\,\phi\chi^2 + 2\,\phi^3 \\ L\chi &= -2r\,\chi + 2r\,(2r+1)\,\phi^2\chi + 2r^2\,\chi^3, \end{aligned} \tag{3.4}$$

and

$$\begin{aligned} \phi &= c_0 + c_1 x - 2L^{-1}\phi + \beta L^{-1}\phi\chi^2 + 2L^{-1}\phi^3 \\ \chi &= \alpha_0 + \alpha_1 x - 2rL^{-1}\chi + \beta L^{-1}\phi^2\chi + 2r^2L^{-1}\chi^3, \end{aligned} \tag{3.5}$$

where,

$$\begin{aligned} \phi_0 &= c_0 + c_1 x \\ \chi_0 &= \alpha_0 + \alpha_1 x. \end{aligned} \tag{3.6}$$

Finally, from (2.7), one obtain $\phi_n(x)$ and $\chi_n(x)$ as follows,

$$\begin{aligned} \phi_n &= -2\int_0^x\int_0^x \phi_{n-1}(x)dxdx + \beta\int_0^x\int_0^x \phi_{n-1}(x)A_{n-1}[\chi(x)^2]dxdx \\ &+ 2\int_0^x\int_0^x A_{n-1}[\phi(x)^3]dxdx \\ \chi_n &= -2r\int_0^x\int_0^x \chi_{n-1}(x)dxdx + \beta\int_0^x\int_0^x A_{n-1}[\phi(x)^2]\chi_{n-1}(x)dxdx \\ &+ 2r^2\int_0^x\int_0^x A_{n-1}[\chi(x)^3]dxdx \qquad n = 1, 2, 3, \ldots \end{aligned} \tag{3.7}$$

where $A_{n-1}[.]$ are the $(n-1)$th Adomian polynomials and $\beta = 2r\,(2r+1)$.

The exact solution is

$$\begin{aligned} \phi &= \lim_{n\to\infty}\sum_{i=0}^{n}\phi_i \\ \chi &= \lim_{n\to\infty}\sum_{i=0}^{n}\chi_i, \end{aligned} \tag{3.8}$$

where we calculate only the first four terms.
Applying the initial conditions $\phi(0) = 0$, $\acute{\phi}(0) = 1$, $\chi(0) = 1$ and $\acute{\chi}(0) = 0$, the constant coefficient from equation (3.6) are,

$$c_0 = 0 \qquad c_1 = 1 \qquad \alpha_0 = 1 \qquad \alpha_1 = 0$$

Hence, using the above results in (3.7) we get,

$$\begin{aligned}\phi(x) = x &+ (\frac{2}{3}r^2 + \frac{1}{3}r - \frac{1}{3})x^3 + (-\frac{1}{15}r^2 - \frac{1}{30}r + \frac{2}{15})x^5 \\ &+(-\frac{8}{63}r^6 - \frac{10}{63}r^4 + \frac{8}{63}r^2 + \frac{1}{21}r - \frac{11}{210})x^7 \\ &+(\frac{2}{81}r^6 + \frac{1}{27}r^5 + \frac{7}{405}r^4 - \frac{2}{135}r^3 - \frac{7}{810}r^2 + \frac{1}{120})x^9 \\ &+(\frac{2}{825}r^4 + \frac{2}{825}r^3 + \frac{1}{1650}r^2)x^{11} \\ &+R^4(x),\end{aligned}$$

and

$$\begin{aligned}\chi(x) = 1 &+ (r^2 - r)x^2 + (\frac{1}{2}r^4 - \frac{2}{3}r^3 + \frac{1}{2}r^2 + \frac{1}{6}r)x^4 \\ &+(\frac{1}{15}r^4 + \frac{1}{90}r^3 - \frac{1}{90}r^2)x^6 + (\frac{2}{21}r^6 - \frac{5}{42}r^4 + \frac{1}{42}r^2)x^8 \\ &+(\frac{8}{405}r^6 + \frac{4}{135}r^5 + \frac{28}{2025}r^4 - \frac{8}{675}r^3 - \frac{14}{2025}r^2)x^{10} \\ &+(\frac{1}{495}r^4 + \frac{1}{495}r^3 + \frac{1}{1980}r^2)x^{12} \\ &+R^4(x),\end{aligned}$$

where $R^4(x)$ is the reminder.

Finally we discuss several cases and draw the $\phi(x)$ and $\chi(x)$ for different values of the real parameter r. The solutions of the nonlinear equations are sensitive to the initial conditions and parameters, and hence we should choose them carefully, else the result will be very different from our purposes. In our case straightforward calculation for $r = 0$ leads to

$$\begin{aligned}\phi(x) &= x - \frac{1}{3}x^3 + \frac{2}{15}x^5 - \frac{17}{315}x^7 + \frac{11}{504}x^9 - \frac{307}{46200}x^{11} + \\ &\quad + \frac{11}{15600}X^{13} + R^4(x) = tanh(x) \\ \chi(x) &= 1. \end{aligned} \tag{3.9}$$

The plots of $\phi(x)$ and $\chi(x)$ in terms of x, for $r = 0$, $r = 0.0001$, $r = 0.01$ and $r = 0.1$ are presented in Fig.[1], Fig.[2] and Fig.[3] respectively

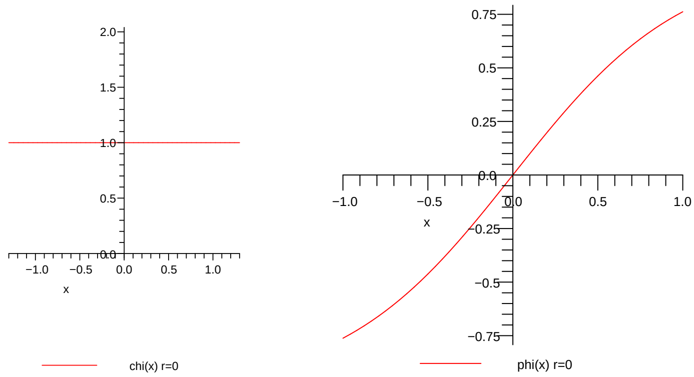

Fig.1. Plots of $\phi(x)$ and $\chi(x)$ fields for r=0

and for r=0.0001, we have

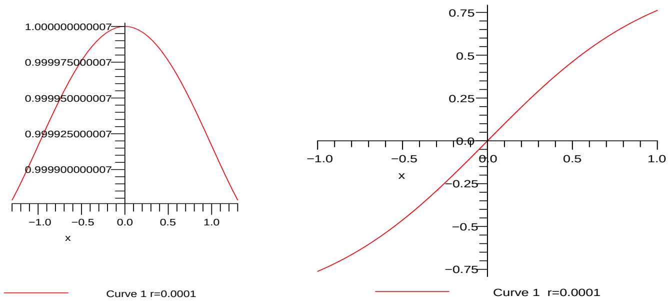

Fig.2. Plots of $\phi(x)$ and $\chi(x)$ fields for r=0.0001

Finally, in the case r=0.1, we obtain

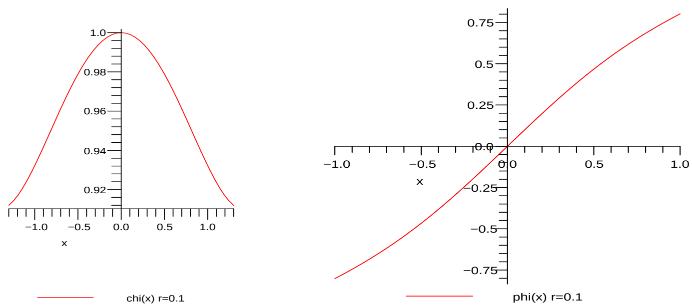

Fig.3. Plots of $\phi(x)$ and $\chi(x)$ fields for r=0.1

4 Conclusion

In this article we have considered the theory based on two scalar fields. By using the ADM method we obtained the soliton solutions for two scalar fields. We have studied the problem for a certain range for the parameter r, which plays an important role in our calculation. When $r \in (0, 1)$, the results are be soliton solutions for both $\phi(x)$ and $\chi(x)$. The graph of $\phi(x)$ in r=0 is non-solitonic and the graph of $\chi(x)$ in $r = 1$ exhibits a minimum.

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