Finsler structures associated to control systems

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Abstract. We show how a control system induces a definite Finsler structure on a subset of the tangent bundle, such that the cost value of a curve, which is solution of the control system, is the length of the curve. In other words, Finslerian geodesics are solutions of the optimal control problem - curves, which decrease the cost. Further, we provide an original example of Finsler structures assigned to the control problem determined by a spacecraft motion in the gravitational field.

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1 Introduction

Let M be an n-dimensional C^∞ manifold. Denote by T_xM the tangent space at $x \in M$, and by $TM = \bigcup_{x\in M} T_xM$ the tangent space of M. Each point of TM has the form (x, y), where $x \in M$, $y \in T_xM$. Denote by F a Finsler structure globally defined on M.

Consider a function $f\colon TM \to \mathbf{R}$ of class C^∞ on $TM \setminus 0$, whose partial function $y \to f(x, y)$ has no critical point. Then, for each point $x \in M$, the set $I_x\colon f(x, y) = 0$ features a hypersurface of T_xM. On the other hand, there exists a function $F\colon TM \setminus 0 \to \mathbf{R}$, implicitly defined, which is C^∞ and positive homogeneous of degree one with respect to y, such that $I_x\colon F(x, y) = 1$. Therefore I_x becomes an indicatrix (the set of unit vectors). Applying the Okubo method [2], one derives that the function F is a solution of implicit equation $\left(x, \dfrac{y}{F(x,y)}\right) = 0$.

The algorithm which provides a new Finsler structure was developed in [6] and relies on the following remark: let V_x be a cone in T_xM, regarded as a subset of T_xM, which is invariant with respect to positive homotheties. If $f(x, y) = \begin{cases} f_1(x,y), & y \in V_x \subset T_xM \\ f_2(x,y), & y \in T_xM \setminus V_x \end{cases}$ has suitable properties, then we get a Finsler function $F = \sqrt{F_1^2 + F_2^2}$, which satisfies $f\left(x, \dfrac{y}{F(x,y)}\right) = 0$, $\forall y \in T_xM$ via

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$$f_1\left(x, \frac{y}{F_1(x,y)}\right) = 0, \quad \forall y \in V_x,$$
$$f_2\left(x, \frac{y}{F_2(x,y)}\right) = 0, \quad \forall y \in T_xM \setminus V_x.$$

2 Finsler structures induced by time optimal problems

Following the algorithm of [6], we will introduce the Finsler structure produced by time optimal problems.

To this aim, let us consider M an n-dimensional C^∞ manifold (U, η, M) as control fiber bundle, and (TM, τ, M) the tangent bundle. M is called *state manifold*, and the components x^i, $i = \overline{1,n}$ of the point $x \in M$ are called *state variables*. Then (x^i, u^a), $i = \overline{1,n}$, $a = \overline{1,k}$ are adapted coordinates in U, and (x^i, y^i) are natural coordinates in TM. The components u^a of the point u are called controls.

Let us consider a fibered mapping $X\colon U \to TM$, $X = X^i(x,u)\dfrac{\partial}{\partial x^i}$ over the identity in the state manifold M, which produces a continuous control system $\dfrac{dx^i}{dt} = X^i(x,u)$, $i = \overline{1,n}$, where t is the parameter of evolution. The evolution of the state manifold M is characterized by the image set $S = \mathrm{Im}\,(X) \subset TM$, which is described by the control equations $x^i = x^i$, $y^i = X^i(x,u)$, $i = \overline{1,n}$. Consider also the set $S_x = \{y \mid \exists u \in U_x = \eta^{-1}(x),\ y = f(x,u)\}$.

Having a cost functional $\displaystyle\int X^0(x(t), u(t))\,dt$, the optimal control theory requires allowed curves of the control system complying some boundary conditions and minimizing the cost functional. In this way, the Pontryagin maximum principle offers a set of necessary conditions for a curve $(x(t), u(t))$ to be optimal, using a Hamiltonian $H(x, p_0, p, u) = p_0X^0(x,u) + p_iX^i(x,u)$, where (p_0, p_i) are called moment coordinates. The optimal curves $(x(t), u^*(t))$ have to satisfy:

- the control system, $\dfrac{dx^i}{dt} = \dfrac{\partial H}{\partial p_i} = X^i(x(t), u^*(t))$;
- the adjoint differential equations,

$$\frac{dp_i}{dt} = -\frac{\partial H}{\partial x^i} = \left(-p_0\frac{\partial X^0}{\partial x^i} - p_j\frac{\partial X^j}{\partial x^i}\right)(x(t), u^*(t));$$

- the maximally condition $H(x, p_0, p, u^*) \geq H(x, p_0, p, u)$, $\forall u \in U_x$.

For a time optimal problem, we shall minimize the final time T (taking $t = 0$ as initial value) such that $X^0(x,u) = 1$ and $H(x,p,u) = p_iX^i(x,u) - 1$ (for the normal case $p_0 = -1$). Therefore, the optimal curves take values on the subset $H = 0$ of $W = U \times MT^*M$.

In order to define a Finsler structure $F = \sqrt{F_1^2 + F_2^2}$, it is required to have a set of allowed directions, a cone bundle $V \subset TM$, an indicatrix included in $V_x \subset T_xM$, a homogeneous function F_1 on V_x, an indicatrix included in $T_xM \backslash V_x$ and a homogeneous function F_2 on $T_xM \setminus V_x$.

Therefore, it is considered:

- the subset of possible optimal controls,

$$U_x^* = \{(x, u_0), \exists (x, p) \in T^*M,\ H(x, p, u_0) \geq H(x, p, u),\ \forall u \in U_x\};$$

- the indicatrix

$$S_{1x} = \{y \in S_x,\ \text{with } ty \notin S_x,\ \forall t > 1\};$$

- the cone

$$V_x = \{y \in T_xM,\ \exists y_0 \in S_{1x} \text{ such that } y = \lambda y_0,\ \lambda > 0\};$$

- the positive homogeneous functions of degree one

$$F_{1x} \colon V_x \to \mathbf{R}_+,\ F_{1x}(y) = \lambda,\ \text{where } y = \lambda y_0,\ y_0 \in S_{1x}$$

(λ is the factor between y with $y \in V_x$ and y_0 with $y_0 \in S_{1x}$ in the same ray).

If there exists a suitable function $f_{1x} \colon V_x \to \mathbf{R}$ with $S_{1x} = f_{1x}^{-1}(0)$, then F_1 is a solution of $f_1\left(x, \dfrac{y}{F_1(x,y)}\right) = 0$. In this context, an optimal curve has the cost

$$T = \int_0^T dt = \int_{\tau_i}^{\tau_f} \frac{dt}{d\tau} d\tau = \int_{\tau_i}^{\tau_f} \lambda(\tau) d\tau = \int_{\tau_i}^{\tau_f} F(x(\tau), y(\tau))\, d\tau$$

(Finslerian length of the curve);

- an arbitrary indicatrix $S_{2x} \colon f_2(x, y) = 0$ in $T_xM \setminus V_x$ and the positive homogeneous function of degree one $F_2(x, y)$ produced via the equation

$$f_2\left(x, \frac{y}{F_2(x,y)}\right) = 0.$$

3 Finsler structures associated to an optimal control problem

In the following we give an original example of getting a Finsler structure assigned to an optimal control problem.

We consider a spacecraft whose motion is influenced by central gravity field. Its motion equations are:

$$\ddot{\vec{r}} = -\frac{\mu}{r^3}\vec{r} + \frac{\vec{F}}{m},$$

where:

- $\vec{r}$ is the position vector of the spacecraft from the center and $r = |\vec{r}|$ is the distance of the spacecraft from the center of gravity;

- $\mu = GM$ is the gravitational parameter of the central body, where G is the universal gravity constant and M is the mass of the central body;

- $\vec{F}$ is the non-gravitational control force;

- m is the mass of the spacecraft, which is assumed constant.

First we fix the origin of the inertial frame at the center of gravity and introduce another coordinate frame, which is rotating along a circular orbit at a constant angular velocity:

- $\delta\vec{r} = (xyz)^T$ is the position vector of the spacecraft whose components represent radial, tangential and normal displacements from the origin of the rotating frame, respectively;
- $\vec{R} = R\mathbf{i}$ is the position vector of the origin of the rotating frame from the origin of the inertial frame;
- $\vec{\omega} = \omega\mathbf{k}$ is the constant angular velocity equal to the mean motion of the circular orbit;
- $\vec{u} = \dfrac{\vec{F}}{m} = u_x i + u_y j + u_z k$ is the control acceleration whose components are given in the rotating frame.

The position vector has the expression

$$\vec{r} = \vec{R} + \delta\vec{r} = (R + x)i + yj + zk.$$

Using the formula ([1])

$$\dot{\vec{r}} = (\dot{R} + \dot{x})i + \dot{y}j + \dot{z}k + \vec{\omega} \times \vec{r},$$

we find the velocity vector expression

$$\dot{\vec{r}} = (\dot{x} - \omega y)i + (\dot{y} + \omega(R + x))j + \dot{z}k.$$

Using the formula ([1])

$$\ddot{\vec{r}} = (\ddot{x} - \dot{\omega}y - \omega\dot{y})\, i + \Big(\ddot{y} + \dot{\omega}(R + x) + \omega(\dot{R} + \dot{x})\Big)\, j + \ddot{z}k + \dot{\vec{\omega}} \times \vec{r} + \vec{\omega} \times \dot{\vec{r}} + \vec{\omega} \times (\vec{\omega} \times \vec{r})$$

we derive that the acceleration vector has the expression

$$\ddot{\vec{r}} = \left(\ddot{x} - 2\omega\dot{y} - \omega^2(R + x)\right) i + \left(\ddot{y} + 2\omega\dot{x} - \omega^2 y\right) j + \ddot{z}k.$$

From Newton's equation we get the following component-wise equations of motion in the rotating frame:

$$\begin{aligned}
&\ddot{x} - 2\omega\dot{y} - \omega^2(R + x) = -\frac{\mu}{r^3}(R + x) + u_x \\
&\ddot{y} + 2\omega\dot{x} - \omega^2 y = -\frac{\mu}{r^3} y + u_y \\
&\ddot{z} = -\frac{\mu}{r^3} z + u_z,
\end{aligned}$$

where $r = \sqrt{(R + x)^2 + y^2 + z^2}$.

If non-dimensionalized with reference length R and reference time $\dfrac{1}{\omega}$, we get the simplified formula

$$\begin{aligned}
&\ddot{x} - 2\dot{y} + (1 + x)\left(\frac{1}{r^3} - 1\right) = u_x \\
&\ddot{y} + 2\dot{x} + y\left(\frac{1}{r^3} - 1\right) = u_y \\
&\ddot{z} + \frac{1}{r^3} z = u_z,
\end{aligned}$$

where, this time, $r = \sqrt{(1 + x)^2 + y^2 + z^2}$.

We further define the states in the following way: $\vec{x} = (x_1 x_2 x_3 x_4)^T = (xy\dot{x}\dot{y})^T$ and get the equations of planar motion in the state space, of the form

$$\begin{pmatrix} \dot{x_1} \\ \dot{x_2} \\ \dot{x_3} \\ \dot{x_4} \end{pmatrix} = \begin{pmatrix} x_3 \\ x_4 \\ 2x_2 - (1+x_1)\left(\dfrac{1}{r_3} - 1\right) + u_x \\ -2x_3 - x_2\left(\dfrac{1}{r_3} - 1\right) + u_y \end{pmatrix},$$

where $r = \sqrt{(x_1+1)^2 + x_2^2}$.

We specify that in [4] the Taylor series expansion was employed, in order to define a dynamic system from gravity center field against which a cost quadratic function is minimized.

Further on, let us consider the planar motion such that $u_x = u_1 \cos u_2$, $u_y = u_1 \sin u_2$, $u_1 \in [0,1]$, $u_2 \in [0, 2\pi]$.

This new system may be regarded as a control system whose equations are

$$\begin{cases} y^1 = x_3 \\ y^2 = x_4 \\ y^3 = 2x_4 - (1+x_1)\left(\dfrac{1}{r^3} - 1\right) + u_1 \cos u_2 \\ y^4 = -2x_3 - x_2\left(\dfrac{1}{r^3} - 1\right) + u_1 \sin u_2. \end{cases}$$

Remark. In the plane $y^3 O y^4$ consider the circle with the center at $C(\lambda, \mu)$ with $\lambda > 1$ and radius 1.

As $\lambda > 1$, it results that $\lambda^2 + \mu^2 > 1$, hence O is outside the circle.

The equations of the tangents d_1 and d_2, carried out from O to this circle are

$$d_1: \; y^4 = \frac{\lambda\mu - \sqrt{\mu^2 + \lambda^2 - 1}}{\lambda^2 - 1} y^3$$

$$d_2: \; y^4 = \frac{\lambda\mu + \sqrt{\mu^2 + \lambda^2 - 1}}{\lambda^2 - 1} y^3.$$

If we denote by T_1 and T_2 the crossing points of the tangents d_1 and d_2 respectively with the considered circle, then equation of the line T_1T_2 is

$$\mu y^4 + \lambda y^3 - (\lambda^2 + \mu^2 - 1) = 0.$$

Then the half plane determined by T_1T_2 which does not contain O, is described by

$$\mu y^4 + \lambda y^3 - (\lambda^2 + \mu^2 - 1) \geq 0.$$

The cone between d_1 and d_2 for which $y^3 > 0$, is defined by

$$\begin{cases} \dfrac{\lambda\mu - \sqrt{\mu^2 + \lambda^2 - 1}}{\lambda^2 - 1} \leq \dfrac{y^4}{y^3} \leq \dfrac{\lambda\mu + \sqrt{\mu^2 + \lambda^2 - 1}}{\lambda^2 - 1} \\ y^3 > 0. \end{cases}$$

Comming back to the control system took into consideration, we denote

$$\lambda = 2x_4 - (1 + x_1)\left(\frac{1}{r^3} - 1\right) \quad \text{and} \quad \mu = -2x^3 - x_2\left(\frac{1}{r^3} - 1\right).$$

Suppose $\lambda > 1$, which is equivalent to

$$2x_4 - (1 + x_1)\left(\frac{1}{r^3} - 1\right) > 1.$$

Consider the sets

$$S\colon \begin{cases} (y^3 - \lambda)^2 + (y^4 - \mu)^2 \leq 1 \\ \lambda > 1 \end{cases}$$

and

$$S_1\colon \begin{cases} (y^3 - \lambda)^2 + (y^4 - \mu)^2 = 1 \\ \mu y^4 + \lambda y^3 - (\lambda^2 + \mu^2 - 1) \geq 0 \\ \lambda > 1. \end{cases}$$

The cone V_x is defined as in the remark,

$$\begin{cases} \dfrac{\lambda\mu - \sqrt{\mu^2 + \lambda^2 - 1}}{\lambda^2 - 1} \leq \dfrac{y^4}{y^3} \leq \dfrac{\lambda\mu + \sqrt{\mu^2 + \lambda^2 - 1}}{\lambda^2 - 1} \\ y^3 > 0 \\ \lambda > 1. \end{cases}$$

We look for a positive and homogeneous solution of the equation

$$\left(\frac{y^3}{F_1} - \lambda\right)^2 + \left(\frac{y^4}{F_1} - \mu\right)^2 = 1,$$

with respect to F_1.

Equivalently,

$$(\lambda^2 + \mu^2 - 1)F_1^2 - 2F_1(\lambda y^3 + \mu y^4) + (y^3)^2 + (y^4)^2 = 0,$$

whose discriminant $\Delta = 4(y^3)^2 \left[\left(\dfrac{y^4}{y^3}\right)^2 (1 - \lambda^2) + 2\lambda\mu\dfrac{y^4}{y^3} + 1 - \mu^2\right]$ is positive for points inside the cone and which additionally check condition $\lambda > 1$.

The convenient positive homogeneous solution is

$$F_1(y^3, y^4) = \frac{(y^3)^2 + (y^4)^2}{\lambda y^3 + \mu y^4 + \sqrt{(y^3)^2 + (y^4)^2 + 2\lambda\mu y^3 y^4 - \lambda^2 (y^4)^2 - \mu^2 (y^3)^2}}.$$

On the set $T_xM \setminus V_x$ we have built in a general manner a first degree positive and homogeneous function $F_2(y^3, y^4)$; finally, $F = \sqrt{F_1^2 + F_2^2}$ is a first degree homogeneous function on T_xM.

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