Nonregular and degenerated optimal control problems with control-state constrains

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Abstract. The present paper is an introductory paper of the treatment of optimal control problems from applications in the aerospace field. Our purpose is to describe the procedures for numerical solving optimal control problems with local constraints. Methods of finding an admissible trajectory complying with the maximum principle are proposed.

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1 Introduction

The present paper is an introductory paper of the treatment of optimal control problems from applications in the aerospace field. Especially those problems are considered which include control-state variable constraints. An optimal control problem can be solved completely or at least qualitatively with the help of the maximum principle (MP). Difficulties in the numeric solving the optimal control problems with control-state constraints (CSC) might derive from insufficient analysis of model problems and inadequate progress in the MP theory in the case. The MP inself could be a base of the numeric methods. Availability of state variable and CSC of a complex nature leads to a complication of the formulation and properties of the MP. New objects appear: measure and functional Lagrange multipliers. It becomes necessary to examine the properties of these objects and analyze the interconnections of the various parts of the MP. Otherwise it would be impossible to use the MP. It is known that the MP reduces the initial control problem to the solution of the two-point (multi-point boundary value problems). They include: initial value problem (IVP); the problem of linear (LP) or nonlinear (NP) programming; root finding of transcendental functions. The IVP consists of the two groups on nonlinear differential equations (direct and adjoint). A number of investigators have tried and rejected shooting methods for certain classes of two-point problems which are particularly sensitive to the initial

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conditions and are thereby troublesome numerically. On the other hand there are successful examples of shooting methods at the expense of a special parametrisation and continuation (homotory chains) the solution (R. Bulirsch, H.J. Pesch, J. Stoer, J. Bett, H. Maurer, A. Afanasjev; E. Smoljakov and others). One of the most effective numerical techniques for the solution of optimal control problems is discretizing the differential equations. This approach combines a nonlinear programming problems with discretization. The resulting problem is characterized by matrices which arc large and sparse. The iteration routine requires an good initial guess for the adjoint variables. Such treatment of the problem has been successfully utilized for applications. This paper extends the continuation methods for ill-posed problems. Our approach utilized the system dynamics and adjoint differential equations defined by the maximum principle. The resulting boundary value problem is characterized by Jacobi or Hesse matrices which are small (as dimension) and ill-conditioned. The method of introduction of the parameter helps to overcome the difficulties associated with adjoint initial values.

2 Problem Statement

We consider the problem of choice of an angle of attack by a device which is slowed down in the atmosphere for the flight on the maximum and minimum distance with constraints on the value of full loading factor. The solution of mentioned problems allows to determine maneuver abilities of the device. The range of the device flight is determined by the following integral

$$L = \int_0^T \frac{RV\cos\Theta}{R+H}\,dt. \tag{2.1}$$

It is required to determine the control $C_y(t)$ such as to minimize (maximize) $L(t)$ (2.1) under constraints:

$$n_\Sigma = \sqrt{C_x^2 + C_y^2}\, q\frac{S}{G} \le N, \qquad q = \frac{\rho V^2}{2}, \quad G = mg, \tag{2.2}$$

$$C_y^{\min} \le C_y \le C_y^{\max}, \qquad C_x = C_{x_0} + kC_y^2, \tag{2.3}$$

$$\rho = \rho_0 e^{-\beta H}, \qquad g = g_0\frac{R^2}{(R+H)^2}, \qquad \dot V = -C_x q\frac{S}{m} - g\sin\Theta, \tag{2.4}$$

$$\dot\Theta = C_y q\frac{S}{mV} + \left(\frac{V}{R+H} - \frac{g}{V}\right)\cos\Theta, \qquad \dot H = V\sin\Theta, \tag{2.5}$$

where n_Σ — full loading factor, q — dynamic velocity pressure, ρ — atmospheric density, V — velocity of device. Θ — path inclination, H — flight altitude, G — weight of device, m — mass, g_0 — acceleration of gravity on the Earth surface, R — the Earth's radius, C_x — the drag force coefficient, C_y — the lift force coefficient, S — typical device area, C_{x_0}, k, ρ_0, β, $C_y^{\min}$, $C_y^{\max}$, N — constants.

For the system (2.1)–(2.5) the initial conditions are given

(2.6) $$V(0) = V_0, \qquad \Theta(0) = \Theta_0, \qquad H(0) = H_0, \qquad L(0) = 0.$$

with boundary conditions

(2.7) $$V(T) = V_1, \qquad \Theta(T) = \Theta_1, \qquad H(T) = H_1, \qquad T \text{ — is not fixed.}$$

3 Maximum principle (regular case)

Let the descent device come from the initial state (2.6) to the final state (2.7) in optimal way in respect of minimum or maximum of the range. We assume the regularity condition being fulfilled on the optimal trajectory. In our case the regular condition is equal to this condition

(3.1) $$\frac{\partial n_\Sigma}{\partial C_y} \neq 0, \qquad n_\Sigma = N.$$

The maximum principle:

(3.2) $$\begin{aligned} \Pi &= P_\Theta\left[\frac{C_y\rho VS}{2m} + \left(\frac{V}{R+H} - \frac{g}{V}\right)\cos\Theta\right] + P_H V\sin\Theta - \\ &- P_V\left[\frac{C_x\rho V^2 S}{2m} + g\sin\Theta\right] + P_L\frac{RV\cos\Theta}{R+H}, \end{aligned}$$

(3.3) $$\begin{aligned} \dot P_\Theta &= P_\Theta\left[\frac{V}{R+H} - \frac{g}{V}\right]\sin\Theta - P_H V\cos\Theta + P_V g\cos\Theta + P_L\frac{RV\sin\Theta}{R+H}, \\ \dot P_V &= -P_\Theta\left[\frac{C_y\rho S}{2m} + \left(\frac{1}{R+H} - \frac{g}{V^2}\right)\cos\Theta\right] - P_H\sin\Theta + P_V\frac{C_x\rho VS}{m} - \\ &- P_L\frac{R\cos\Theta}{R+H} + \lambda(t)\frac{\rho VS}{mg_0}\sqrt{C_x^2 + C_y^2}, \\ \dot P_H &= P_\Theta\left[\frac{\beta C_y\rho VS}{2m} + \frac{V\cos\Theta}{(R+H)^2} - \frac{2g\cos\Theta}{V(R+H)}\right] - \\ &- P_V\left[\frac{\beta C_x\rho V^2 S}{2m} + \frac{2g\sin\Theta}{R+H}\right] + P_L\frac{RV\cos\Theta}{(R+H)^2} - \\ &- \lambda(t)\frac{\rho V^2 S\beta}{2mg_0}\sqrt{C_x^2 + C_y^2}, \\ \dot P_L &= 0. \end{aligned}$$

Here $\lambda(t)$ — is Lagrange multiplier which can be found from the condition of Bliss.

$$\frac{\partial \Pi}{\partial C_y} - \lambda(t)\frac{\partial n_\Sigma}{\partial C_y} = 0, \qquad \lambda(t) = \frac{2\left(\dfrac{P_\Theta}{2} - kP_V C_y V\right)}{C_y V[1+2kC_x]} g_0\sqrt{C_x^2 + C_y^2}, \tag{3.4}$$

P_Θ, P_V, P_H, P_L — adjoint variables. To confine a type of inequalities (2.2) the condition of complementary slackness.

$$\lambda(t)(n_\Sigma - N) = 0. \tag{3.5}$$

Since the system (2.1), (2.5) is autonomous and there are no constraints on the start moment, the Pontryagin function (3.2) is identically zero, i.e.

$$\Pi(P, x, u) \equiv 0, \qquad u = C_y, \quad x = (\Theta, V, Hy, L), \quad P = (P_\Theta, P_V, P_H, P_L). \tag{3.6}$$

The adjoint variable $P_L(t)$ is normalized by the condition

$$P_L(t) = -1. \tag{3.7}$$

From $\dot{P}_L = 0$ (3.3) it follows that $P_L(t) \equiv -1$ on the whole optimal trajectory.

The initial conditions for the system (3.3) are unknown and they are the parameters of the problem. The conditions $P_L(t) \equiv -1$ and $\Pi(P, x, u) \equiv 0$ (3.6) substantially determine two free parameters

$$P_\Theta(0) = C_1, \qquad P_V(0) = C_2, \tag{3.8}$$

since $P_H(0)$ is given from the condition $\Pi(P, x, u) \equiv 0$.

In this case the number of functions (2.7) controlled in the of trajectory coincide with the number of free parameters of problem (2.1)–(2.7), (3.2), (3.3), as a time T is not fixed and it is a free parameter.

According to the maximum principle the programm of control is chosen from condition

$$\begin{aligned} \Pi \quad &\to \quad \min_{C_y} \quad L(T) \to \max, \\ \Pi \quad &\to \quad \max_{C_y} \quad L(T) \to \min, \end{aligned} \tag{3.9}$$

Consider the part of the Pontryagin function (3.2), which explicitly depends on the control $C_y(t)$

$$\Pi_0 = P_\Theta \frac{C_y \rho V S}{2m} - P_V \frac{C_x \rho V^2 S}{2m}. \tag{3.10}$$

The control $C_y(t)$ can possess not only the end values (2.3), but also an intermediate value which is determined from condition.

$$\frac{\partial \Pi_0}{\partial C_y} = 0, \qquad C_y^* = \frac{P_\Theta}{2kP_V V}, \qquad C_y^{\min} \le C_y^* \le C_y^{\max}. \tag{3.11}$$

Now we calculate three values of Π_0 (3.10)

$$\Pi_1 = \Pi_0(C_y^{\min}), \qquad \Pi_2 = \Pi_0(C_y^{\max}), \qquad \Pi_3 = \Pi_0(C_y^*)$$

and determine corresponding maximum and minimum values of Π_0

$$\Pi_0^{\min} = \min\{\Pi_1, \Pi_2, \Pi_3\}, \qquad \Pi_0^{\max} = \max\{\Pi_1, \Pi_2, \Pi_3\}. \tag{3.12}$$

The relations (3.12) determine character of the optimal control for the Pontryagin problem, i.e. under condition $n_\Sigma < N$. The solution of problem is significantly simplified if the right end of the trajectory is controlled by the condition

$$H(T) = H_1. \tag{3.13}$$

In this case the solution of boundary problem (2.1), (2.7) is determined by boundary conditions

$$\Theta(T) = \Theta_1, \qquad V(T) = V_1 \tag{3.14}$$

and depends on arbitrary constants C_1 and C_2 (3.8) .

Thus the initial problem is reduced to the two parametric boundary problem (2.1), (2.6), (3.3), (3.8), (3.14), and the optimal control $C_y(t)$ is determined at every point t according to the maximum principle (3.12).

4 Parametric continuation of solution

For the Pontryagin problem with $n_\Sigma < N$ the Lagrange multiplier $\lambda(t) = 0$ (3.4). The direct solution of the boundary problem by the well-known Newton's method meets a whole set of fundamental difficulties. The first matter is associated with the choice of good initial approximation. The second one is considered to the conditionality of Jacobi matrix. With ill-conditioned matrix of Jacobi the convergence rate of Newton's method decreases and the region of convergence is narrowed.

At first in this work we consider the problem with free right end. In this case

$$P_\Theta(H_1) = 0, \qquad P_V(H_1) = 0. \tag{4.1}$$

Let $H(t)$ is the strictly decreasing function. Divide the interval of integration $[H_0, H_1]$ to a system of embedded sections

$$[H_0, H_{11}] \subset [H_0, H_{12}] \subset \cdots \subset [H_0, H_{1r}], \qquad H_{1r} = H_1. \tag{4.2}$$

On the interval $[H_0, H_{11}]$ consider the boundary problem

$$\begin{aligned} \dot{x}_i &= f_i(x,u), \qquad x_i(H_0) = x_{i0}, \quad i = \overline{1,4}, \\ \dot{\psi}_j &= -\frac{\partial \Pi}{\partial x_i}, \qquad \psi_j(H_0) = C_{1j}, \quad j = \overline{1,4}, \\ \psi_1(H_{11}) &= P_\Theta(H_{11}) = 0, \qquad \psi_2(H_{11}) = P_V(H_{11}) = 0, \end{aligned} \tag{4.3}$$

where $x = (\Theta, V, H, L)$, $u = C_y$, $\psi = (P_\Theta, P_V, P_H, P_L)$.

Since the interval $[H_0, H_{11}]$ is sufficiently small, the boundary problem (4.3) is easily solved by Newton's method. Indeed, in this case as the initial approximation we can choose

$$\psi_1(H_0) = P_\Theta(H_0) = C_{11}, \qquad \psi_2(H_0) = P_V(H_0) = C_{12} \tag{4.4}$$

close to end values $P_\Theta(H_{11})$ $P_V(H_{11})$. The obtained values C_{11} and C_{12} we designate as $\bar{C}_{11}$ and $\bar{C}_{12}$ and further on the interval $[H_0, H_{12}]$ solve new boundary problem with conditions

$$\psi_1(H_{12}) = P_\Theta(H_{12}) = 0, \qquad \psi_2(H_{12}) = P_V(H_{12}) = 0,$$

assumed $\psi_1(H_0) = \bar{C}_{11}$, $\psi_2(H_0) = \bar{C}_{12}$. Then the solution is specified by Newton's method. The described process can be continued up to the value $H_{1r} = H_1$.

In practice the choice of dividing points of interval $[H_0, H_1]$ is carried on the base of Runge-Kutta principle, which is applied for the integration of the ordinary differential equations. For this purpose they set a certain number of iteration in Newton's method and point H_{11}. If on the interval $[H_0, H_{11}]$ the number of iterations is more than specified one, then the interval $[H_0, H_{11}]$ we divide into halves etc. After the choice of H_{11} we choose H_{12} by doubling the length of $[H_0, H_{11}]$.

In the described method of parametric continuation the matter of choice of good initial approximation for the solution of boundary problem is solved. As a rule with the well-conditioned matrix of Jacoby no calculation problems occur. The situation is considerably complicated with an increase of the conditional number of matrix Jacobi. For example, when the device goes into the atmosphere with the orbital velocity $V(H_0) = 8$ km/sec, $H_0 = 100$ at the altitude $H = 35$ the constructed iterative process practically ceases to converge even if the integration interval is added.

For overcoming arising difficulties on the interval $[H_0, H_{11}]$ we chose the intermediate point H_{21} and the primary interval of integration was divided in two intervals $[H_0, H_{21}]$ and $(H_{21}, H_{11}]$. On the interval $[H_0, H_{21}]$ systems (2.1), (2.5), (3.3) were integrated from left to right and on the interval $[H_{21}, H_{11}]$ the same systems were integrated from right to left. The free parameters H_0 and H_{11} were selected from the condition of continuity of main and adjoint variables in point H_{21}

$$\begin{array}{llllll} \Theta(H_{21}-0) & = & \Theta(H_{21}+0), & V(H_{21}-0) & = & V(H_{21}+0), \\ L(H_{21}-0) & = & L(H_{21}+0), & P_\Theta(H_{21}-0) & = & P_\Theta(H_{21}+0), \\ P_V(H_{21}-0) & = & P_V(H_{21}+0). & & & \end{array} \tag{4.5}$$

At the same time the dimension of the boundary problem is doubled, however the conditionality of matrix Jacobi is improved. With parametric continuation of solution point H_{21} is also current. However its value has to satisfy the condition $H_{21} > 35\,\text{km}$. The described method allows to continue the solution till the value $H = H_1$.

As a result of solution of the boundary problem with free right end we get certain values $\Theta(H_1)$ and $V(H_1)$, which generally are not coincide with necessary boundary conditions.

For the further continuation of the solution we introduce the following functional

$$F(H_1) = \left\{[\Theta(H_1) - \Theta_1]^2 + [V(H_1) - V_1]^2\right\}^{1/2}, \tag{4.6}$$

where Θ_1 V_1 are the given boundary conditions.

Minimum of the functional (4.6) is searched by Newton's method

$$\bar{C}_{k+1} = \bar{C}_k - [F''(\bar{C}_k)]^{-1} F'(\bar{C}_k), \qquad \bar{C} = (C_1, C_2). \tag{4.7}$$

Here $F'(\bar{C}_k)$ — gradient of function (4.6) in point $\bar{C}_k$, $F''(\bar{C}_k)$ — Hessian matrix. Subjects of functional (4.6) minimum searching with ill-conditioned Hessian matrix are considered in other works.

The other approach for researching of different classes of variational problems including optimal control problems is connected with homotopy methods. The idea of these methods is connected with a minimum point and homotopy invariant. If an extremal under test is uniformly parametrically isolated in the process of deformation, then its property to be the minimum point is the homotopy invariant.

5 Constraints on loading factor

Taking into account the constraints on overload (2.2) is significantly increases difficulties of the solution even in regular case. The first problem is connected with calculating of the Lagrange multiplier $\lambda(t)$ (3.4). When use an iterative search of the optimal trajectory we see considerable growth of the Lagrange multipliers provided $C_y(t) \to 0$. The specified difficulty can be overcome by the means of the theory of singularly perturbed systems.

The second difficulty in optimal control problem with inequality constraints is connected with an estimation of geometry of the optimal trajectory or in other words the set of active indexes. To a certain degree this question is solved for the linear in control problems. At the same time the initial problem is discretized and then we solve the linear programming problem of high dimension. This solution make possible to estimate geometry of the optimal trajectory. At the same time a number of possible alternatives in character of optimal trajectory is decreased. On the base of the obtained solution we can form hypothesis about geometry of the optimal trajectory. Then we can verify expected trajectory on optimality using the maximum principle.

We should note that considerable computational difficulties bounded with given ill-posed problem appear during solution of the linear programming problem.

In the assigned problem the difficulty of estimation of geometry of the optimal trajectory is connected with the difficulty of determining the moment of departure from constraints $n_\Sigma = N$ (2.2).

We should note that the full load factor n_Σ (2.2) has two components n_x and n_y. The first one is called the longitudinal load factor, and the second one is the normal load factor.

$$n_y = \frac{\rho V^2 S}{2mg_0} C_y, \qquad n_x = \frac{\rho V^2 S}{2mg_0} C_x, \qquad n_\Sigma = \sqrt{n_x^2 + n_y^2}. \tag{5.1}$$

Instead of constraint (2.2) we introduce a new constraint

$$|n_y| + n_x \leq N_1, \qquad |n_y| + n_x - N_1 = \varphi(x, u) \leq 0. \tag{5.2}$$

With proper choice of N_1 from the validity of inequality (5.2) we trivially obtain the constraint (2.2). The mentioned fact is followed from inequality

$$N_1 \geq [|n_y| + |n_x|] \geq \sqrt{n_x^2 + n_y^2}, \tag{5.3}$$

and an equality is obtained when $C_y = 0$.

Now calculate the derivative $\varphi(x,u)$ (5.2) provided respect to C_y

$$\frac{\partial\varphi}{\partial C_y} = \frac{\rho V^2 S}{2mg_0}[\operatorname{sign} C_y + 2kC_y]. \tag{5.4}$$

In this case the Lagrange multiplier $\lambda(t)$ for constraint $\varphi(x,u) \le 0$ (5.2) is given by formula

$$\lambda(t) = \frac{2\left(\frac{P_\Theta}{2} - kP_V C_y V\right) g_0}{V[\operatorname{sign} C_y + 2kC_y]}. \tag{5.5}$$

From (5.5) we can see that the Lagrange multiplier $\lambda(t)$ is constrained for any values of C_y. We redefine by continuity the value $\operatorname{sign} C_y$ provided $C_y = 0$ from the condition $\operatorname{sign} C_y$ provided $C_y \to 0$.

The stated approach make possible to carry out the continuous iterative process of searching the optimal trajectory with values of $C_y(t)$ close to zero.

For a motion on constraint like (2.2) the control C_y^2 is obtained from the condition of connection $n_\Sigma = N$

$$\begin{aligned} C_y^2 &= \frac{(1+2kC_{x_0}) + \left[(1+2kC_{x_0})^2 + 4\left(\frac{Nmg_0}{Sq} - C_{x_0}\right)k^2\right]^{1/2}}{2k^2} = a_0, \\ C_{y_1} &= \sqrt{a_0}, \qquad C_{y_2} = -\sqrt{a_0}. \end{aligned} \tag{5.6}$$

The optimal control is chosen according to the maximum principle from (3.10). For this purpose we calculate

$$\Pi_4 = \Pi_0(C_{y_1}), \qquad \Pi_5 = \Pi_0(C_{y_2}), \qquad \Pi_0^{\max} = \max\{\Pi_4, \Pi_5\}. \tag{5.7}$$

We replace the obtained value of the control from (5.7) in the equations of motion.

For constraints like (5.2) we have

$$|n_y| + n_x = N_1, \qquad \frac{\rho V^2 S}{2mg_0}(|C_y| + C_{x_0} + kC_y^2) = N_1. \tag{5.8}$$

Then we can yield $C_y(t)$ from the equation (5.8)

$$kC_y^2 + |C_y| - \frac{N_1 2mg_0}{\rho V^2 S} + C_{x_0} = 0,$$

$$\begin{aligned} C_{y_3} &= \frac{-1 + \sqrt{1 + 4k\left(\frac{2N_1 mg_0}{\rho V^2 S} - C_{x_0}\right)}}{2k}, \qquad C_y > 0, \\ C_{y_4} &= \frac{1 - \sqrt{1 + 4k\left(\frac{2N_1 mg_0}{\rho V^2 S} - C_{x_0}\right)}}{2k}, \quad C_y < 0. \end{aligned} \tag{5.9}$$

The necessary control function is obtained from the maximum principle (3.10). Here the following inequality is assumed valid.

$$2N_1mg_0 - C_{x_0}\rho V^2S > 0.$$

Otherwise extraneous roots is also obtained from the maximum principle.

The values N and N_1 is obtained from the solution of Pontryagin problem. The maximum value of N is evaluated from the condition of maximum of function n_Σ on the optimal trajectory with fixed initial data (2.6) for phase coordinates. Similarly we obtain the parameter N_1.

Geometry of the optimal trajectory in the problem $L(T) \to \min$ is bounded with the following theorem.

THEOREM 1. *The moment of departure from constraint* $n_\Sigma = N$ *in the problem* $L(T) \to \min$ *is determined by the equality*

$$C_{y_2} = -\sqrt{a_0} = C_y^{\min}, \tag{5.10}$$

where C_{y_2} is calculated from (5.6).

6 Necessary conditions of the extremum in nonregular case

Now consider a situation when the optimal trajectory contains the interval where $n_\Sigma = N$ and on this interval in some point $\dfrac{\partial n_\Sigma}{\partial C_y} = 0$.

The set of points determined by equations

$$\frac{\partial n_\Sigma}{\partial C_y} = 0, \qquad n_\Sigma = N, \tag{6.1}$$

we call **nonregular points**. For the problem under consideration $\dfrac{\partial n_\Sigma}{\partial C_y} = 0$ for $C_y = 0$.

For the solution of prescribed problem in our case we can use the results of A. Milyutin works.

According to his works in the presence of nonregular points the system of adjoint equations is given by

$$\begin{aligned}
\dot{P}_\Theta &= -\frac{\partial \Pi}{\partial \Theta}, \\
\dot{P}_H &= -\frac{\partial \Pi}{\partial H} + \lambda(t)\frac{\partial n_\Sigma}{\partial H} + \frac{d\mu}{dt}\frac{\partial n_\Sigma}{\partial H}, \\
\dot{P}_V &= -\frac{\partial \Pi}{\partial V} + \lambda(t)\frac{\partial n_\Sigma}{\partial V} + \frac{d\mu}{dt}\frac{\partial n_\Sigma}{\partial V}, \\
\dot{P}_L &= 0.
\end{aligned} \tag{6.2}$$

Here $\lambda(t)$ — the Lagrange multiplier, $\dfrac{d\mu}{dt}$ — the ideal function. For specified objects the conditions of complementary slackness are fulfilled

$$\lambda(t)(n_\Sigma - N) = 0, \qquad C_y \frac{d\mu}{dt} = 0. \tag{6.3}$$

The situation when nonregular point is the end of trajectory is not excluded.

From (6.2) it follows that in nonregular point (6.1) the adjoint variables P_H and P_V will jump by values $\mu\frac{\partial n_\Sigma}{\partial H}$ and $\mu\frac{\partial n_\Sigma}{\partial V}$ respectively, and $\mu > 0$. That is the significant difference between the non regular case and regular case, where adjoint variables are the continuous functions for mixed constraints of class $\varphi(x,u) \le 0$.

Besides the conditions (6.1)–(6.3) on the optimal trajectory the conditions of integrability of Lagrange multipliers and the conditions of normalization (conditions of nontriviality of maximum principle) have to be fulfilled

$$\int_0^T \lambda(t)\,dt0, \qquad \lambda(t) > 0. \tag{6.4}$$

Substantially the condition of integrability result from the condition of normalization.

7 Structure of the set of nonregular points

THEOREM 2. *The optimal trajectory in the case of constraint $n_\Sigma = N$ contain a finite number of non regular points.*

THEOREM PROVING. Let $n_\Sigma = q\frac{S}{G}\sqrt{C_x^2 + C_y^2} = N$. Suppose that $w = \sqrt{C_x^2 + C_y^2}$. Then

$$w = \frac{NG}{qS}, \qquad w(t_*) = w(C_y = 0) = w_*. \tag{7.1}$$

Let us show that $\frac{d^2 w_*}{dt^2} > 0$. Assume the contrary, i.e. $\lim\limits_{t\to t_*} \frac{w - w_*}{(t-t_*)^2} = 0$. Then $\lim\limits_{t\to t_*} \frac{w - w_*}{(t-t_*)^3}$ exists and is also equal zero. Prove it.

$C_y^2 = 0$ for $t = t_*$ and is continuous in this point. In this case

$$\lim_{t\to t_*} \frac{w - w_*}{t - t_*} = \dot{w}|_{t=t_*} = 0,$$

$$\lim_{t\to t_*} \frac{w - w_*}{(t-t_*)^2} \cong \lim_{t\to t_*} \frac{\dot{w}}{t - t_*} = -\lim_{t\to t_*} \frac{Sw^2}{NG}\,\frac{\dot{q}}{t-t_*} =$$

$$= -\frac{Sw_*^2}{NG}\lim \frac{\dot{q}}{t-t_*}; \qquad \dot{q} = \frac{\dot{\rho}V^2}{2} + \dot{V}V\rho, \qquad \dot{\rho} = -\rho V \sin\Theta\rho,$$

$$\dot{V} = -\frac{C_x \rho V^2 S}{2m} - g\sin\Theta.$$

But $\dot{\rho}$, $\dot{V}$, ρ, V are continuously differentiable in point t_*. Therefore, $\lim\limits_{t\to t_*} \frac{\dot{q}}{t-t_*}$ exists. As a result we have

$$\ddot{q} = \frac{\ddot{\rho} V^2}{2} + 2\dot{V}\dot{\rho}V + \ddot{V}V\rho\Big|_{t=t_*} = 0.$$

Let us show that $\lim\limits_{t\to t_*} \dfrac{w - w_*}{(t-t_*)^3}$ exists.

$$\lim \frac{w - w_*}{(t-t_*)^3} \cong \lim \frac{\dot{w}}{(t-t_*)^2} - \frac{Sw_*^2}{NG} \lim \frac{\dot{q}}{(t-t_*)^2} \cong \lim \frac{\ddot{q}}{t-t_*}.$$

Since $\ddot{q}$ — continuously differentiable function in point t_*, then $\lim\limits_{t\to t_*} \dfrac{\ddot{q}}{t-t_*}$ exists and equal zero, because w_* — minimum point w, i.e. $\dddot{q}\ (t_*) = 0$.

As a result we obtain a system

$$C_{x_0} = \frac{NG}{Sq}, \qquad \dot{q}(t_*) = 0, \qquad \ddot{q}(t_*) = 0, \qquad \dddot{q}\ (t_*) = 0. \tag{7.2}$$

The system (7.2) has unique solution relative to H, V, Θ, N with fixed constraints C_{x_0}, S, G. With all other values of H, V, Θ, N we come to contradiction.

Thus $\lim \dfrac{w - w_*}{(t-t_*)^2} > 0$, i.e. the point t_* is isolated. Consequently on the optimal trajectory as on any other trajectory with the constraints $n_\Sigma = N$ a number of points w_* is finite.

8 Continuity $|\dot{C}_y|$ at the point t_*

THEOREM 3. *The function $|\dot{C}_y|$ is is continuous for t_*.*

THEOREM PROVING. If the optimal trajectory satisfies the constraint $n_\Sigma = N$ on the interval of null measure, then $\dot{n}_\Sigma = \ddot{n}_\Sigma = \dddot{n}_\Sigma = 0$.

$$\begin{aligned} \dot{n}_\Sigma &= \dot{q}(C_x^2 + C_y^2) + C_y\dot{C}_y[1 + 2kC_{x_0}]q = 0, \\ \ddot{n}_\Sigma &= \ddot{q}(C_x^2 + C_y^2) + C_y\ddot{C}_y[1 + 2kC_{x_0}]q + \dot{C}_y^2[1 + 2kC_{x_0}]q + \\ &+ \ 3\dot{q}C_y\dot{C}_y[1 + 2kC_{x_0}] + 4k^2C_y^2\dot{C}_y^2 q = 0. \end{aligned}$$

Assume the contrary. Let $C_y = A(t-t_*)^\alpha$, $0 < \alpha < 1$. Then $\dot{C}_y A\alpha(t-t_*)^{\alpha-1}$, $C_y\dot{C}_y = A^2\alpha(t-t_*)^{2\alpha-1}$. Since $\dot{q}(t_*) = 0$, then from $\dot{n}_\Sigma = 0$ follows $C_y\dot{C}_y = 0$. Then $2\alpha - 1 > 0$, $\alpha > \frac{1}{2}$. Now we calculate $C_y\ddot{C}_y = A(t-t_*)^\alpha A\alpha(\alpha-1)(t-t_*)^{\alpha-2}$. On the other hand $\dot{C}_y^2 = A^2\alpha^2(t-t_*)^{2(\alpha-1)}$. Since $\ddot{q}(t_*)$ is bounded, then from $\ddot{n}_\Sigma = 0$ it follows $[C_y\ddot{C}_y + \dot{C}_y^2](1 + 2kC_{x_0})q = 0$. Hence we get $-\alpha^2 = \alpha(\alpha - 1)$, $2\alpha^2 - \alpha = 0$, $\alpha_1 = 0$, $\alpha_2 = \dfrac{1}{2}$. We have got a contradiction. Thus $\alpha \geq 1$. In this case

$$\dot{C}_y^2 = -\frac{\ddot{q}(C_x^2 + C_y^2)}{q[1 + 2kC_{x_0}]}. \tag{8.1}$$

According to (8.1) from the continuity of $\ddot{q}$ at the point t_* it follows the continuity of $|\dot{C}_y|$ at the point t_*.

9 Nonregular optimal trajectory

From continuity of $|\dot{C}_y|$ in nonregular point t_* and from the condition of integrability of Lagrange multiplier $\lambda(t)$ (6.4) follows

$$P_\Theta(t_*) = 0. \tag{9.1}$$

As a result the whole set of conditions in nonregular point is given by

$$\dot{q}(t_*) = 0, \qquad P_\Theta(t_*) = 0, \qquad C_y = 0, \qquad n_\Sigma = N. \tag{9.2}$$

We can fulfil the mentioned conditions by means of choice of arbitrary constants for adjoint variables $P_V(0)$, $P_\Theta(0)$ (3.8). In this case, the left side of the optimal trajectory including the nonregular point in the right end is uniquely determined.

Now consider the problem of continuation of the optimal trajectory through nonregular point. For easy analysis suppose that we have single nonregular point on the whole of optimal trajectory.

For determining the right side of trajectory in the problem with free right end, it is necessary to fulfil two boundary conditions (4.1). However we have only one free parameter $\mu > 0$ which determines a value of jump for the adjoint variables in nonregular point.

$$\begin{aligned} P_V(t_* + 0) - P_V(t_* - 0) &= \mu \frac{\partial n_\Sigma}{\partial V}, \\ P_H(t_* + 0) - P_H(t_* - 0) &= \mu \frac{\partial n_\Sigma}{\partial H}, \qquad \mu > 0. \end{aligned} \tag{9.3}$$

Impossibility to fulfil the boundary conditions (4.1) on the right end of the optimal trajectory argues a breach of necessary conditions of the extremum on the whole optimal trajectory.

Let us turn our attention to the condition of normalization (6.4). It is completely obvious that the Lagrange multiplier $\lambda(t)$ will be of primary importance under this condition. Now consider a possibility of existence of the nonregular optimal trajectory provided

$$P_L(t) \equiv 0. \tag{9.4}$$

In this case the left side of the nonregular optimal trajectory is also uniquely determined by the means of choice of $P_\Theta(0)$ and $P_V(0)$. At the same time we are able to fulfil the boundary conditions (4.1) on the right end if we suppose

$$P_V(t_* + 0) = 0. \tag{9.5}$$

From (9.4), (9.5) (9.2) follows

$$P_V(t) \equiv 0, \qquad P_H(t) \equiv 0, \qquad P_\Theta(t) \equiv 0, \qquad t \in (t_*, T].$$

Thus on the right end of nonregular trajectory the maximum principle is trivially fulfilled. Since the left end of the trajectory is uniquely determined then on the interval $(t_*, T]$ we can consider any optimal control problem. Described problems will contain nonregular point t_* on the left end in which conditions (9.2) are fulfilled.

It begs the question of construction of the optimal trajectory with degenerate maximum principle.

10 Regularization of degenerate maximum principle

One of the possible way of constructing the optimal trajectory is change of the structure of constraint (5.1). We used the constraint similar to (5.2) before for the stable iterative search of an optimal trajectory for small values of $C_y(t)$. And the Lagrange multiplier was calculated from (5.5). Change of the structure of mixed constraint (5.1) does not impose additional conditions on the function $P_\Theta(t)$ in nonregular point ($P_\Theta(t_*) = 0$). However for continuation of trajectory through nonregular point t_* it is necessary to fulfil condition $\dot{q}(t_*) = 0$. As a result on the nonregular optimal trajectory we obtain three conditions

$$\dot{q}(t_*) = 0, \qquad P_V(T) = 0, \qquad P_\Theta(T) = 0, \tag{10.1}$$

which can be fulfilled by the means of choice of jumps of adjoint variables like (9.3) and arbitrary constants $P_V(0)$ $P_\Theta(0)$.

By this approach we get the nondegenerate maximum principle on the whole optimal trajectory.

The presence of several nonregular point does not lead to degeneration of the maximum principle, however it complicates a search of the optimal trajectory.

Now consider another approach to construction of the nondegenerate maximum principle. For this purpose when we construct the Pontryagin function (3.2) assume that the term $P_\Theta \dfrac{C_y \rho V S}{2m}$ is a small parameter for sufficiently small $C_y(t)$. Then expression for the Lagrange multiplier $\lambda(t)$ is given by

$$\lambda(t) = -\frac{2kP_V}{1+2kC_x}\, g_0 \sqrt{C_x^2 + C_y^2}. \tag{10.2}$$

In this case the conditions of integrability of $\lambda(t)$ are fulfilled automatically.

As a result we obtain the nondegenerate maximum principle with nonregular points. In addition the expression (10.2) allows to carry out the stable iterative search of an optimal trajectory for small values of $C_y(t)$.

Let us describe another method of regularization of degenerate maximum principle. Let on the optimal trajectory the condition $n_\Sigma = N$ (2.2) is fulfilled. Then we have

$$\frac{1}{2}\ln\,(C_x^2 + C_y^2) + \ln\frac{\rho V^2 S}{2mg_0} = \ln N. \tag{10.3}$$

Now consider separately components of (10.3) which is connected with the control.

$$\begin{aligned}
\frac{1}{2}\ln\,(C_x^2 + C_y^2) &= \frac{1}{2}\ln\,[(C_x + C_y)^2 - 2C_yC_x] = \\
&= \frac{1}{2}\ln\,(C_x + C_y)^2\left[1 - \frac{2C_yC_x}{(C_y + C_x)^2}\right] = \\
&= \ln\,(C_x + C_y) + \frac{1}{2}\ln\left[1 - \frac{2C_yC_x}{(C_y + C_x)^2}\right].
\end{aligned} \tag{10.4}$$

A differentiation (10.4) on C_y gives

$$[\ln (C_x + C_y)]'_{C_y} = \frac{1 + 2kC_y}{C_x + C_y}, \quad \left\{ \frac{1}{2} \ln \left[1 - \frac{2C_yC_x}{(C_y + C_x)^2} \right] \right\}'_{C_y} -$$

$$- \frac{[(C_x + 2kC_y)(C_y + C_x)^2 - 2(C_y + C_x)(1 + 2kC_y)C_yC_x]}{(C_y + C_x)^4 \left[1 - \frac{2C_yC_x}{(C_y + C_x)^2} \right]}.$$

These expressions do not have singularities for $C_y = 0$. The last implies that in this case the Lagrange multiplier $\lambda(t)$ for the constraint (10.3) will be finite. Thus the nonregular point imposes no constraints on the adjoint variable $P_\Theta(t)$.

As a result we obtain the nondegenerate maximum principle.

From given reasoning we can easily get an algorithm of construction of a nonregular trajectory. At the first stage we determine left side of nonregular trajectory from the conditions (9.2). In this process we use the conditions of regularization of the Lagrange multiplier $\lambda(t)$. At the second stage we determine right side of the trajectory from boundary conditions on the right end similar to (2.7) or (4.1). Here we suggest an existence of a single nonregular point.

During a search of right side of the trajectory we introduce new free parameters $P_\Theta(t_*{+}0)$ and $P_V(t_*{+}0)$. Then we choose them to fulfil specified boundary conditions.

The presence of several nonregular point does not change our considerations since each side of the trajectory is determined independently.

Remark 1. *Note that in a nonregular point t_* we allow the discontinuity of the adjoint variable $P_\Theta(t)$.*

The trajectory determined by the sides serves as the first approximation for solution on the base of the nondegenerate maximum principle. Here $P_\Theta(t)$ will be a continuous function on the whole optimal trajectory.

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