

The Gauss-Weingarten formulae of second order

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Abstract. It is the purpose of the present paper to outline an introduction in theory of embeddings in the 2-osculator bundle. First, we recall the notion of 2-osculator bundle ([1],[2]) and the notion of submanifolds in the 2-osculator bundle. A moving frame is constructed. The induced connections and the relative covariant derivation are discussed in third and fourth sections. The Gauss-Weingarten formulae are present in the next section.

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Introduction

Generally, the geometries of higher order defined as the study of the category of bundles of jet $(J_0^k M, \pi^k, M)$ is based on a direct approach of the properties of objects and morphisms in this category, without local coordinates.

But, many mathematical models from Lagrangian Mechanics, Theoretical Physics and Variational Calculus used multivariate Lagrangians of higher order acceleration.

From here one can see the reason of construction of the geometry of the total space of the bundle of higher accelerations (or the osculator bundle of higher order) in local coordinates.

As far we know the theory of Finsler submanifolds is far from being settled. In [8] and [9] R. Miron and M. Anastasiei give the theory of subspaces in generalized Lagrange spaces. Also, in [4] R. Miron presented the theory of subspaces in higher order Lagrange spaces. This article is drawn upon the original construction of the higher order geometry given by R. Miron and Gh. Atanasiu ([4], [5],[6],[7]) and new aspects given by Gh. Atanasiu in [1].

If \check{M} is an immersed manifold in manifold M , a non linear connection on $Osc^2 M$ induce a nonlinear connection \check{N} on $Osc^2 \check{M}$. We take the metrical N-linear connection D on the manifold $Osc^2 M$. This allows obtain of the induced tangent and normal connections and introduction of the relative covariant derivation in the algebra of d-tensor fields ([10]). If in [4] R. Miron use the canonical metrical N-linear connection of the space $L^{(2)n}$ having the coefficients $(F_{jk}^i, C_{1jk}^i, C_{2jk}^i)$, in this article we take the

canonical metrical N-linear connection of the manifold Osc^2M having the coefficients

$$\left(\begin{matrix} L \\ (p0) \end{matrix} \begin{matrix} a \\ bc \end{matrix}, \begin{matrix} C \\ (p1) \end{matrix} \begin{matrix} a \\ bc \end{matrix}, \begin{matrix} C \\ (p2) \end{matrix} \begin{matrix} a \\ bc \end{matrix} \right),$$

$(p = 0, 1, 2)$.

In this paper we get the Gauss-Weingarten formulae (Theorem 5.1) generalizing the same problem solved by R. Miron ([4]) and A. Bejancu ([3]).

Of course, the study of the higher geometry in this direction will be continued by autors.

1 The 2-osculator bundle (Osc^2M, π^2, M)

Let M be a real differentiable manifold of dimension n . A point of M will be denoted by x and its local coordinate system by (U, φ) , $\varphi(x) = (x^a)$. The indices a, b, \dots run over set $\{1, 2, \dots, n\}$ and Einstein convention of summarizing is adopted all over this work.

Let us consider two curves $\rho, \sigma : I \rightarrow M$, having images in a domain of local chart $U \subset M$. We say that ρ and σ have a "contact of order 2" in a point $x_0 \in U$ if: $\rho(0) = \sigma(0) = x_0$, $(0 \in I)$, and for any function $f \in \mathcal{F}(U)$:

$$(1.1) \quad \frac{d^\beta}{dt^\beta} (f \circ \rho)(t) |_{t=0} = \frac{d^\beta}{dt^\beta} (f \circ \sigma)(t) |_{t=0}, (\beta = 1, 2)$$

The relation "contact of order 2" is an equivalence on the set of smooth curves in M , which pas through the point x_0 . Let $[\rho]_{x_0}$ be a class of equivalence. It will be called a "2-osculator space" in a point $x_0 \in M$. The set of 2-osculator spaces in the point $x_0 \in M$ will be denoted by $Osc_{x_0}^2M$, and we put

$$Osc^2M = \bigcup_{x_0 \in M} Osc_{x_0}^2M$$

One considers the mapping $\pi^2 : Osc^2M \rightarrow M$ define by $\pi^2([\rho]_{x_0}) = x_0$. Obviously, π^2 is a surjection.

The set Osc^2M is endowed with a natural differentiable structure, induced by that of the manifold M , so that π^2 is a differentiable mapping. It will be descrieb bellow.

The curve $\rho : I \rightarrow M$ ($\text{Im } \rho \subset U$) is analytically represented in the local chart (U, φ) by $x_0 = x_0^a (= x^a(0))$. Taking the function f from 1.1, succesively equal to the coordinate functions x^a , then a representative of the class $[\rho]_{x_0}$ is given by

$$x^{*a}(t) = x^a(0) + t \frac{dx^a}{dt}(0) + \frac{1}{2} t^2 \frac{d^2x^a}{dt^2}(0), t \in (-\varepsilon, \varepsilon) \subset I.$$

The previous polynomials are determined by the coefficients

$$(1.2) \quad x_0^a = x^a(0), y^{(1)a} = \frac{dx^a}{dt}(0), y^{(2)a} = \frac{1}{2} \frac{d^2x^a}{dt^2}(0)$$

Hence, the pair $\left((\pi^2)^{-1}(U), \Phi \right)$, with $\Phi([\rho]_{x_0}) = (x_0^a, y^{(1)a}, y^{(2)a}) \in R^{3n}$, $\forall [\rho]_{x_0} \in (\pi^2)^{-1}(U)$ is a local chart on Osc^2M . Thus a differentiable atlas \mathcal{A}_M of the diferentiable structure on the manifold M determines a differentiable atlas A_{Osc^2M} on Osc^2M

and therefore the triple (Osc^2M, π^2, M) is a differentiable bundle. We will identify the 2-osculator bundle (Osc^2M, π^2, M) with 2-tangent bundle (T^2M, π^2, M) .

By 1.2, a transformation of local coordinates $(x^a, y^{(1)a}, y^{(2)a}) \rightarrow (\tilde{x}^a, \tilde{y}^{(1)a}, \tilde{y}^{(2)a})$ on the manifold Osc^2M is given by

$$(1.3) \quad \begin{cases} \tilde{x}^a = \tilde{x}^a(x^1, \dots, x^n), \det \left(\frac{\partial \tilde{x}^a}{\partial x^b} \right) \neq 0 \\ \tilde{y}^{(1)a} = \frac{\partial \tilde{x}^a}{\partial x^b} y^{(1)b} \\ 2\tilde{y}^{(2)a} = \frac{\partial \tilde{y}^{(1)a}}{\partial x^b} y^{(1)b} + 2 \frac{\partial \tilde{y}^{(1)a}}{\partial y^{(1)b}} y^{(2)b} \end{cases}$$

One can see that Osc^2M is of dimension $3n$.

Let us consider the 2-tangent structure \mathbb{J} on Osc^2M

$$\mathbb{J} \left(\frac{\partial}{\partial x^a} \right) = \frac{\partial}{\partial y^{(1)a}}, \quad \mathbb{J} \left(\frac{\partial}{\partial y^{(1)a}} \right) = \frac{\partial}{\partial y^{(2)a}}, \quad \mathbb{J} \left(\frac{\partial}{\partial y^{(2)a}} \right) = 0$$

where $\left(\frac{\partial}{\partial x^a} \Big|_u, \frac{\partial}{\partial y^{(1)a}} \Big|_u, \frac{\partial}{\partial y^{(2)a}} \Big|_u \right)$ is the natural basis of tangent space

$T_u Osc^2M, u \in Osc^2M$. If N is a nonlinear connection on Osc^2M , then $N_0 = N, \mathbb{J}(N_0) = N_1$ are two distributions geometrically defined on Osc^2M , all of dimension n . Let us consider the distributions V_2 on Osc^2M locally generated by the vector fields $\left\{ \frac{\partial}{\partial y^{(2)a}} \right\}$. Consequently, the tangent bundle to Osc^2M at the point $u \in Osc^2M$ is given by a direct sum of the vector space:

$$(1.5) \quad T_u Osc^2M = N_0(u) \oplus N_1(u) \oplus V_2(u), \forall u \in Osc^2M.$$

We consider $\left\{ \frac{\delta}{\delta x^a}, \frac{\delta}{\delta y^{(1)a}}, \frac{\delta}{\delta y^{(2)a}} \right\}$ an adapted basis to the decomposition 1.5 and its dual basis denoted by $(dx^a, \delta y^{(1)a}, \delta y^{(2)a})$, where

$$(1.6) \quad \begin{cases} \frac{\delta}{\delta x^a} = \frac{\partial}{\partial x^a} - N_{(1)a}^b \frac{\delta}{\delta y^{(1)b}} - N_{(2)a}^b \frac{\partial}{\partial y^{(2)b}} \\ \frac{\delta}{\delta y^{(1)a}} = \frac{\partial}{\partial y^{(1)a}} - N_{(1)a}^b \frac{\partial}{\partial y^{(2)b}} \\ \frac{\delta}{\delta y^{(2)a}} = \frac{\partial}{\partial y^{(2)a}} \end{cases}$$

respectively

$$(1.7) \quad \begin{cases} dx^a = & dx^a \\ \delta y^{(1)a} = & dy^{(1)a} + M_{(1)b}^a dx^b \\ \delta y^{(2)a} = & dy^{(2)a} + M_{(1)b}^a \delta y^b + M_{(2)b}^a \delta y^{(2)b} \end{cases}$$

where

$$M_{(1)b}^a = N_{(1)b}^a, \quad M_{(2)b}^a = N_{(2)b}^a + N_{(1)c}^a N_{(1)b}^c$$

Let h, v_1, v_2 be the projectors defined by the distributions N_0, N_1 and V_2 . If $X \in \mathcal{X}(Osc^2 M)$ we denote:

$$X^H = hX, \quad X^{V_1} = v_1X, \quad X^{V_2} = v_2X.$$

Therefore we have the unique decomposition:

$$X = X^H + X^{V_1} + X^{V_2}.$$

Each of the components X^H, X^{V_1}, X^{V_2} is called *d-vector field* on $Osc^2 M$. If $\omega \in \mathcal{X}^*(Osc^2 M)$ we can write

$$\omega = \omega^H + \omega^{V_1} + \omega^{V_2}$$

where

$$\omega^H = h\omega, \quad \omega^{V_1} = v_1\omega, \quad \omega^{V_2} = v_2\omega.$$

The quantities $\omega^H, \omega^{V_1}, \omega^{V_2}$ are called *d-1 forms*.

Similarly, a *tensor field* $T \in T_s^r(Osc^2 M)$ can be split with respect to 1.5 into components, which will be called *d-tensor field*.

Definition 1.1 A linear connection D on $Osc^2 M$ is called *N-linear connection* if it preserves by parallelism the horizontal and vertical distribution N_0, N_1 and V_2 on $Osc^2 M$.

Any N-linear connection D can be represented by an unique system of functions $D\Gamma(N) = \left(L_{(p0)bd}^a, C_{(p1)bd}^a, C_{(p2)bd}^a \right), (p = 0, 1, 2)$ where

$$\begin{cases} D_{\delta_c} \delta_b = L_{(00)bc}^a \delta_a, & D_{\delta_c} \delta_{1b} = L_{(10)bc}^a \delta_{1a}, & D_{\delta_c} \delta_{2b} = L_{(20)bc}^a \delta_{2a} \\ D_{\delta_{1c}} \delta_b = C_{(01)bc}^a \delta_a, & D_{\delta_{1c}} \delta_{1b} = C_{(11)bc}^a \delta_{1a}, & D_{\delta_{1c}} \delta_{2b} = C_{(21)bc}^a \delta_{2a} \\ D_{\delta_{2c}} \delta_b = C_{(02)bc}^a \delta_a, & D_{\delta_{2c}} \delta_{1b} = C_{(12)bc}^a \delta_{1a}, & D_{\delta_{2c}} \delta_{2b} = C_{(22)bc}^a \delta_{2a}, \end{cases}$$

These functions are called *the coefficients* of the N-linear connection D .

Let us consider vector fields $X, Y \in \mathcal{X}(Osc^2 M)$ and a N-linear connection D . It follows that

$$D_X Y = \sum_{p=0}^2 (D_{X^H} Y^{V_p} + D_{X^{V_1}} Y^{V_p} + D_{X^{V_2}} Y^{V_p}) (V_0 = H).$$

We have

$$\left\{ \begin{array}{l} D_0^H X Y = D_{X^H} Y^H, D_0^{V_1} X Y = D_{X^{V_1}} Y^H, D_0^{V_2} X Y = D_{X^{V_2}} Y^H \\ D_p^H X Y = D_{X^H} Y^{V_p}, D_p^{V_1} X Y = D_{X^{V_1}} Y^{V_p}, D_p^{V_2} X Y = D_{X^{V_2}} Y^{V_p}, (p = 1, 2). \end{array} \right.$$

If on the manifold $Osc^2 M$ is given a N-linear connection D then there exists a h_p -, v_{1p} - and v_{2p} -covariant derivatives in local adapted basis ($p = 0, 1, 2$).

Any d-tensor T, of type (r, s) can be represented in the adapted basis and its dual basis in the form

$$T = T_{b_1 \dots b_s}^{a_1 \dots a_r} \delta_{a_1} \otimes \dots \otimes \dot{\partial}_{2a_r} \otimes dx^{b_1} \otimes \dots \otimes \delta y^{(2)b_s}.$$

For $X = \overset{(0)}{X}^a \delta_a + \overset{(1)}{X}^a \delta_{1a} + \overset{(2)}{X}^a \delta_{2a}$ using the properties of the operators $D_p^H, D_p^{V_1}$ and $D_p^{V_2}$ we get

$$\begin{aligned} D_p^H T &= \overset{(0)}{X}^d T_{b_1 \dots b_s |_{pd}}^{a_1 \dots a_r} \delta_{a_1} \otimes \dots \otimes \dot{\partial}_{2a_r} \otimes dx^{b_1} \otimes \dots \otimes \delta y^{(2)b_s}, \\ D_p^{V_1} T &= \overset{(1)}{X}^a T_{b_1 \dots b_s}^{a_1 \dots a_r} \Big|_d \delta_{a_1} \otimes \dots \otimes \dot{\partial}_{2a_r} \otimes dx^{b_1} \otimes \dots \otimes \delta y^{(2)b_s}, \\ D_p^{V_2} T &= \overset{(2)}{X}^a T_{b_1 \dots b_s}^{a_1 \dots a_r} \Big|_{pd} \delta_{a_1} \otimes \dots \otimes \dot{\partial}_{2a_r} \otimes dx^{b_1} \otimes \dots \otimes \delta y^{(2)b_s} \end{aligned}$$

where

$$\begin{aligned} T_{b_1 \dots b_s |_{pd}}^{a_1 \dots a_r} &= \delta_a T_{b_1 \dots b_s}^{a_1 \dots a_r} + L_{(p0)cd}^{a_1} T_{b_1 \dots b_s}^{ca_2 \dots a_r} + \dots + \\ &+ L_{(p0)cd}^{a_r} T_{b_1 \dots b_s}^{a_1 \dots a_{r-1}c} - L_{(p0)b_1 d}^c T_{cb_2 \dots b_s}^{a_1 \dots a_r} - \dots - L_{(p0)b_s d}^c T_{cb_2 \dots b_{s-1}c}^{a_1 \dots a_r} \end{aligned}$$

$$\begin{aligned} T_{b_1 \dots b_s}^{a_1 \dots a_r} \Big|_{pd} \overset{(1)}{=} & \delta_{1a} T_{b_1 \dots b_s}^{a_1 \dots a_r} + C_{(p1)cd}^{a_1} T_{b_1 \dots b_s}^{ca_2 \dots a_r} + \dots + \\ &+ C_{(p1)cd}^{a_r} T_{b_1 \dots b_s}^{a_1 \dots a_{r-1}c} - C_{(p1)b_1 d}^c T_{cb_2 \dots b_s}^{a_1 \dots a_r} - \dots - C_{(p1)b_s d}^c T_{cb_2 \dots b_{s-1}c}^{a_1 \dots a_r} \end{aligned}$$

$$\begin{aligned} T_{b_1 \dots b_s}^{a_1 \dots a_r} \Big|_{pd} \overset{(2)}{=} & \delta_{2a} T_{b_1 \dots b_s}^{a_1 \dots a_r} + C_{(p2)cd}^{a_1} T_{b_1 \dots b_s}^{ca_2 \dots a_r} + \dots + \\ &+ C_{(p2)cd}^{a_r} T_{b_1 \dots b_s}^{a_1 \dots a_{r-1}c} - C_{(p2)b_1 d}^c T_{cb_2 \dots b_s}^{a_1 \dots a_r} - \dots - C_{(p2)b_s d}^c T_{cb_2 \dots b_{s-1}c}^{a_1 \dots a_r} \end{aligned}$$

$$\left(\delta_{1a} = \frac{\delta}{\delta y^{(1)a}}, \delta_{2a} = \frac{\delta}{\delta y^{(2)a}}; p = 0, 1, 2 \right)$$

The operators " $|_{pd}$ ", " $\Big|_{pd}$ " and " $\Big|_{pd}$ " are called the h_p -, v_{1p} - and v_{2p} -covariante derivatives with respect to $D\Gamma(N)$.

Definition 1.2 A N-linear connection D on $Osc^2 M$ endowed with a metric g is said to be a *metric N-linear connection* if $D_X g = 0$ for every $X \in X(Osc^2 M)$.

Proposition 1.1 A N-linear connection on $Osc^2 M$ is a metric N-linear connection if and only if

$$g_{ab|_{pd}} = 0, g_{ab} \Big|_{pd} \overset{(1)}{=} 0, g_{ab} \Big|_{pd} \overset{(2)}{=} 0.$$

2 Submanifolds in the 2-osculator bundle

Let M be a C^∞ real, n -dimensional manifold and \check{M} be a real, m -dimensional manifold, immersed in M through the immersion $i : \check{M} \rightarrow M$. Locally, i can be given in the form

$$(2.1) \quad x^a = x^a(u^1, \dots, u^m), \quad \text{rank} \left\| \frac{\partial x^a}{\partial u^\alpha} \right\| = m$$

The indices a, b, c, \dots run over the set $\{1, \dots, n\}$ and $\alpha, \beta, \gamma, \dots$ run on the set $\{1, \dots, m\}$. We assume $1 < m < n$. If i is an embedding, then we identify \check{M} to $i(\check{M})$ and say that \check{M} is a submanifold of the manifold M . Therefore 2.1 will be called the parametric equations of the submanifold M in the manifold M .

The embedding $i : \check{M} \rightarrow M$ determines an immersion $Osc^2 i : Osc^2 \check{M} \rightarrow Osc^2 M$, defined by the covariant functor $Osc^2 : Man \rightarrow Man$. ([4])

The mapping $Osc^2 i : Osc^2 \check{M} \rightarrow Osc^2 M$ has the parametric equations:

$$(2.2) \quad \begin{cases} x^a = x^a(u^1, \dots, u^m), \text{rank} \left\| \frac{\partial x^a}{\partial u^\alpha} \right\| = m \\ y^{(1)a} = \frac{\partial x^a}{\partial u^\alpha} v^{(1)\alpha} \\ 2y^{(2)a} = \frac{\partial y^{(1)a}}{\partial u^\alpha} v^{(1)\alpha} + 2 \frac{\partial y^{(1)a}}{\partial v^{(1)\alpha}} v^{(2)\alpha} \end{cases}$$

where

$$(2.3) \quad \begin{cases} \frac{\partial x^a}{\partial u^\alpha} = \frac{\partial y^{(1)a}}{\partial v^{(1)\alpha}} = \frac{\partial y^{(2)a}}{\partial v^{(2)\alpha}} \\ \frac{\partial y^{(1)a}}{\partial u^\alpha} = \frac{\partial y^{(2)a}}{\partial v^{(1)\alpha}} \end{cases}$$

The Jacobian matrix of 2.2 is $J(Osc^2 i)$ and it has the rank equal to $3m$. So, $Osc^2 i$ is an immersion. The differential i_* of the mapping $Osc^2 i : Osc^2 \check{M} \rightarrow Osc^2 M$ leads to the relation between the natural basis of the modules $\mathcal{X}(Osc^2 \check{M})$ and $\mathcal{X}(Osc^2 M)$ given by

$$i_* \left\| \frac{\partial}{\partial u^\alpha} \frac{\partial}{\partial v^{(1)\alpha}} \dots \frac{\partial}{\partial v^{(2)\alpha}} \right\| = \left\| \frac{\partial}{\partial x^a} \frac{\partial}{\partial y^{(1)a}} \dots \frac{\partial}{\partial y^{(2)a}} \right\| J(Osc^2 i).$$

This differential i_* maps the cotangent space $T^*(Osc^2 M)$ in a point of $Osc^2 M$, into the cotangent space $T^*(Osc^2 \check{M})$ in a point of $Osc^2 \check{M}$ by the rule:

$$(2.4) \quad \begin{aligned} dx^a &= \frac{\partial x^a}{\partial u^\alpha} du^\alpha \\ dy^{(1)a} &= \frac{\partial y^{(1)a}}{\partial u^\alpha} du^\alpha + \frac{\partial y^{(1)a}}{\partial v^{(1)\alpha}} dv^{(1)\alpha} \\ dy^{(2)a} &= \frac{\partial y^{(2)a}}{\partial u^\alpha} du^\alpha + \frac{\partial y^{(2)a}}{\partial v^{(1)\alpha}} dv^{(1)\alpha} + \frac{\partial y^{(2)a}}{\partial v^{(2)\alpha}} dv^{(2)\alpha} \end{aligned}$$

We used the previous theory for study the induced geometrical object fields from Osc^2M to $Osc^2\check{M}$.

Let us consider

$$(2.5) \quad B_\alpha^a = \frac{\partial x^a}{\partial u^\alpha}.$$

We take g_{ab} a nondegenerate metric on the manifold M and $\mathbb{G} = g_{ab}dx^a \otimes dx^b + g_{ab}\delta y^{(1)a} \otimes \delta y^{(1)a} + g_{ab}\delta y^{(2)a} \otimes \delta y^{(2)a}$ a metric on manifold Osc^2M .

Thus, $\{B_1^a, B_2^a, \dots, B_m^a\}$ are m-linear independent d-vector fields on $Osc^2\check{M}$. Also, $\{B_\alpha^1, B_\alpha^2, \dots, B_\alpha^n\}$ are d-covector fields, with respect to the next transformations of coordinates

$$(2.6) \quad \begin{cases} \bar{u}^\alpha = \bar{u}^\alpha(u^1, \dots, u^m), \text{rank} \left\| \frac{\partial \bar{u}^\alpha}{\partial u^\beta} \right\| = m \\ \bar{v}^{(1)\alpha} = \frac{\partial \bar{u}^\alpha}{\partial u^\beta} v^{(1)\beta} \\ 2\bar{v}^{(2)\alpha} = \frac{\partial \bar{v}^{(1)\alpha}}{\partial u^\beta} v^{(1)\beta} + 2 \frac{\partial \bar{v}^{(1)\alpha}}{\partial v^{(1)\beta}} v^{(2)\beta} \end{cases}$$

Of course, d-vector fields $\{B_1^a, \dots, B_m^a\}$ are tangent to the submanifold \check{M} .

We say that a d-vector field $\xi^a(x, y^{(1)}, y^{(2)})$ is *normal* to $Osc^2\check{M}$ if, on $\tilde{U} \subset Osc^2\check{M}$, we have

$$\begin{aligned} g_{ab}(x(u), y^{(1)}(u, v^{(1)}, v^{(2)}), y^{(2)}(u, v^{(1)}, v^{(2)})) B_\alpha^a(u) \cdot \\ \cdot \xi^b(x(u), y^{(1)}(u, v^{(1)}, v^{(2)}), y^{(2)}(u, v^{(1)}, v^{(2)})) = 0 \end{aligned}$$

Consequently, on $\tilde{U} \subset Osc^2\check{M}$ there exist $n - m$ unit vector fields $B_{\bar{\alpha}}^a$, ($\bar{\alpha} = 1, \dots, n - m$) normal to

As usual $(x^a, y^{(1)a}, y^{(2)a})$, ($a = 1, \dots, n$) will denote the local coordinates on Osc^2M and $Osc^2\check{M}$ and orthogonal to each other. Namely, we have:

$$(2.7) \quad g_{ab} B_{\bar{\alpha}}^a B_{\bar{\beta}}^b = 0, \quad g_{ab} B_{\bar{\alpha}}^a B_{\bar{\beta}}^b = \delta_{\bar{\alpha}\bar{\beta}}, \quad (\bar{\alpha}, \bar{\beta} = 1, \dots, n - m)$$

The system of d-vectors $B_{\bar{\alpha}}^a$ ($\bar{\alpha} = 1, \dots, n - m$) is determined up to orthogonal transformations of the form

$$(2.8) \quad B_{\bar{\alpha}'}^a = A_{\bar{\alpha}'}^{\bar{\beta}} B_{\bar{\beta}}^a, \quad \|A_{\bar{\alpha}'}^{\bar{\beta}}\| \in \mathcal{O}(n - m),$$

where $\bar{\alpha}, \bar{\beta}, \dots$ run over the set $(1, 2, \dots, n - m)$.

We say that the system of d-vectors $\{B_{\bar{\alpha}}^a, B_{\bar{\alpha}}^a\}$ determines a frame in Osc^2M along to $Osc^2\check{M}$.

Its dual frame will be denoted by $\{B_{\bar{\alpha}}^\alpha(u, v^{(1)}, \dots, v^{(2)}), B_{\bar{\alpha}}^{\bar{\alpha}}(u, v^{(1)}, \dots, v^{(2)})\}$. This is also defined on an open set $\tilde{U} \subset Osc^2\check{M}$, \tilde{U} being a domain of a local chart on the submanifold \check{M} .

The conditions of duality are given by:

$$(2.9) \quad B_{\beta}^a B_a^{\alpha} = \delta_{\beta}^{\alpha}, \quad B_{\beta}^a B_a^{\bar{\alpha}} = 0, \quad B_a^{\alpha} B_{\beta}^a = 0, \quad B_a^{\bar{\alpha}} B_{\beta}^a = \delta_{\beta}^{\bar{\alpha}}$$

$$(2.9') \quad B_{\alpha}^a B_b^{\alpha} + B_{\bar{\alpha}}^a B_b^{\bar{\alpha}} = \delta_b^a$$

Using 2.7, we deduce:

$$(2.10) \quad g_{\alpha\beta} B_a^{\alpha} = g_{ab} B_{\beta}^a, \quad \delta_{\bar{\alpha}\bar{\beta}} B_b^{\bar{\beta}} = g_{ab} B_{\bar{\alpha}}^a.$$

So, we can look to the set

$$\mathcal{R} = \left\{ \left(u, v^{(1)}, v^{(2)} \right); B_{\alpha}^a(u), B_{\bar{\alpha}}^a \left(u, v^{(1)}, v^{(2)} \right) \right\}$$

$(u, v^{(1)}, v^{(2)}) \in \tilde{\pi}^{-1}(\tilde{U})$ as a moving frame. Now, we shall represent in \mathcal{R} the d-tensor fields from the space $Osc^2 M$, restricted to the open set $\tilde{\pi}^{-1}(\tilde{U})$.

In a next section, we will study the variation of the moving frame \mathcal{R} along to $\tilde{\pi}^{-1}(\tilde{U})$.

3 Induced nonlinear connection

Now, let us consider the canonical nonlinear connection N on the $Osc^2 M$. Then its dual coefficients $M_{(1)b}^a, M_{(2)b}^a$ depends only by the metric g . We will prove that the restriction of the of the nonlinear connection N to $Osc^2 \tilde{M}$ uniquely determines an induced nonlinear connection \check{N} on $Osc^2 \tilde{M}$. Of course, \check{N} is well determined by means of its dual coefficients $\left(\check{M}_{(1)\beta}^{\alpha}, \check{M}_{(2)\beta}^{\alpha} \right)$ or by means of its adapted cobasis $(du^{\alpha}, \delta v^{(1)\alpha}, \delta v^{(2)\alpha})$.

Definition 3.1 A non-linear connection \check{N} on $Osc^2 \tilde{M}$ is called *induced* by the nonlinear connection N if we have

$$(3.1) \quad \delta v^{(1)\alpha} = B_a^{\alpha} \delta y^{(1)a}, \quad \delta v^{(2)\alpha} = B_a^{\alpha} \delta y^{(2)a}$$

Proposition 3.1 *The dual coefficients of the non-linear connection \check{N} are*

$$(3.2) \quad \begin{aligned} \check{M}_{(1)\beta}^{\alpha} &= B_a^{\alpha} \left(B_{0\beta}^a + M_{(1)b}^a B_{\beta}^b \right) \\ \check{M}_{(2)\beta}^{\alpha} &= B_a^{\alpha} \left(\frac{1}{2} \frac{\partial B_{\delta\gamma}^a}{\partial u^{\beta}} v^{(1)\delta} v^{(1)\gamma} + B_{\delta\beta}^a v^{(2)\delta} + M_{(1)b}^a B_{0\beta}^b + M_{(2)b}^a B_{\beta}^b \right) \end{aligned}$$

where $M_{(1)b}^a, M_{(2)b}^a$ are the dual coefficients of the non-linear connection N .

Theorem 3.1 *The cobasis $(dx^a, \delta y^{(1)a}, \delta y^{(2)a})$ restricted to $Osc^2 \tilde{M}$ is uniquely represented in the moving frame \mathcal{R} in the following form:*

$$(3.5) \quad \begin{cases} dx^a = B_\beta^a du^\beta \\ \delta y^{(1)a} = B_\alpha^a \delta v^{(1)\alpha} + B_{\bar{\alpha}}^a K_{(1)\beta}^{\bar{\alpha}} du^\beta \\ \delta y^{(2)a} = B_\alpha^a \delta v^{(2)\alpha} + B_{\bar{\beta}}^a K_{(1)\alpha}^{\bar{\beta}} \delta v^{(1)\alpha} + B_{\bar{\beta}}^a K_{(2)\alpha}^{\bar{\beta}} du^\alpha \end{cases}$$

where

$$(3.6) \quad \begin{aligned} K_{(1)\beta}^{\bar{\alpha}} &= B_a^{\bar{\alpha}} \left(B_{0\beta}^a + M_{(1)b}^a B_\beta^b \right) \\ K_{(2)\beta}^{\bar{\alpha}} &= B_a^{\bar{\alpha}} \left(\frac{1}{2} \frac{\partial B_{\delta\gamma}^a}{\partial u^\beta} v^{(1)\delta} v^{(1)\gamma} + B_{\delta\beta}^b v^{(2)\delta} + M_{(1)b}^a B_{0\beta}^b + M_{(2)b}^a B_\beta^b - \right. \\ &\quad \left. - B_f^{\bar{\alpha}} B_d^\gamma \left(B_\gamma^f + M_{(1)b}^f B_\gamma^b \right) \left(B_{0\beta}^d + M_{(1)g}^d B_\beta^g \right) \right) \end{aligned}$$

are mixed d-tensor fields.

Proof. The first relation is obviously. From 2.2 and 3.2 we obtain 3.5. \square

Generally, a set of functions $T_{j\dots\beta\dots\bar{\beta}}^{i\dots\alpha\dots\bar{\alpha}}(u, v^{(1)}, v^{(2)})$ which are d-tensors in the index i, j, \dots , and d-tensors in the index α, β, \dots and tensors with respect to the transformations 2.8 in the index $\bar{\alpha}, \bar{\beta}, \dots$ is called a *mixed* d-tensor field on $\text{Osc}^2 \check{M}$.

4 The relative covariant derivatives

We shall construct the operators $\nabla_{(p)}$ of relative (or mixed) covariant derivation in the algebra of mixed d-tensor fields. It is clear that $\nabla_{(p)}$ will be known if its action of functions and on the vector fields of the form

$$(4.1) \quad \begin{aligned} X^a(x(u), y^{(1)}(u, v^{(1)}), y^{(2)}(u, v^{(1)}, v^{(2)})) \\ X^\alpha(u, v^{(1)}, v^{(2)}), X^{\bar{\alpha}}(u, v^{(1)}, v^{(2)}) \end{aligned}$$

are known.

Definition 4.1 We call a *coupling* of the canonical metrical N-connection D to the induced nonlinear connection \check{N} along $\text{Osc}^2 \check{M}$ the operators $\check{D}_{(p)}$ with the property

$$(4.2) \quad \check{D}_{(p)} X^a = D_{(p)} X^a \text{ (modulo 3.5)}$$

Here

$$(4.3) \quad D_{(p)} X^a = dX^a + X^b \omega_{(p)b}^a,$$

and

$$(4.4) \quad \check{D}_{(p)} X^a = dX^a + X^b \check{\omega}_{(p)b}^a$$

The 1-forms $\check{\omega}_{(p)b}^a$ are the connection 1-forms of \check{D} ($p=0,1,2$).

We have

Theorem 4.1 *The connection 1-forms $\check{\omega}_{(p)b}^a$ of \check{D} are given by*

$$\check{\omega}_{(p)b}^a = \check{L}_{(p0)b\delta}^a du^\delta + \check{C}_{(p1)b\delta}^a \delta v^{(1)\delta} + \check{C}_{(p2)b\delta}^a \delta v^{(2)\delta}$$

where

$$(4.5) \quad \begin{aligned} \check{L}_{(p0)b\delta}^a &= L_{(p0)bd}^a B_\delta^d + C_{(p1)bd}^a B_\delta^d K_{(1)\delta}^\delta + C_{(p2)b\delta}^a B_\delta^d K_{(2)\delta}^\delta \\ \check{C}_{(p1)b\delta}^a &= C_{(p1)bd}^a B_\delta^d + C_{(p2)bd}^a B_\delta^d K_{(1)\delta}^\delta \quad (p = 0, 1, 2) \\ \check{C}_{(p2)b\delta}^a &= C_{(p2)bd}^a B_\delta^d \end{aligned}$$

Proof. From 4.2, 4.4, 4.3, and 3.5 we obtain 4.5. \square

Corollary 4.1 *The coupling \check{D} of the canonical metrical N-connection D to \check{N} depends on the metric g_{ab} and on the embedding $i: \check{M} \rightarrow M$, only.*

Of course, we can write $\check{D}_{(p)} X^a$ in the form

$$\check{D}_{(p)} X^a = X^a \mid_{p\delta} du^\delta + X^a \mid_{p\delta}^{(1)} \delta v^{(1)\delta} + X^a \mid_{p\delta}^{(2)} \delta v^{(2)\delta}$$

where

$$\begin{aligned} X^a \mid_{p\delta} &= \frac{\delta X^a}{\delta u^\delta} + X^b \check{L}_{(p0)b\delta}^a, & X^a \mid_{p\delta}^{(1)} &= \frac{\delta X^a}{\delta v^{(1)\delta}} + X^b \check{C}_{(p1)b\delta}^a, \\ X^a \mid_{p\delta}^{(2)} &= \frac{\partial X^a}{\partial v^{(2)\delta}} + X^b \check{C}_{(p2)b\delta}^a \end{aligned}$$

Definition 4.2 We call the induced tangent connection on $Osc^2 \check{M}$ by the canonical metrical N-connection D the operators $D_{(p)}^\top$ given by

$$(4.6) \quad D_{(p)}^\top X^\alpha = B_b^\alpha \check{D}_{(p)} X^b, \quad \text{for } X^\alpha = B_\gamma^\alpha X^\gamma$$

We have

$$(4.7) \quad D_{(p)}^\top X^\alpha = dX^\alpha + X^\beta \omega_{(p)\beta}^\alpha$$

where $\omega_{(p)\beta}^\alpha$ are the connection 1-forms of D^\top ($p = 0, 1, 2$).

We have

Theorem 4.2 *The connection 1-forms of the induced tangent connection D^\top are given by:*

$$(4.8) \quad \omega_{(p)\beta}^\alpha = L_{(p0)\beta\delta}^\alpha du^\delta + C_{(p1)\beta\delta}^\alpha \delta v^{(1)\delta} + C_{(p2)\beta\delta}^\alpha \delta v^{(2)\delta}$$

where

$$(4.9) \quad \begin{aligned} L_{(p0)\beta\delta}^\alpha &= B_d^\alpha \left(B_{\beta\delta}^d + B_{\beta}^f \check{L}_{(p0)f\delta}^d \right) \\ C_{(p1)\beta\delta}^\alpha &= B_d^\alpha B_{\beta}^f \check{C}_{(p1)f\delta}^d \quad (p = 0, 1, 2) \\ C_{(p2)\beta\delta}^\alpha &= B_d^\alpha B_{\beta}^f \check{C}_{(p2)f\delta}^d \end{aligned}$$

Proof. From 4.4, 4.7 and 4.6 we have 4.9. □

Corollary 4.2 *The induced tangent connection D^\top depends on the metric g_{ab} and the embedding $i: \tilde{M} \rightarrow M$, only.*

As in the case of \check{D} we may write

$$D^\top X^\alpha = X_{|p\delta}^\alpha du^\delta + X^\alpha \Big|_{p\delta}^{(1)} \delta v^{(1)\delta} + X^\alpha \Big|_{p\delta}^{(2)} \delta v^{(2)\delta}$$

where

$$\begin{aligned} X_{|p\delta}^\alpha &= \frac{\delta X^\alpha}{\delta u^\delta} + X^\beta \check{L}_{(p0)\beta\delta}^\alpha, \quad X^\alpha \Big|_{p\delta}^{(1)} = \frac{\delta X^\alpha}{\delta v^{(1)\delta}} + X^\beta \check{C}_{(p1)\beta\delta}^\alpha \\ X^\alpha \Big|_{p\delta}^{(2)} &= \frac{\partial X^\alpha}{\partial v^{(2)\delta}} + X^\beta \check{C}_{(p2)\beta\delta}^\alpha \quad (p = 0, 1, 2) \end{aligned}$$

Definition 4.3 We call the induced normal connection on $Osc^2 \tilde{M}$ by the canonical metrical N-connection D the operators $D_{(p)}^\perp$ given by

$$(4.10) \quad D_{(p)}^\perp X^{\bar{\alpha}} = B_{\bar{b}}^{\bar{\alpha}} \check{D}_{(p)} X^{\bar{b}}, \quad \text{for } X^{\bar{a}} = B_{\bar{\gamma}}^{\bar{a}} X^{\bar{\gamma}}$$

As before, we set

$$(4.11) \quad D_{(p)}^\perp X^{\bar{\alpha}} = dX^{\bar{\alpha}} + X^{\bar{\beta}} \omega_{(p)\bar{\beta}}^{\bar{\alpha}}$$

where $\omega_{(p)\bar{\beta}}^{\bar{\alpha}}$ are the connection 1-forms of $D_{(p)}^\perp$ ($p = 0, 1, 2$).

We deduce

Theorem 4.3 *The connection 1-forms of the induced normal connection D^\perp are as follows:*

$$(4.12) \quad \omega_{(p)\bar{\beta}}^{\bar{\alpha}} = L_{(p0)\bar{\beta}\delta}^{\bar{\alpha}} du^\delta + C_{(p1)\bar{\beta}\delta}^{\bar{\alpha}} \delta v^{(1)\delta} + C_{(p2)\bar{\beta}\delta}^{\bar{\alpha}} \delta v^{(2)\delta}$$

where

$$(4.13) \quad \begin{aligned} L_{(p0)\bar{\beta}\delta}^{\bar{\alpha}} &= B_d^{\bar{\alpha}} \left(\frac{\delta B_{\bar{\beta}}^d}{\delta u^\delta} + B_{\bar{\beta}(p0)f\delta}^f \right) \\ C_{(p1)\bar{\beta}\delta}^{\bar{\alpha}} &= B_d^{\bar{\alpha}} \left(\frac{\delta B_{\bar{\beta}}^d}{\delta v^{(1)\delta}} + B_{\bar{\beta}(p1)f\delta}^f \check{C}_{f\delta}^d \right) \quad (p = 0, 1, 2) \\ C_{(p2)\bar{\beta}\delta}^{\bar{\alpha}} &= B_d^{\bar{\alpha}} \left(\frac{\partial B_{\bar{\beta}}^d}{\partial v^{(2)\delta}} + B_{\bar{\beta}(p2)f\delta}^f \check{C}_{f\delta}^d \right) \end{aligned}$$

Proof. From 4.4, 4.10, 4.11 and 3.5 we have 4.13. \square

Corollary 4.3 *The induced normal connection D^\perp depends on the metric g_{ab} and the embedding $i : \check{M} \rightarrow M$, only.*

We may set

$$D_{(p)}^\perp X^{\bar{\alpha}} = X_{|p\delta}^{\bar{A}} du^\delta + X^{\bar{\alpha}} \Big|_{p\delta}^{(1)} \delta v^{(1)\delta} + X^{\bar{\alpha}} \Big|_{p\delta}^{(2)} \delta v^{(2)\delta}$$

where

$$\begin{aligned} X_{|p\delta}^{\bar{\alpha}} &= \frac{\delta X^{\bar{\alpha}}}{\delta u^\delta} + X^{\bar{\beta}} L_{(p0)\bar{\beta}\delta}^{\bar{\alpha}}, \quad X^{\bar{\alpha}} \Big|_{p\delta}^{(1)} = \frac{\delta X^{\bar{\alpha}}}{\delta v^{(1)\delta}} + X^{\bar{\beta}} C_{(p1)\bar{\beta}\delta}^{\bar{\alpha}}, \\ X^{\bar{\alpha}} \Big|_{p\delta}^{(2)} &= \frac{\partial X^{\bar{\alpha}}}{\partial v^{(2)\delta}} + X^{\bar{\beta}} C_{(p2)\bar{\beta}\delta}^{\bar{\alpha}} \quad (p = 0, 1, 2). \end{aligned}$$

Now, we can define the relative (or mixed) covariant derivation ∇ enounced at the beginning of this section.

Theorem 4.4 *A relative (mixed) covariant derivation in the algebra of mixed d -tensor fields is an operator $\nabla_{(p)}$ for which the following properties hold:*

$$\nabla_{(p)} f = df, \quad \forall f \in \mathcal{F}(Osc^2 \check{M})$$

$$\nabla_{(p)} X^a = \check{D}_{(p)} X^a, \quad \nabla_{(p)} X^\alpha = D_{(p)}^\top X^\alpha, \quad \nabla_{(p)} X^{\bar{\alpha}} = D_{(p)}^\perp X^{\bar{\alpha}} \quad (p = 0, 1, 2)$$

The connection 1-forms $\check{\omega}_{(p)b}^a$, $\omega_{(p)\bar{\beta}}^\alpha$, $\omega_{(p)\bar{\beta}}^{\bar{\alpha}}$ will be called the connection 1-forms of $\nabla_{(p)}$.

As a direct consequence of the Theorem 4.1, 4.2 and 4.3 we can obtain:

Proposition 4.1 *The following equations hold:*

$$\nabla_{(p)} g_{ab} = 0, \quad \nabla_{(p)} \delta_{\bar{\alpha}\bar{\beta}} = 0, \quad (p = 0, 1, 2).$$

5 The Gauss-Weingarten formulae

As usual in the theory of the submanifolds we are interested in finding the moving equations of the moving frame \mathcal{R} along $Osc^2\tilde{M}$.

These equations, called also Gauss-Weingarten formulae, are obtained when the relative covariant derivatives of the vector fields from \mathcal{R} are expressed again in the frame \mathcal{R} .

Thus we have

Theorem 5.1 *The following Gauss-Weingarten formulae hold:*

$$(5.1) \quad \nabla_{(p)} B_{\alpha}^a = B_{\delta}^a \Pi_{(p)}^{\delta}$$

$$(5.2) \quad \nabla_{(p)} B_{\alpha}^a = -B_{\delta}^a \Pi_{(p)}^{\delta}$$

Proof. From 4.13 we have

$$\begin{aligned} \nabla_{(p)} B_{\alpha}^a &= B_{\alpha|p\beta}^a du^{\beta} + B_{\alpha}^a \Big|_{p\beta}^{(1)} \delta v^{(1)\delta} + B_{\alpha}^a \Big|_{p\beta}^{(2)} \delta v^{(2)\delta} \\ &= \left(\frac{\delta B_{\alpha}^a}{\delta u^{\beta}} + \check{L}_{(p0)}^a{}_{b\beta} B_{\alpha}^b - L_{(p0)}^{\delta}{}_{\alpha\beta} B_{\delta}^a \right) du^{\beta} + \\ &\quad + \left(\frac{\delta B_{\alpha}^a}{\delta v^{(1)\beta}} + \check{C}_{(p1)}^a{}_{b\beta} B_{\alpha}^b - C_{(p1)}^{\delta}{}_{\alpha\beta} B_{\delta}^a \right) \delta v^{(1)\beta} + \\ &\quad + \left(\frac{\delta B_{\alpha}^a}{\delta v^{(1)\beta}} + \check{C}_{(p2)}^a{}_{b\beta} B_{\alpha}^b - C_{(p2)}^{\delta}{}_{\alpha\beta} B_{\delta}^a \right) \delta v^{(2)\beta} \\ &= B_{\alpha\beta}^a du^{\beta} + B_{\alpha}^b \left(\check{L}_{(p0)}^a{}_{b\beta} du^{\beta} + \check{C}_{(p1)}^a{}_{b\beta} \delta v^{(1)\beta} + \check{C}_{(p2)}^a{}_{b\beta} \delta v^{(2)\beta} \right) - \\ &\quad - B_{\delta}^{\delta} \left[B_{\alpha}^{\delta} \left(B_{\alpha\beta}^d + B_{\alpha}^f \check{L}_{(p0)}^d{}_{f\beta} \right) du^{\beta} + B_{\alpha}^{\delta} B_{\alpha}^f \check{C}_{(p1)}^d{}_{f\beta} \delta v^{(1)\beta} + \right. \\ &\quad \left. + B_{\alpha}^{\delta} B_{\alpha}^f \check{C}_{(p2)}^d{}_{f\beta} \delta v^{(2)\beta} \right] \end{aligned}$$

Use the notations

$$\Pi_{(p)}^{\delta} = H_{(p0)}^{\delta}{}_{\alpha}{}^{\beta} du^{\beta} + H_{(p1)}^{\delta}{}_{\alpha}{}^{\beta} \delta v^{(1)\beta} + H_{(p2)}^{\delta}{}_{\alpha}{}^{\beta} \delta v^{(2)\beta}$$

where

$$H_{(p0)\alpha}^{\bar{\delta}\beta} = B_d^{\bar{\delta}} \left(B_{\alpha\beta}^d + B_{\alpha(p0)}^f \check{L}_{f\beta}^d \right)$$

$$H_{(p1)\alpha}^{\bar{\delta}\beta} = B_d^{\bar{\delta}} B_{\alpha(p1)}^f \check{C}_{f\beta}^d$$

$$H_{(p2)\alpha}^{\bar{\delta}\beta} = B_d^{\bar{\delta}} B_{\alpha(p2)}^f \check{C}_{f\beta}^d$$

we obtain 5.1.

Now, by applying $\nabla_{(p)}$ to the both sides of the equations

$$g_{ab} B_{\alpha}^a B_{\bar{\beta}}^b = 0$$

one gets

$$g_{ab} B_{\bar{\delta}}^a \Pi_{\alpha}^{\bar{\delta}} B_{\bar{\beta}}^b + g_{ab} B_{\alpha}^a \nabla_{(p)} B_{\bar{\beta}}^b = 0.$$

Multiplying with B_d^{α} we obtain

$$g_{bd} \nabla_{(p)} B_{\bar{\beta}}^b - B_{\bar{\delta}}^a B_d^{\bar{\delta}} g_{ab} \nabla_{(p)} B_{\bar{\beta}}^b = -B_d^{\alpha} \delta_{\bar{\beta}\bar{\gamma}} \Pi_{(p)\alpha}^{\bar{\gamma}}.$$

But $B_{\bar{\delta}}^a B_d^{\bar{\delta}} g_{ab} \nabla_{(p)} B_{\bar{\beta}}^b = 0$. Consequently, we obtain 5.2. □

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