BH-mean curvature in Randers spaces with anisotropically scaled metric

Vladimir Balan

Abstract. Within the framework of Randers (α, β)-Finsler spaces with anisotropically scaled metric, the volume element and the mean curvature of hypersurfaces are obtained, extending thus the corresponding results from [6, 4]. As a particular case, the mean curvature form of hypersurfaces in conformally deformed Randers spaces are derived.

M.S.C. 2000: 53C60, 53B40.
Key words: Finsler structure, Randers metric, conformal deformation, mean curvature, minimality PDE.

1 Introduction

Within the framework of Finsler submanifolds, significant advances have been obtained by Z. Shen ([12]) by deriving the Busemann - Hausdorff (BH) mean curvature of a Finsler submanifold. Further, for (α, β)-Finsler submanifolds with constant β-form, M.Souza, K. Tenenblat and J. Spruck ([15, 14]) and the author ([3, 5]) have investigated Randers minimal hypersurfaces, and constant mean curvature (and in particular, minimal) Kropina hypersurfaces, respectively.

We should note that, as alternative, a special attention has been devoted to the Holmes - Thompson (HT) approach ([1, 2]), substantial progress being provided by Q. He and Y.-B. Shen ([8, 9]) and recently by B.Y. Wu ([16, 17, 18]).

In the present work, we determine the the volume element and the BH-mean curvature of isometrically immersed hypersurfaces in Randers (α, β)-Finsler spaces with anisotropically scaled metric with bounded indicatrix. In particular, these lead to the explicit formulas for the BH-mean curvature form in conformally deformed Randers spaces, which extend the already known results ([6, 4, 13, 15, 14, 16, 17, 18]).

Let (M, F) and $(\tilde{M}, \tilde{F})$ be Finsler structures, and $\varphi : (M, F) \to (\tilde{M}, \tilde{F})$ be an isometric immersion, with F induced by $\tilde{F}$. Then the following result holds true:

BSG Proceedings 15. The International Conference "Differential Geometry - Dynamical Systems" DGDS-2007, October 5-7, 2007, Bucharest-Romania, pp. 34-39.

Theorem 1.1. *The mean curvature of M is given by ([12, (57), p.563])*

$$\tilde{H}_\varphi(X) = G^{-1}\left(G_{;x^i} - G_{;z^i_a z^j_b}\varphi^j_{;u^a u^b} - G_{;x^j z^i_a}\varphi^j_{;u^a}\right)X^i, \tag{1.1}$$

where lower indices stand for corresponding partial derivatives and:

- $(u^a, v^b)_{a,b\in\overline{1,n}}$ *are local coordinates in* TM *(*$\dim M = n$*);*
- $(x^i, y^j)_{i,j\in\overline{1,m}}$ *are local coordinates in* $T\tilde{M}$ *(*$\dim \tilde{M} = m$*);*
- z^i_a *are the entries of the Jacobian matrix* $[J(\varphi)] = (\frac{\partial\varphi^i}{\partial u^a})_{a=\overline{1,n},i=\overline{1,m}}$*;*
- $\varphi_t : M \to \tilde{M}, t \in (-\varepsilon,\varepsilon), \varphi_0 = \varphi$*, is the variation of the surface;*
- X *is the variation vector field* $X_x = \left.\dfrac{\partial\varphi_t}{\partial t}\right|_{t=0}(x), \forall x \in M$*;*
- G *is the coefficient of the induced Finsler volume form*

$$G(z) = \frac{vol[B^n]}{vol\{(v^a)\in\mathbb{R}^n \mid \tilde{F}(v^a z^i_a \tilde{e}_i) \le 1\}}, \tag{1.2}$$

where $z = (z^i_a)_{a=\overline{1,n},i=\overline{1,m}} \in GL_{m\times n}(\mathbb{R})$*,* $\tilde{e} = \{\tilde{e}_i\}_{i=\overline{1,m}}$ *is an arbitrary basis in* $\mathbb{R}^m$ *and* $B^n \subset \mathbb{R}^n$ *is the standard Euclidean ball.*

It has been shown ([12]) that the variation of the volume of M reaches a minimum for $\tilde{H}_\varphi = 0$.

2 The scaled Randers case

We shall further obtain the explicit formulas for the mean curvature form in the scaled Randers case, which extends the known free-potential case ([6, 4]).

In the following we consider the Finsler hypersurfaces ($\dim \tilde{M} = m = n+1$) for the scaled Randers structure ([11, 7, 10]) with the fundamental function

$$\tilde{F}(x,y) = \left(\sum_{i=1}^{n+1} a_i^2(x)(y^i)^2\right)^{1/2} + a_{n+1}(x)b_{n+1}(x)y^{n+1}, \tag{2.1}$$

with $b_{n+1}(x) \in [0,1)$ and $a_1,\ldots,a_{n+1}$ positive functions of x. We note that for $a_1 = \cdots = a_{n+1} = 1$ this metric reduces to the case studied in [6, 4] and further, for $b_{n+1} = const.$, to the one studied by M. Souza, K. Tenenblat and J.Spruck ([15, 14]).

Let $M = Im\ \varphi$, $\varphi : D \subset \mathbb{R}^n \to \tilde{M} = \mathbb{R}^{n+1}$ be a isometrically imbedded simple hypersurface. We denote the Jacobian entries by $z^i_\alpha = \dfrac{\partial\varphi^i}{\partial u^\alpha}$, $u = (u^1,\ldots,u^n) \in D$. We shall further use the notations:

$$\begin{cases} C = \sqrt{\det(h_{\alpha\beta})}, \text{ where } h_{\alpha\beta} = \sum_{i=1}^{n+1} z^i_\alpha z^i_\beta \\ \tilde{C} = \sqrt{\det(\tilde{h}_{\alpha\beta})}, \text{ where } \tilde{h}_{\alpha\beta} = \sum_{i=1}^{n+1} \tilde{z}^i_\alpha \tilde{z}^i_\beta, \end{cases}$$

with $\tilde{z}^i_\alpha = a_i z^i_\alpha$ (with no summation after i), and

$$B = b^2 z^{n+1}_\alpha z^{n+1}_\beta h^{\alpha\beta}, \qquad \tilde{B} = b^2 \tilde{z}^{n+1}_\alpha \tilde{z}^{n+1}_\beta \tilde{h}^{\alpha\beta}.$$

Then, using

$$\Delta = \det(g_{\alpha\beta}) \quad \Rightarrow \frac{\partial\sqrt{\Delta}}{\partial\tau} = \frac{\sqrt{\Delta}}{2} g^{\alpha\beta}\frac{\partial g_{\alpha\beta}}{\partial\tau}$$
$$g_{\alpha\beta}g^{\beta\gamma} = \delta^\gamma_\alpha \quad \Rightarrow \frac{\partial g^{\alpha\beta}}{\partial\tau} = -g^{\alpha\gamma}g^{\beta\delta}\frac{\partial g_{\gamma\delta}}{\partial\tau}$$

we easily infer that denoting $\tilde{C}_{x^i} = \dfrac{\partial\tilde{C}}{\partial x^i}$, $\tilde{B}_{*i} = b^2\dfrac{\partial}{\partial x^i}(\tilde{z}^{n+1}_\alpha \tilde{z}^{n+1}_\beta \tilde{h}^{\alpha\beta})$, we get

$$\begin{cases} \tilde{C}_{x^i} = \tilde{C}\cdot\tilde{h}^{\alpha\beta}\displaystyle\sum_{j=1}^{n+1}\tilde{z}^j_\alpha\tilde{z}^j_\beta\theta_{j,i} \\ \tilde{B}_{*i} = 2\left(\tilde{B}\theta_{n+1,i} - b^2\tilde{z}^{n+1}_\alpha\tilde{z}^{n+1}_\beta\tilde{h}^{\alpha\gamma}\tilde{h}^{\beta\delta}\displaystyle\sum_{j=1}^{n+1}\tilde{z}^j_\gamma\tilde{z}^j_\delta\theta_{j,i}\right) \end{cases}$$

where $\theta_{j,i} = \dfrac{\partial \ln a_i}{\partial x^i}$. Then, applying the previous Theorem and the expression of the induced volume density for hypersurfaces $\tilde{G} = \tilde{C}\cdot(1-\tilde{B})^{(n+1)/2}$ in Randers spaces, we obtain the following result:

Theorem 2.1. *The components of the mean curvature 1-form of the hypersurface M, isometrically immersed in the Randers space $\tilde{M} = \mathbb{R}^{n+1}$ endowed with the fundamental function (2.1), are given by*

$$\tilde{H}_i = H_i + H_{*i}, \quad i = \overline{1, n+1} \tag{2.2}$$

with

$$\begin{aligned} H_i = &-\frac{(n+1)\tilde{B}\omega_i}{1-\tilde{B}} - \frac{1}{\tilde{C}}\cdot\left[\frac{(n^2-1)}{4(1-\tilde{B})^2}\frac{\partial\tilde{B}}{\partial z^i_\varepsilon}\frac{\partial\tilde{B}}{\partial z^j_\eta}\tilde{C} + \frac{\partial^2\tilde{C}}{\partial z^i_\varepsilon\partial z^j_\eta} - \right. \\ &\left. - \frac{n+1}{2(1-\tilde{B})}\cdot\left(\frac{\partial^2\tilde{B}}{\partial z^i_\varepsilon\partial z^j_\eta}\tilde{C} + \frac{\partial\tilde{B}}{\partial z^j_\eta}\frac{\partial\tilde{C}}{\partial z^i_\varepsilon} + \frac{\partial\tilde{B}}{\partial z^i_\varepsilon}\frac{\partial\tilde{C}}{\partial z^j_\eta}\right)\right]\varphi^j_{\varepsilon\eta} + \\ &+ \frac{(n+1)\omega_j}{\tilde{C}(1-\tilde{B})}\left[\tilde{B}\frac{\partial\tilde{C}}{\partial z^i_\varepsilon} + \left(1 - \frac{\tilde{B}(n-1)}{2(1-\tilde{B})}\right)\tilde{C}B_{z^i_\varepsilon}\right]\cdot\frac{\partial\varphi^j}{\partial u^\varepsilon}, \\ H_{*i} = &\frac{\tilde{C}_{x^i}}{\tilde{C}} - \frac{n+1}{2(1-\tilde{B})}\tilde{B}_{*i} - \left[\frac{\tilde{C}_{x^i z^i_\alpha}}{\tilde{C}} - \frac{n+1}{2(1-\tilde{B})}\left(\tilde{C}_{x^j}\cdot\tilde{B}_{z^i_\alpha}\cdot\frac{1}{\tilde{C}} + \right.\right. \\ &\left.\left. + \tilde{C}_{z^i_\alpha}\tilde{B}_{x^j}\cdot\frac{1}{\tilde{C}} - \frac{n-1}{2(1-\tilde{B})}\tilde{B}_{z^i_\alpha}\tilde{B}_{*j} + \frac{\partial}{\partial z^i_\alpha}\tilde{B}_{*j}\right)\right]\varphi^j_\alpha, \end{aligned} \tag{2.3}$$

where $\omega_i = \partial \ln b/\partial x^i$, $\varphi^j_{\varepsilon\eta} = \frac{\partial^2\varphi^j}{\partial u^\varepsilon\partial u^\eta}$ and $\varphi^j_\varepsilon = \frac{\partial\varphi^j}{\partial u^\varepsilon}$.

Proof. We have

$$\tilde{G}_{x^i} = -(n+1)\tilde{B}\tilde{C}(1-\tilde{B})^{(n-1)/2}\cdot\omega_i + \tilde{G}_{*i},$$

where $\tilde{G}_{*i} = \tilde{C}_{x^i}(1-\tilde{B})^{(n+1)/2} - \dfrac{n+1}{2}\tilde{C}(1-\tilde{B})^{(n-1)/2}\tilde{B}_{*i}$ and

$$\tilde{G}_{x^j z^i_\alpha} = \frac{\partial}{\partial z^i_\alpha}\left(-(n+1)\tilde{B}\tilde{C}(1-\tilde{B})^{(n-1)/2}\cdot\omega_i\right) + \tilde{G}_{*ji\alpha},$$

with

$$\begin{aligned}\tilde{G}_{*ji\alpha} = \quad & \tilde{C}_{x^j z^i_\alpha}(1-\tilde{B})^{(n+1)/2} - \frac{(n+1)(1-\tilde{B})^{(n-1)/2}}{2}\cdot \\ & \cdot\left[\tilde{C}_{x^j}\tilde{B}_{z^i_\alpha} + \tilde{C}_{z^i_\alpha}\tilde{B}_{*j} - \frac{n-1}{2(1-\tilde{B})}\tilde{C}\tilde{B}_{z^i_\alpha}\tilde{B}_{*j} + \frac{\partial}{\partial z^i_\alpha}\tilde{B}_{*j}\right].\end{aligned}$$

Further, using (1.1), the claim follows. □

3 The conformally deformed Randers case

As a particular case, for $a_1(x) = \cdots = a_{n+1}(x) = a(x) = e^{\sigma(x)}$, the previous theorem provides the mean curvature for the homothetic (conformal) deformation of the Finsler metric investigated in [4, 6], as follows

Corollary 3.1. *The components of the mean curvature 1-form of the hypersurface M, isometrically immersed in the Randers space $\tilde{M} = \mathbb{R}^{n+1}$ endowed with the fundamental function*

$$\tilde{F}(x,y) = a(x)\cdot\left[\left(\sum_{i=1}^{n+1}(y^i)^2\right)^{1/2} + b_{n+1}(x)y^{n+1}\right], \tag{3.1}$$

are given by

$$\tilde{H}_i = H_i + H_{*i}, \quad i=\overline{1,n+1} \tag{3.2}$$

with

$$\begin{aligned}H_i = \quad & -\frac{(n+1)B\omega_i}{1-B} - \frac{1}{C}\cdot\left[\frac{(n^2-1)}{4(1-B)^2}B_{z^i_\varepsilon}B_{z^j_\eta}C + \right. \\ & \left. + C_{z^i_\varepsilon z^j_\eta} - \frac{n+1}{2(1-B)}\cdot\left(B_{z^i_\varepsilon z^j_\eta}C + B_{z^j_\eta}C_{z^i_\varepsilon} + B_{z^i_\varepsilon}C_{z^j_\eta}\right)\right]\varphi^j_{\varepsilon\eta} + \\ & + \frac{(n+1)\omega_j}{C(1-B)}\left[BC_{z^i_\varepsilon} + \left(1 - \frac{B(n-1)}{2(1-B)}\right)CB_{z^i_\varepsilon}\right]\cdot\frac{\partial\varphi^j}{\partial u^\varepsilon}, \\ H_{*i} = \quad & \frac{n}{2}\left[\theta_i - \left(\frac{1}{C}C_{z^i_\alpha} - \frac{n+1}{2(1-B)}B_{z^i_\alpha}\right)\theta_j\varphi^j_\alpha\right],\end{aligned} \tag{3.3}$$

where $\omega_i = \partial\ln b/\partial x^i, \theta_i = \frac{\partial\sigma}{\partial x^i}$, $\varphi^j_{\varepsilon\eta} = \frac{\partial^2\varphi^j}{\partial u^\varepsilon\partial u^\eta}$ and $\varphi^j_\varepsilon = \frac{\partial\varphi^j}{\partial u^\varepsilon}$.

Proof. Denoting $a = e^{\sigma(x)}$ and $\theta_i = \frac{\partial\ln a}{\partial x^i} = \frac{\partial\sigma}{\partial x^i}$, we use the relations: $\tilde{h}_{\alpha\beta} = a^2 h_{\alpha\beta}$, $\tilde{h}^{\alpha\beta} = a^{-2}h^{\alpha\beta}$, $\tilde{B} = B$, $\tilde{C} = a^n C$, $\tilde{C}_{z^i_\alpha} = a^n C_{z^i_\alpha}$, $\tilde{C}_{x^i} = n\theta_i\tilde{C}$, $\tilde{C}_{x^j z^i_\alpha} = n\theta_j\tilde{C}_{z^i_\alpha}$. □

Corollary 3.2. *The two constituents of the mean curvature 1-form (3.2) rewrite in terms of C and $E = BC^2$ as follows:*

$$\begin{aligned}H_i = & -\frac{1}{C(C^2-E)}\Big[(C^2)_{z^i_\varepsilon z^j_\eta}\cdot k_{c_2} - E_{z^i_\varepsilon z^j_\eta}\cdot k_{e_2} + E_{z^i_\varepsilon}E_{z^j_\eta}\cdot k_{e_{12}} + \\ & + \left(C_{z^i_\varepsilon}E_{z^j_\eta} + C_{z^j_\eta}E_{z^i_\varepsilon}\right)\cdot k_{ce} + C_{z^i_\varepsilon}C_{z^j_\eta}\cdot k_{cc}\Big]\varphi^j_{\varepsilon\eta} + \\ & + \frac{(n+1)}{C(C^2-E)}\left[EC_{z^i_\varepsilon} + k_m\cdot\left(CE_{z^i_\varepsilon} - 2EC_{z^i_\varepsilon}\right)\right]\omega_j\varphi^j_\varepsilon - \frac{(n+1)E}{C^2-E}\omega_i, \\ H_{*i} = & \frac{n}{2}\left[\theta_i + \frac{1}{C^2-E}\left(\frac{n+1}{2}E_{z^i_\alpha} - \frac{C^2+nE}{C}C_{z^i_\alpha}\right)\theta_j\varphi^j_\alpha\right]\end{aligned} \tag{3.4}$$

where

$$k_{e_2} = \frac{(n+1)C}{2},\quad k_{e_{12}} = \frac{(n^2-1)C}{4(C^2-E)},\quad k_{ce} = \frac{(n+1)(C^2-nE)}{2(C^2-E)},$$

$$k_{c_2} = \frac{C^2+nE}{2C},\quad k_{cc} = \frac{n(n+2)E^2 - 2nEC^2 - C^4}{C(C^2-E)},\quad k_m = 1 - \frac{(n-1)E}{2(C^2-E)}$$

Remark 3.3. a) We note that for $a(x) \equiv 1$, one obtains the case of nonconstant potential studied in [6, 4]. Moreover, for $b = const.$ (i.e., for $\omega_i = 0$, $\forall i \in \overline{1, n+1}$), $H_i X^i = 0$ leads to the minimality characterization obtained by Souza-Tenenblat ([15, Theorem 2, p. 629]).

b) For $n = 2, b_{n+1} \equiv 0, a \equiv 1$ and $\varphi(u,v) = (u, v, f(u,v))$, $X = \varphi_u \times \varphi_v$, (3.1) becomes the Euclidean norm, and the equation $H_i X^i = 0$ becomes the classical minimal surface equation

$$(1 + f_u^2)f_{vv} - 2f_u f_v f_{uv} + (1 + f_v^2)f_{uu} = 0.$$

Acknowledgements. The author is grateful to Yi-Bing Shen for the useful discussions and support on the present topic. The study of existence results of special CMC (α, β) hypersurfaces, rigidity and Bernstein-type theorems, is in progress.

References

[1] J.C. Álvarez Paiva, G. Breck, *What is wrong with the Hausdorff measure in Finsler spaces*, arXiv.math.DG/0408413, 2004; Adv. Math. 204 (2006), 647-663.

[2] J.C. Álvarez Paiva, A.C. Thompson, *Volumes on normed and Finsler spaces*, In: Riemann-Finsler Geometry, MSRI Publications, 49 (2004), 1-46.

[3] V. Balan, *CMC and minimal surfaces in Berwald-Moor spaces*, Hypercomplex Numbers in Geometry and Physics, 2(6), 3 (2006), 113-122.

[4] V. Balan, *Constant mean curvature submanifolds in (α, β)−Finsler spaces*, An.Şt. Univ. "Al.I.Cuza" Iaşi, Tom LIII, s.I, Mathematics, suppliment, 2007, 61-66.

[5] V. Balan, *DPW method for CMC surfaces in Randers spaces*, in: "Modern Trends in Geometry and Topology" (Eds: D. Andrica, P.A. Blaga, S. Moroianu), Proc. of The VII-th Int. Workshop on Diff. Geom. Appl., 5-11 September 2005, Deva, Romania, Cluj Univ. Press 2006, 79-86.

[6] V. Balan, *Euler-Lagrange characterization of area-minimizing graphs in Randers spaces with non-constant potential*, WSEAS Transactions on Mathematics 1, 7 (2008), 1-5.

[7] A.Bejancu, *Finsler Geometry and Applications*, E.Horwood, 1990.

[8] Q. He, Y.-B. Shen, *On Bernstein type theorems in Finsler spaces with the volume form induced from the projective sphere bundle*, Proc. of the AMS, 134 (3) (2006), 871-880.

[9] Q. He, Y.-B. Shen, *On the mean curvature of Finsler submanifolds*, Chinese Jour. of Contemp. Math., 27 (2006), 4, 1-12.

[10] M. Matsumoto, *Theory of Y-extremal and minimal hypersurfaces in a Finsler space*, J. Math. Kyoto Univ. 26 (1986), 647-665.

[11] R.Miron, M.Anastasiei, *The Geometry of Vector Bundles. Theory and Applications*, Kluwer Acad. Publishers, FTPH, no. 59, 1994.

[12] Z. Shen, *On Finsler geometry of submanifolds*, Mathematische Annalen, Springer-Verlag 311 (1998), 549-576.

[13] M. Souza, *Superficies Minimas en Espacos de Finsler com Una Metrica de Randers*, Ph.D. Thesis, University of Brasil, 2001.

[14] M. Souza, J. Spruck, K. Tenenblat, *A Bernstein type theorem on a Randers space*, Math. Annalen 329 (2004), 291-305.

[15] M. Souza, K. Tenenblat, *Minimal surface of rotation in Finsler space with a Randers metric*, Math. Ann. 325 (2003), 625-642.

[16] B.Y. Wu, *A local rigidity theorem for minimal surfaces in Minkowski 3-space of Randers type*, Ann. Glob. Anal. Geom. 31 (2007), 375-384.

[17] B.Y. Wu, *A Reilly type inequality for the first eigenvalue of Finsler submanifolds in Minkowski space*, Ann. Glob. Anal. Geom. 29 (2006), 95-102.

[18] B.Y. Wu, *On the volume forms and submanifolds in Finsler geometry*, Chinese J. Contemp. Math. 27 (2006), 61-72.

Author's address:

Vladimir Balan
University Politehnica of Bucharest, Faculty of Applied Sciences,
Department of Mathematics I, Splaiul Independentei 313,
RO-060042, Bucharest, Romania.
E-mail address: vbalan@mathem.pub.ro