Some fractional dynamic systems which describe economic processes

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Abstract. Using the Caputo fractional derivative of order α, we build the fractional tangent bundle of order k on $\mathbb{R}$. Here are analyzed the Philipps fractional model and the business-cycle model with inventories.

M.S.C. 2000: 26A33, 58A05, 70H05.
Key words: Caputo fractional derivative, fractional Euler-Lagrange equation, fractional bundle.

1 Introduction

Generally speaking, three definitions for fractional derivatives, i.e. Gr$\ddot{u}$nwald-Letnikov fractional derivative, Riemann-Liouville derivative and Caputros's fractional derivative, are mostly used. Here we only discuss the Caputo derivative: $D_t^\alpha x(t) = I^{m-\alpha}\left(\frac{d}{dt}\right)x(t)$, $\alpha > 0$, where $m-1 < \alpha \leq m$, $\left(\frac{d}{dt}\right)^m = \frac{d}{dt} \circ ... \circ \frac{d}{dt}$, $x : \mathbb{R} \to \mathbb{R}$ and I^β is expressed as follows:$I^\beta x(t) = \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}x(s)ds$, $\beta > 0$. $\Gamma(\beta)$ is Gamma function. Using the Caputo fractional derivative of order α, we build the fractional tangent bundle of order k on $\mathbb{R}$ and the fractional Euler-Lagrange equation. Here are described the fractional Philipps model and the business-cycle model with inventories. These models are analyzed and then numerically simulated.

2 The fractional tangent bundle of order k on $\mathbb{R}$. The Euler-Lagrange fractional equation.

Let $\alpha \in (0,1)$ be fixed. Two curves $C_1, C_2 : I \to \mathbb{R}$ with $C_1(0) = C_2(0) = x_0 \in \mathbb{R}$, $0 \in I$, have a fractional contact α of order $k \in \mathbb{N}^*$ in x_0, if for any $f \in \mathcal{F}(U)$, $x_0 \in U \subset \mathbb{R}$, it holds:

$$D_t^{\alpha a}(f \circ C_1) \mid_{t=0} = D_t^{\alpha a}(f \circ C_2) \mid_{t=0},\ a = 1, ..., k. \tag{2.1}$$

BSG Proceedings 15. The International Conference "Differential Geometry - Dynamical Systems" DGDS-2007, October 5-7, 2007, Bucharest-Romania, pp. 61-67.

The relation (2.1) is an equivalence relation. The equivalence class $[C]_{x_0}^{\alpha k}$ is called fractional k-tangent space of $\mathbb{R}$ in x_0 and it will be denoted by $T_{x_0}^{\alpha k}(\mathbb{R})$. If the curve $C : I \to \mathbb{R}$ is given by x=x(t), $t \in I$, the class $[C]_{x_0}^{\alpha k}$, may be written as:$x(t) = x(0) + \sum_{a=1}^{k} \frac{t^{\alpha a}}{\Gamma(1+\alpha a)}\Big|_{t=0}$ where $t \in (-\varepsilon, \varepsilon)$, $\varepsilon > 0$. We shall use the notations:

$$x(0) = x,\ y^{(\alpha a)} = \frac{1}{\Gamma(1+\alpha a)} D_t^{\alpha a} x(t)\Big|_{t=0},\ a = 1, ..., k. \tag{2.2}$$

The fractional tangent bundle of order k, on $\mathbb{R}$, is the fibre bundle $(T^{\alpha a}(R), \pi_0^{\alpha k}, \mathbb{R})$, where $T^{\alpha a}(R) = \bigcup_{x_0 \in \mathbb{R}} T_{x_0}^{\alpha a}(R)$ and $\pi_0^{\alpha k} : T^{\alpha a}(R) \to \mathbb{R}$ is defined by $\pi_0^{\alpha k}([C]_{x_0}^{\alpha k}) = x_0$, $\forall [C]_{x_0}^{\alpha k} \in T^{\alpha k}(R)$. From the definition of the fractional tangent bundle of order k we infer that if $\lim_{k\to\infty} \alpha_n = 1$, then $\lim_{n\to\infty} T^{\alpha_n k}(\mathbb{R}) = T(\mathbb{R})$. The bundle space $(T^{\alpha k}(M), \pi_0^{\alpha k}, M)$, was defined and studied in ([2], [5]). Let $C : t \in [0,1] \to x(t) \in \mathbb{R}$ be a parameterized curve. The extension of C to $T^{\alpha k}(\mathbb{R})$ is the curve $C^{\alpha k} : t \in [0,1] \to (x(t), y^{(\alpha a)}(t)) \in T^{\alpha k}(\mathbb{R})$, $a = 1, ..., k$. Let $L : T^{\alpha k}(\mathbb{R}) \to \mathbb{R}$ be a Lagrange fractional function. The action of L along the curve $C^{\alpha k}$ is given by

$$\mathcal{A}(C^{\alpha k}) = \int_0^1 L(t, x(t), y^{(\alpha a)}(t))dt. \tag{2.3}$$

From (2.3) results ([1], [2]):

Proposition 2.1 *A necessary condition for the action (2.3) to reach the fractional extremal value is that $C(t)$ satisfies the fractional Euler-Lagrange equation*

$$D_x^{\alpha} L + \sum_{a=1}^{k} (-1)^a d_t^{\alpha a} (D_{y^{(\alpha a)}}^{\alpha} L) = 0, \tag{2.4}$$

where $d_t^{\alpha a} = \sum_{b=1}^{a} y^{(\alpha b)} D_{y(b-1)}^{\alpha}$, $D_{y(0)}^{\alpha} = D_x^{\alpha}$. □

3 Some fractional dynamic system which describe economic processes.

Mathematical models for some economical processes [6] are described by two or three order differential equations of the form:

$$\ddot{x}(t) + a_1 \dot{x}(t) + \varphi_1(x(t)) = 0,\ x(0) = x_0,\ \dot{x}(0) = x_1 \tag{3.1}$$

$$\dddot{x}(t) + a_2 \ddot{x}(t) + a_1 \dot{x}(t) + \varphi_2(x(t)) = 0,\ x(0) = x_0,\ \dot{x}(0) = x_1,\ \ddot{x}(0) = x_2. \tag{3.2}$$

With the notations (2.2), the equations (3.1) and (3.2) become:

$$\Gamma(3) y^{(2)}(t) + a_1 \Gamma(2) y^{(1)}(t) + \varphi_1(x(t)) = 0 \tag{3.3}$$

$$\Gamma(4) y^{(3)}(t) + a_2 \Gamma(3) y^{(2)}(t) + a_1 \Gamma(2) y^{(1)}(t) + \varphi_2(x(t)) = 0. \tag{3.4}$$

From (2.4), results the following:

Proposition 3.1 *a) Let the Lagrange function $L : T^{2\alpha}(\mathbb{R}) \to \mathbb{R}$ be given by:*

$$L(x, y^{(\alpha)}, y^{(2\alpha)}) = -\psi_1(x) + \frac{1}{2} a_1 \Gamma(1+2\alpha)(y^{(\alpha)})^2 - \frac{1}{2}\Gamma(1+4\alpha)(y^{(2\alpha)})^2, \tag{3.5}$$

where $\varphi_1(x) = \frac{d\psi_1(x)}{dx}$. The fractional Euler-Lagrange equation for L has the form

$$\Gamma(1+4\alpha) y^{(4\alpha)} + a_1 \Gamma(1+2\alpha) y^{(2\alpha)} + \varphi_1(x) = 0. \tag{3.6}$$

If $\alpha = \frac{1}{2}$, then the equation (3.6) is the equation (3.1).

b) Let the Lagrange function $L : T^{3\alpha}(\mathbb{R}) \to \mathbb{R}$ be given by:

$$\begin{aligned} L(x, y^{(\alpha)}, y^{(2\alpha)}, y^{(3\alpha)}) &= -\psi_2(x) + \tfrac{1}{2} a_1 \Gamma(1+2\alpha)(y^{(\alpha)})^2 - \\ &- \tfrac{1}{2} a_2 \Gamma(1+4\alpha)(y^{(2\alpha)})^2 + \tfrac{1}{2}\Gamma(1+6\alpha)(y^{(3\alpha)})^2, \end{aligned} \tag{3.7}$$

where $\varphi_2(x) = \frac{d\psi_2(x)}{dx}$. The fractional Euler-Lagrange equation for L is

$$\Gamma(1+6\alpha) y^{(6\alpha)} + a_2 \Gamma(1+4\alpha) y^{(4\alpha)} + a_1 \Gamma(1+2\alpha) y^{(2\alpha)} + \varphi_2(x) = 0. \tag{3.8}$$

If $\alpha = \frac{1}{2}$, the equation (3.8) is the equation (3.2). □

From Proposition 3.1, it results that the equations (3.3) and (3.4) are the Euler-Lagrange equations associated with some fractional Lagrange functions. There exist no classical Lagrange functions, whose Euler-Lagrange equations are described by these equations.

3.1 The fractional Phillips model

Consider the income $Y(t)$, the capital stock $K(t)$, the consumption $C(t)$, and the investment $I(t)$ as variables for this model. The fractional Phillips model is described by:

$$C(t) = cY(t),\ I(t) = \beta(vY(t) - K(t)),\ D_t^\alpha K(t) = I(t),\ D_t^\alpha Y(t) = a(c(t) + I(t) - Y(t)), \tag{3.9}$$

where $c \in (0, 1]$, $v > 0$, $\beta > 0$, $a > 0$, and $\alpha \in (0, 1]$. From (3.9), results the fractional equation:

$$D_t^{2\alpha} Y(t) + a_1 D_t^\alpha Y(t) + b_1 Y(t) = 0 \tag{3.10}$$

where $a_1 = a(1-c) + \beta(1-av)$, $b_1 = a\beta(1-c)$. For $\alpha = 1$, the equation (3.9) are the classical Phillips model equations [6]. Let $Y(t) = x(t)$. From (3.10) results the equation (3.6), where $\varphi_1(x) = -b_1 x$. The fractional Lagrangian function is:

$$L(x, y^{(\alpha)}, y^{(2\alpha)}) = -\frac{b_1 x^2}{2} - \frac{a_1}{2}\Gamma(1+2\alpha)(y^{(\alpha)})^2 - \frac{1}{2}\Gamma(1+4\alpha)(y^{(2\alpha)})^2 \tag{3.11}$$

where $y^{(\alpha)} = \frac{1}{\Gamma(1+\alpha)} D_t^\alpha Y(t), \quad y^{(2\alpha)} \frac{1}{\Gamma(1+2\alpha)} D_t^{(2\alpha)} Y(t)$.

3.2 The fractional business-cycle model with inventories

Let $Y(t)$ be the national product, $B^\alpha(t)$ and $B(t)$ the desired and the actual inventory stock, $S(t)$ and $I(t)$ savings and investments respective, $Y^e(t)$ the expected national product, $t \in \mathbb{R}$. The fractional business-cycle model with inventory is given by:

$$\begin{gathered} D_t^\alpha Y(t) = a(B^\alpha(t) - B(t)), \quad D_t^\alpha B(t) = S(t) - I(t), \\ B^\alpha(t) = KY^e(t), \quad Y^e(t) = Y(t) + a_1 D_t^\alpha Y(t) + a_2 D_t^{2\alpha} Y(t) \end{gathered} \tag{3.12}$$

where $a > 0$, $K > 0$, $a_1, a_2 \in \mathbb{R}_{\geq 0}$, $\alpha \in (0, 1]$. For $\alpha = 1$, the equations (3.12) are the classical business-cycle model equations [6].

From (3.12), results the fractional equation:

$$D_t^{3\alpha} Y(t) + A_2 D_t^{2\alpha} Y(t) - D_t^\alpha Y(t) - \beta(S(t) - I(t)) = 0, \tag{3.13}$$

where $A_1 = \frac{1}{a_2}$, $\beta = \frac{1}{ka_2}$, $A_2 = \frac{aka_1 - 1}{aka_2}$. We consider the situation in which $S(t) = \varphi_2(Y(t))$ and $I(t) = I_0$.

Let $Y(t) = x(t)$. From (3.13), results:

$$\Gamma(1 + 3\alpha) y^{(3\alpha)}(t) + A_2 \Gamma(1 + 2\alpha) y^{(2\alpha)}(t) + A_1 \Gamma(1 + \alpha) y^{(\alpha)}(t) - \beta(\varphi_2(x(t)) - I_0) = 0. \tag{3.14}$$

4 Numerical simulations

Within the study of the dynamics of the fractional Phillips model (3.9), we numerically simulate the solution of the fractional equation (3.10). Let $y_0 = x(t)$, $y_1(t) = D_t^\alpha y_0(t)$, $y_2(t) = D_t^\alpha y_1(t)$, $y_3(t) = D_t^\alpha y_2(t)$. The fractional equation (3.10) becomes:

$$\begin{gathered} D_t^\alpha y_0(t) = y_1(t), \quad D_t^\alpha y_1(t) = y_2(t) \\ D_t^\alpha y_2(t) = y_3(t), \quad D_t^\alpha y_3(t) = -\frac{\varphi_1(y_0(t))}{\Gamma(1 + 4\alpha)} - a_1 \frac{\Gamma(1 + 2\alpha)}{\Gamma(1 + 4\alpha)} y_2(t). \end{gathered} \tag{4.1}$$

For $\varphi_1(y_0(t)) = b_1 y_0(t)$, with the notation $Z(t) = (y_0(t), y_1(t), y_2(t), y_3(t))^T$ the matrix form of the system (4.1) is:

$$D_t^\alpha Z(t) = A_4 Z(t) \tag{4.2}$$

where

$$A_4 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{b_1}{\Gamma(1+4\alpha)} & 0 & -\frac{a_1 \Gamma(1+2\alpha)}{\Gamma(1+4\alpha)} & 0 \end{bmatrix}. \tag{4.3}$$

The solution of system (4.2) with initial condition $Z(0)$ is

$$Z(t) = \sum_{k=0}^{\infty} E_\alpha((-1)^{\alpha k} A_4^k, t)) Z(0) = (E + \sum_{k=1}^{\infty} (-1)^k \frac{t^{\alpha k}}{\Gamma(1 + \alpha k)} A_4^k) Z(0). \tag{4.4}$$

The solution of (4.1) is: $Y(t) = (1, 0, 0, 0)Z(t)$, $C(t) = cY(t)$, $I(t) = D_t^\alpha Y(t) + a(1-c)Y(t)$, $K(t) = \frac{v}{\beta}Y(t) - \frac{1}{\beta}I(t)$.

For $a = 0.3$, $c = 0.2$, $v = 0.5$, $\beta = 0.4$ and using Maple 11, we have the graphics $(t, I(t))$, $(t, C(t))$, $(t, Y(t))$, $(t, K(t))$, for $\alpha = 0.1$, 0.5, and 1.

Fig.1.(t,I(t,alpha)),alpha:0.1,0.5,1

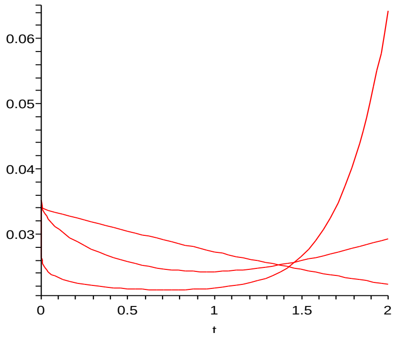


Fig.2.(t,C(t,alpha)),alpha:0.1,0.5,1

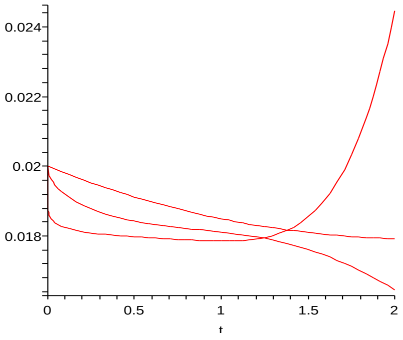


Fig.3.(t,Y(t,alpha)),alpha:0.1,0.5,1

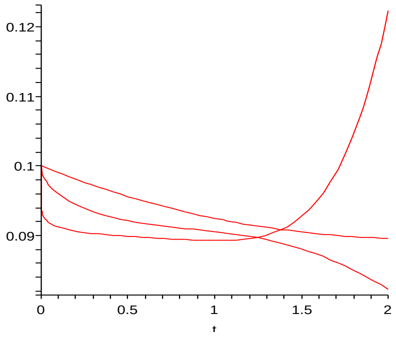
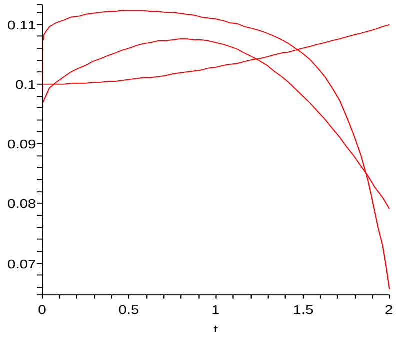


Fig.4.(t,K(t,alpha)),alpha:0.1,0.5,1



From the graphics can be seen that the solution of (4.1) depends on the value of α.

For the system (3.12), we simulate numerically the fractional equation (3.13). Consider the notations $y_0(t) = x(t)$, $y_i(t) = D_t^\alpha y_{i-1}(t)$, $i = 1, 2$ and taking $\varphi_2(y_0(t)) = b_1 y_0(t)$, the equation (3.13) has the following matrix form:

$$D_t^\alpha Z(t) = A_3 Z(t) + V_0, \tag{4.5}$$

where $Z(t) = (y_0(t), y_1(t), y_2(t))^T$, $V_0 = (0, 0, -\frac{\beta I_0}{\Gamma(1+2\alpha)})^T$. The solution of (4.4) is given by:

$$Z(t) = U(t) - A_3^{-1} V_0 \tag{4.6}$$

where

$$U(t) = \sum_{k=0}^{\infty} E_\alpha((-1)^{\alpha k} A_3^k, t) V_0,$$

where A_3 is given by $A_3 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{\beta b_1}{\Gamma(1+3\alpha)} & -\frac{A_1\Gamma(1+\alpha)}{\Gamma(1+3\alpha)} & -\frac{A_2\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)} \end{bmatrix}$.

The solution of (3.12) is:

$$\begin{aligned} &Y(t) = (1,0,0)Z(t), \quad Y^e(t) = (1,a_1,a_2)Z(t), \quad S(t) = (b_1,0,0)Z(t) \\ &B^\alpha(t) = (k,ka_1,ka_2)Z(t), \quad B(t) = (ak, kaa_1 - \frac{1}{a}, kaa_2)Z(t). \end{aligned} \tag{4.7}$$

The numerical simulation leads to the following diagram:

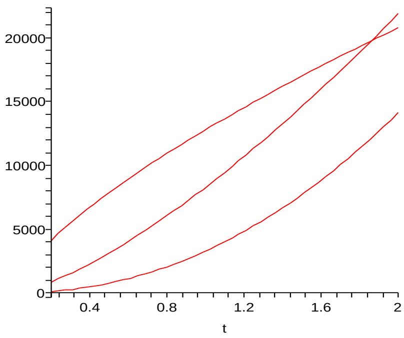


Fig.5.(t,Y(t,alpha)),alpha:0.1,0.2,0.3

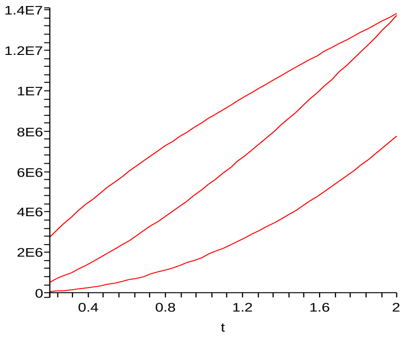


Fig.6.(t,Ye(t,alpha)),alpha:0.1,0.2,0.3

5 Conclusions

The paper presents the modeling of two economical processes using the Caputo fractional derivative. The fractional solutions of these systems are numerically simulated. For other economical models there were used fractional equations from [3] and [4].

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