Lateral extrema and convexity

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Abstract. The paper studies the t_0-convexity and lateral extrema using the restrictions of a given function to the parametrized curves in a suitable pencil Γ_a. The main result is: if $f : D \to \mathbf{R}$ admits the point $a \in D$ as a strict extremum point constrained by each straight line passing through the point a and if for any $\alpha \in \Gamma_a$, $\alpha(t_0) = a$, the restriction $f \circ \alpha$ is strictly convex in a neighborhood of t_0, then the point a is a strict minimum point.

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1 Introduction

The theory of extrema with nonholonomic constraints ([11, 10, 9, 8, 7, 19, 17, 18, 4, 15, 3, 2, 21, 20, 16, 14, 12, 13]) suggests to study the extrema constrained by curves in a family with vertex (pencil of curves) whose images decompose the space. The family can include all the curves contained in a holonomic space or all integral curves of a Pfaff system (nonholonomic case). In this way the holonomic or nonholonomic constraints in an optimiztion problem can be approached unitary. Particularly, this idea is useful for establishing the position of a holonomic or nonholonomic hypersurface around a point with respect to the tangent plane at that point.

Classically, the shape of a hypersurface around a point is given by the second fundamental form, in case that it does not vanish at that point. Even in this situation, the surface behavior around a point can be unexpected, as the next example shows. In fact, we can get information about the way a hypersurface deviates from the tangent plane at a point by pursuing the values of the linear approximation function along various curves on the hypersurface which pass through that point.

Example. We consider the surface $\Sigma : z = g(x) - y^2$, where

$$g(x) = \begin{cases} x^9 \sin \dfrac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

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is a C^4 function. The tangent plane to Σ at $P\,(0,0,0)$ is $xOy : z = 0$. The second fundamental form does not vanish at P, and $k_1\,(P) = 0$ and $k_2\,(P) = -2$ say that the point P is a parabolic point. Consequently the fundamental form cannot decide the shape of the surface around the origin. On the other hand, the values of the function $g(x) - y^2$ along the curve $\alpha(t) = (t, 0)$ show that our surface is not locally convex.

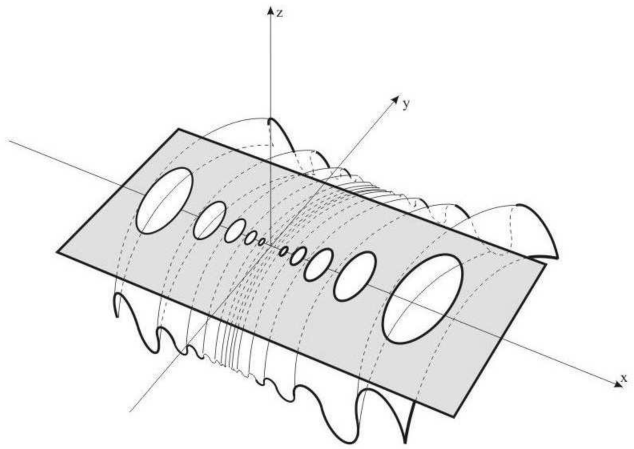

Fig. 1. Graph dilated along Oy

2 Extrema constrained by curves

Let D be an open set in R^p and $f : D \to R$ be a real function. Let Γ_a be a family of parametrized curves $\alpha : I \to D$ passing through the point $a = \alpha(t_0) \in D$.

2.1. Definition. We say that a is an extremum point of f constrained by a parametrized curve $\alpha \in \Gamma_a$ if t_0 is an extremum point of the composed function $f \circ \alpha$. We say that a is a minimum (maximum) point of f constrained by Γ_a if a is a minimum (maximum) point of the function f constrained by any $\alpha \in \Gamma_a$.

2.1. Theorem. ([4, 14]) *Let $f : D \to R$ and $a \in D$. Let Γ_a be either the set of all C^1 parametrized curves passing through a and regular at a or the family of all C^2 parametrized curves passing through a, such that either a is a regular point for α, or a is a singular point of the second order for α. The point a is a minimum (maximum) point of f if and only if a is a minimum (maximum) point of f constrained by Γ_a.*

The previous theorem can be further refined by

2.2. Theorem. [6]) *Let $f : D \to R$ be continuous. The point $a \in D$ is a strict extremum point of the function f if and only if a is a strict extremum point of f constrained by any $\alpha \in \Gamma_a$.*

We can renounce at the continuity of f by modifying the family Γ_a.

2.3. Theorem. [5]) *Let $f : D \to \mathbf{R}$ and $a \in D$. Let Γ_a be the family of all C^1 parametrized curves passing through the point a and regular at a. The point a is a* *strict extremum point of the function f if and only if it is a strict extremum point of f constrained by each α, $\alpha \in \Gamma_a$.*

It is interesting to notice the fact that the last two theorems do no more hold true if we replace "strict extremum point" with "extremum point". More precisely, the point a can be an extremum point of f constrained by any $\alpha \in \Gamma_a$ without being an extremum point of f, even if f is of class C^∞.To build an example, we fix the subsets

$$D_1 : y^2 - xy < 0,\ D_2 : x^2 - xy \le 0,\ D_3 : x^2 + xy < 0,\ D_4 : y^2 + xy \le 0$$

in R^2. Then we define the C^∞ function

$$f : R^2 \to R,\ f(x,y) = \begin{cases} e^{\dfrac{1}{y^2 - xy}} & \text{if} \quad (x,y) \in D_1 \\ 0 & \text{if} \quad (x,y) \in D_2 \cup D_4 \\ -e^{\dfrac{1}{x^2 + xy}} & \text{if} \quad (x,y) \in D_3. \end{cases}$$

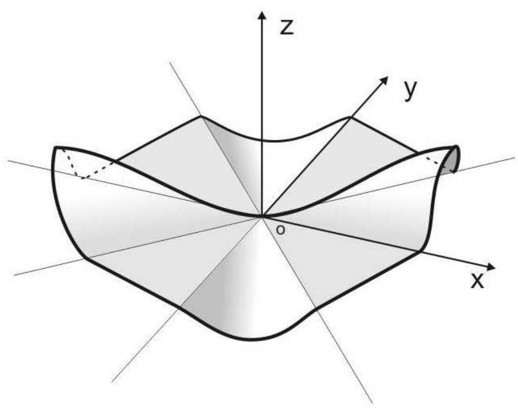

Fig. 2. Graph

The point $a = (0,0)$ is not an extremum point of the function f. On the other hand, since the subsets $D_i, i = \overline{1,4}$ are star-shaped at the point a, it follows that any parametrized curve α of Γ_a, passing through a, rests, in a neighborhood of the point a, in one of the sets $D_1 \cup D_2 \cup D_4$ or $D_2 \cup D_3 \cup D_4$. In this way, the point a is an extremum point (non-strict) for f constrained by any $\alpha \in \Gamma_a$.

By the previous results, a multi-variable problem of extremum can be reduced at a single-variable problem of extremum (on curves).

3 Lateral extrema and convexity

In the following we present some sufficient conditions of extremum, by using the notions of "lateral extremum" and of "t_0-convexity".

Let $\varphi : I \subset \mathbf{R} \to \mathbf{R}$.

3.1. Definition. Let $t_0 \in I$ We say that the function φ is *strictly t_0-convex* if there exists a neighborhood I_{t_0} of t_0 such that

$$\varphi(ut + (1-u)t_0) < u\varphi(t) + (1-u)\varphi(t_0)$$

for $\forall t \in I_{t_0}$ and $\forall u \in (0,1)$.

If φ is strictly convex in a neighborhood I_{t_0} of t_0 i.e,

$$\varphi(ut_2 + (1-u)t_1) < u\varphi(t_2) + (1-u)\varphi(t_1)$$

for $\forall t_1, t_2 \in I_{t_0}$ and $\forall u \in (0,1)$, then φ is strictly t_0-convex. The converse is not true. For this it is sufficient to consider the function

$$\varphi : \mathbf{R} \to \mathbf{R},\ \varphi(t) = \begin{cases} t^2 \ \ if \ \ t \in (-\infty, 0] \cup [1, +\infty), \\ \frac{1}{n}t^2 \ \ if \ \ t \in [\frac{1}{n+1}, \frac{1}{n}),\ n \in \mathbf{N}^*, \end{cases}$$

which is strictly t_0-convex ($t_0 = 0$), but is not strictly convex in any neighborhood of t_0, because there exist points of discontinuity in any neighborhood of t_0.

3.2. Definition. We say that $t_0 \in I$ is *a strict lateral extremum point* of the function φ if "$\varphi(t_0) < \varphi(t)$ or $\varphi(t_0) > \varphi(t)$, $\forall t \in (t_0, t_0 + \varepsilon)$" or "$\varphi(t_0) < \varphi(t)$ or $\varphi(t_0) > \varphi(t)$, $t \in (t_0 - \varepsilon, t_0)$".

3.1. Lemma. *Suppose that the map $\varphi : I \subset R \to R$ is strictly t_0-convex. Then t_0 is a strict lateral extremum point of φ. Moreover, if φ is strictly convex in a neighborhood of t_0, then t_0 cannot be a maximum point.*

Proof. Let $I_{t_0} \subset I$ the neighborhood from definition 3.1.

Step 1. First we will prove that if $t_1 \in I_{t_0}$ is such that $t_0 < t_1$ and $\varphi(t_0) < \varphi(t_1)$, then $\varphi(t_0) < \varphi(t)$, $\forall t \in I_{t_0} \cap [t_1, +\infty)$. Indeed, because $t_1 = ut + (1-u)t_0$, $u \in (0,1)$ and φ is locally strict convex in t_0, it follows that $\varphi(t_1) < u\varphi(t) + (1-u)\varphi(t_0)$. As $\varphi(t_0) < \varphi(t_1)$, we get $\varphi(t_0) < u\varphi(t) + (1-u)\varphi(t_0)$, that is $\varphi(t_0) < \varphi(t)$.

Step 2. There are two possible cases.

a) There exists a sequence (t_n) such that $t_n \to t_0$, $t_0 < t_n$ and $\varphi(t_0) < \varphi(t_n)$, $\forall n \in \mathbf{N}^*$. By applying the step 1 it follows that $\varphi(t_0) < \varphi(t)$, $\forall t \in I_{t_0}$, that is t_0 is a point of strict lateral minimum of φ.

b) In the neighborhood I_{t_0} there exists $t_1 > t_0$ such that $\varphi(t_0) \geq \varphi(t_1)$. Then for $\forall t \in (t_0, t_1)$ we get $t = ut_1 + (1-u)t_0$, $u \in (0,1)$. Therefore $\varphi(t) < u\varphi(t_1) + (1-u)\varphi(t_0) \leq \varphi(t_0)$, i.e. t_0 is a point of strict lateral maximum of φ.

In the case of φ strictly convex in a neighborhood I_{t_0}, let us suppose, per absurdum, that t_0 is a point of maximum of φ. Therefore, there exists t_1, $t_2 \in I_{t_0}$ such that $t_1 < t_0 < t_2$, $\varphi(t_1) \leq \varphi(t_0)$ and $\varphi(t_2) \leq \varphi(t_0)$. Because $t_0 = ut_2 + (1-u)t_1$, $u \in (0,1)$, we get $\varphi(t_0) < u\varphi(t_2) + (1-u)\varphi(t_1) \leq u\varphi(t_0) + (1-u)\varphi(t_0) = \varphi(t_0)$, which is a contradiction.

3.3. Definition. The point $a \in D$ is *a strict lateral extremum point* of the function f constrained by $\alpha \in \Gamma_a$ $\alpha(t_0) = a$ if t_0 is a strict lateral extremum point of $f \circ \alpha$.

3.1. Remark. Let us suppose that a is a strict lateral extremum point constrained by any $\alpha \in \Gamma_a$. Then, it follows that for $\forall \alpha \in \Gamma_a$, $a = \alpha(t_0)$ either a is a strict extremum point of f constrained by α or $f(\alpha(t)) < f(a)$ $(f(\alpha(t)) > f(a))$, $\forall t \in (t_0 - \varepsilon, t_0)$ and $f(\alpha(t)) > f(a)$ $(f(\alpha(t)) < f(a))$, $\forall t \in (t_0, t_0 + \varepsilon)$.

3.4. Definition. If $a \in D$, we say that $f : D \to \mathbf{R}$ is strictly a-convex constrained by the parametrized curve $\alpha \in \Gamma_a$ $(\alpha(t_0) = a)$ if the function $f \circ \alpha$ is strictly t_0-convex.

3.2. Lemma. ([5]) *Let (x_n), (y_n), (u_n), (v_n) be sequences of real numbers that satisfy $x_n > 0$, $u_n > 0$, $\forall n \in N$, $x_n \to 0$, $u_n \to 0$, $\frac{y_n}{x_n} \to 0$ and $\frac{v_n}{u_n} \to 0$. Then there exist the subsequences (x_{n_k}), (y_{n_k}), (u_{n_k}) and (v_{n_k}) and a C^1 function $f : D \to \mathbf{R}$ such that $f(x_{n_k}) = y_{n_k}$, $f(u_{n_k}) = v_{n_k}$, $\forall k \in N^*$, $f(0) = f'(0) = 0$.*

In the sequel Γ_a will denote the family of all C^1 parametrized curves passing through the point a and regular at a.

3.3. Lemma. *Let α, $\beta \in \Gamma_a$ such that $\alpha(0) = \beta(0)$ and $\alpha'(0) = \beta'(0)$. Let (u_n) and (v_n) two sequences of strictly positive real numbers, having the limit 0. There exists a parametrized curve $\gamma \in \Gamma_a$ and a sequence (t_n) of strictly positive real numbers, having the limit 0, such that the sequence $\gamma(t_n)$ contains both a subsequence of $(\alpha(u_n))$ and a subsequence of $(\beta(v_n))$.*

Proof. We can suppose that $\alpha(0) = \beta(0) = 0$ and $\alpha'(0) = \beta'(0) = e_1 = (1, 0, ..., 0)$.

Let $x_n = (x_n^1, ..., x_n^p) = \alpha(u_n)$ and $y_n = (y_n^1, ..., y_n^p) = \beta(v_n)$. It follows that $\frac{x_n}{\|x_n\|} \to e_1$ and $\frac{y_n}{\|y_n\|} \to e_1$. Therefore for n large enough we get $x_n^1 > 0$ and $y_n^1 > 0$. Moreover, $\frac{x_n^i}{\|x_n\|} \to 0$ and $\frac{y_n^i}{\|y_n\|} \to 0$ for $\forall i = \overline{2, p}$. If we successively apply the lemma 3.1 for the sequences (x_n^1), (x_n^i), (y_n^1), (y_n^i), $i = \overline{2, p}$, we get the subsequences $x_{n_k} = (x_{n_k}^1, ..., x_{n_k}^p)$, $y_{n_k} = (y_{n_k}^1, ..., y_{n_k}^p)$ and the maps $f_i : \mathbf{R} \to \mathbf{R}$ of C^1 class, such that $f_i(0) = f_i'(0) = 0$, $f_i(x_{n_k}^1) = x_{n_k}^i$, $f_i(y_{n_k}^1) = y_{n_k}^i$, $i = \overline{2, p}$. Then the parametrized curve $\gamma(t) = (t, f_2(t), ..., f_p(t))$, $t \in \mathbf{R}$, satisfies the required conditions.

3.1. Theorem. *Let $f : D \to R$ and $a \in D$. Suppose that for $\forall \alpha \in \Gamma_a$ the point a is a strict extremum point of f constrained by α if the image of α is contained into a straight line or a strict lateral extremum point of f , otherwise. Then a is a strict extremum point of f.*

Proof. We shall prove that a is a strict extremum point of f constrained by α, for $\forall \alpha \in \Gamma_a$. Applying theorem 2.3, we get the conclusion of the theorem. Suppose, per absurdum, that there exists $\alpha \in \Gamma_a$ $(\alpha(0) = 0)$ such that a is not a strict extremum point of f constrained by α. In accordance with remark 3.1 there exists two sequences of real numbers (t_n) and (u_n) which converge to 0 such that $t_n > 0$, $u_n < 0$, $f(\alpha(t_n)) > 0$ and $f(\alpha(u_n)) < 0$, $\forall n \in \mathbf{N}$. Let $\beta \in \Gamma_a$, $\beta(t) = \alpha'(0)\, t$. By hypothesis, a is a strict extremum point of f constrained by β, because the image of β is contained into a straight line. We get $f(\beta(t_n)) > 0$, $\forall n \in \mathbf{N}$, because, in the opposite case, in accordance with the previous lemma applied to the parametrized curves α and β, we would find a curve γ such that a is not a strict lateral extremum point of f constrained by γ. Applying the same reasoning for $\alpha_1(t) = \alpha(-t)$ and $\beta_1(t) = \beta(-t)$, we obtain $f(\beta_1(-u_n)) < 0$, $\forall n \in \mathbf{N}$, that is $f(\beta(u_n)) < 0$, $\forall n \in \mathbf{N}$, which contradicts the fact that a is a strict extremum point of f constrained by β.

3.2. Theorem. *Let $f : D \to R$ and $a \in D$ such that:*

i) a is a strict extremum point of f constrained by each straight line passing through the point a;

ii) For $\forall\alpha \in \Gamma_a$*, it follows that* f or $-f$ *is strictly a-convex constrained by* α*.*
Then a *is a strict extremum point of* f*.*

This theorem is a direct consequence of lemma 3.1 and of the previous theorem.

3.3. Theorem *Let* $f : D \to R$ *and* $a \in D$ *such that:*

i) a *is a strict extremum point of the function* f *constained by each straight line passing through the point* a*;*

ii) For $\forall\alpha \in \Gamma_a$ $(a = \alpha(t_0))$*, it follows that* $f \circ \alpha$ *is strictly convex in a neighborhood of* t_0*.*

Then the function f is *continuous at* a *and* a *is a strict minimum point of* f*.*

Proof. Because $f \circ \alpha$ is strictly convex in a neighborhood of t_0, we get that $f \circ \alpha$ is continuous in t_0. Since the parametrized curve α is a arbitrary one in the family Γ_a, it results that f is continuous in a ([6], [1]]). From the previous theorem one obtains that a is a strict extremum point of f and from lemma 3.1 it follows that a is a strict minimum point of f.

3.1. **Corollary.** *Let* $f : D \to R$ *and* $a \in D$ *such that* f *is strictly convex in a neighborhood of* a *and* a *is a strict exremum point of the restriction of* f *to any straight line passing through* a*. Then* a *is a strict minimum point of* f*.*

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