Symmetries of the curvature, Weyl conformal and Weyl projective tensors on 4-dimensional Lorentz manifolds

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Abstract. This paper explores the Lie algebras of symmetry vector fields of the curvature tensor (curvature collineations) and the Weyl conformal tensor (Weyl conformal collineations) for a 4-dimensional Lorentz manifold. It discusses their interrelations and their associations with other symmetries. A brief investigation is also given of the symmetry of projective structure for such manifolds. Finally, a few remarks are made regarding the global extension of such symmetries.

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1 Introduction

Let (M, g) be a smooth connected 4-dimensional manifold with smooth Lorentz metric g (that is, a *space-time*). Let ∇ be the associated Levi-Civita connection of g, $\Re$ the corresponding type (1,3) curvature tensor of ∇, Ricc, the Ricci tensor, C the type (1,3) Weyl conformal tensor and W the type (1,3) Weyl projective tensor. Conformally related metrics have the same tensor C whilst projectively related metrics have the same tensor W. Then, in component form and with $R^a{}_{bcd}$ denoting the components of $\Re$, $R_{ab} \equiv R^c{}_{acb}$ are the components of $Ricc$, $R \equiv R_{ab}g^{ab}$ the Ricci scalar and with indices raised and lowered using g, one has (see e.g. [9])

$$R_{abcd} = C_{abcd} + E_{abcd} + \frac{1}{12}R(g_{ac}g_{bd} - g_{ad}g_{bc}) \tag{1.1}$$

where

$$E_{abcd} = \frac{1}{2}(\widetilde{R}_{ac}g_{bd} - \widetilde{R}_{ad}g_{bc} + \widetilde{R}_{bd}g_{ac} - \widetilde{R}_{bc}g_{ad})$$

$$(\widetilde{R}_{ab} = R_{ab} - \frac{1}{4}Rg_{ab}) \tag{1.2}$$

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and

$$W^a{}_{bcd} = R^a{}_{bcd} - \frac{1}{3}(\delta^a{}_c R_{bd} - \delta^a{}_d R_{bc}) \tag{1.3}$$

Let X be a global smooth vector field on M with local flow diffeomorphisms φ_t. Then, with $\mathcal{L}$ denoting a Lie derivative, X is called *conformal* if $\mathcal{L}_X g = \psi g$ for some (smooth) function ψ on M (the *conformal* function). If ψ is constant on M, X is called *homothetic* and if $\psi \equiv 0$ on M, X is called *Killing*. If each local flow φ_t of X preserves the geodesics of ∇, X is called *projective* (and *affine* if each φ_t also preserves affine parameters). If X satisfies $\mathcal{L}_X \Re = 0$, it is called a *curvature collineation*, if X satisfies $\mathcal{L}_X C = 0$, it is called a *Weyl conformal collineation*, if X satisfies $\mathcal{L}_X W = 0$, it is called a *Weyl projective collineation* and if X satisfies $\mathcal{L}_X Ricc = 0$ it is called a *Ricci collineation*. The set of all Killing vector fields on M is denoted by $K(M)$, all homothetic vector fields by $H(M)$, all conformal vector fields by $C(M)$, all affine vector fields by $A(M)$, all projective vector vector fields by $P(M)$, all curvature collineations by $CC(M)$, all Weyl conformal collineations by $WC(M)$, all Weyl projective collineations by $W(M)$ and all Ricci collineations by $RC(M)$. Each of these sets is a Lie algebra under the usual Lie bracket for vector fields and $K(M)$, $H(M)$, $C(M)$ $A(M)$ and $P(M)$ are each finite-dimensional. The Lie algebras $CC(M)$, $WC(M)$, $W(M)$ and $RC(M)$ may be finite- or infinite-dimensional. It is then straightforward to check that

$$\begin{aligned} K(M) \subset H(M) \subset C(M) \subset WC(M) \\ H(M) \subset A(M) \subset P(M) \subset W(M) \\ A(M) \subset CC(M) \subset RC(M) \\ CC(M) \subset W(M) \end{aligned} \tag{1.4}$$

This paper discusses, mainly, the interrelations between the Lie algebras $CC(M)$ and $WC(M)$ but will also include $RC(M)$ and $W(M)$. More details may be found in [11] and a further general discussion of symmetries is available in [9].

2 Curvature and Weyl conformal structure

Let $p \in M$ and let Λ_p denote the vector space of type (2,0) skew-symmetric tensors (bivectors or 2-forms) at p. If $F \in \Lambda_p$ then either $F = 0$ or F has matrix rank equal to two or four. If rank $F = 2$, F is called *simple* and, in components, F may be written as $F^{ab} = r^a s^b - s^a r^b$ for r, s in the tangent space T_pM to M at p. The 2-dimensional subspace of T_pM spanned by r and s is uniquely determined by F and is called the *blade* of F. A simple bivector F is called *spacelike (respectively, timelike, null)* if its blade is a spacelike (respectively timelike, null) subspace of T_pM. The curvature tensor $\Re$ on M leads to a linear map $f : \Lambda_p \to \Lambda_p$ according to $F^{ab} \to R^{ab}{}_{cd} F^{cd}$, called the *curvature map*. The rank of f at p is called the *curvature rank* at p. This leads to a classification of the curvature at p into five classes; A, B, C, D and O [9]. Curvature class B occurs when the curvature rank is two and the range space of the map f can be spanned by a spacelike (simple) bivector F^1 and a timelike (simple) bivector $F^2 \in \Lambda_p$ such that F^1 and F^2 have orthogonal blades. Class C occurs when rank(f)= 2 or 3 and, in addition, the range of f may be spanned by two or three simple bivectors, the orthogonal complements of whose blades have a 1-dimensional intersection in T_pM. Class D occurs when rank(f)=1, class O when $\Re = 0$ and class A consists of all other possibilities. A space-time which is of the same curvature class at every point is said to be *of that curvature class.*

The Weyl conformal tensor C can be classified according to its Petrov type [16, 15]. As is well-known, six types arise; **I**, **II**, **III**, **D**, **N** and **O**. Also, a map f' may be assigned for this tensor in an analogous fashion to the curvature tensor. However, in this case and since the map f' necessarily has even rank, the situation is simpler. In fact, at $p \in M$, if the Petrov type is **I**, rank(f')=4 or 6, if type **II** or **D**, rank(f')=6, if type **III**, rank(f')=4, if type **N**, rank(f')=2 and if type **O**, $C(p) = 0$ (see e.g.[9]). If the Petrov type is the same at each $p \in M$, then M is said to be *of that (Petrov) type.*

It is remarked here that it is generic for a space-time to be of curvature class A and for its Weyl map f' to have rank at least four at each $p \in M$ [17]. The following quite general theorem and corollary can now be stated. The proof can be found in [9] with the corollary following from the theorem by a consideration of the local flows of the vector fields involved and the definition of the Lie derivative in terms of these flows.

Theorem 1 *Let g and g' be Lorentz metrics on M with associated Levi-Civita connections ∇ and ∇', type (1,3) curvature tensors $\Re$ and $\Re'$ and type (1,3) Weyl conformal tensors C and C', respectively.*

(i) *If $\Re = \Re'$ then $\Re$ and $\Re'$ have the same curvature class at each $p \in M$. If, in addition, (M, g) and (M, g') have (common) curvature class A over some open dense subset of M, then $\nabla = \nabla'$ and $g' = cg$ for some positive real number c.*

(ii) *If $C = C'$ then (M, g) and (M, g') have the same Petrov type at each $p \in M$. If, in addition, (M, g) and (M, g') have (common) Petrov type which is not type* **N** *or* **O** *over any non-empty open subset of M, then g and g' are conformally related.*

Corollary

(i) *Let (M, g) be a space-time which is of curvature class A at each point of some open dense subset of M. Then $CC(M) = H(M)$*

(ii) *Let (M, g) be a space-time which is not of Petrov type* **N** *or* **O** *over any non-empty open subset of M. Then $WC(M) = C(M)$.*

It follows that under the generic situations demanded in the corollary, $CC(M)$ and $WC(M)$ are finite-dimensional, since $H(M)$ and $C(M)$ are. Further, and under the same conditions as in the two parts of the corollary, if $C(M) = H(M)$, then $WC(M) = CC(M)$. Otherwise, and under the same generic situation, it can be shown that $CC(M) \subset WC(M)$ and $CC(M) \neq WC(M)$ [9]. But it is interesting to ask of the relations between $WC(M)$ and $CC(M)$ if either of the conditions in the corollary is relaxed. Here, it will be indicated, briefly, what can be achieved and several examples will be supplied.

The idea is to use the following results together with the previous theorem and corollary as the basis for constructing examples. First, suppose that M is simply connected and admits a non-trivial covariantly constant vector field Y. Then $\nabla Y = 0$ and the associated (dual) covector field $\widetilde{Y}$ obtained from Y and g, is a global gradient, $\widetilde{Y} = dT$ for some global real-valued function T on M. It also follows from the Ricci identity that Y annihilates the curvature tensor, $R^a{}_{bcd}Y^d = 0$. Thus, since $\mathcal{L}_Y \Re = 0$, it can be checked that $\mathcal{L}_{TY}\Re = 0$, $\mathcal{L}_{T^2Y}\Re = 0$, $\cdots$, $\mathcal{L}_{T^nY}\Re = 0$, for any positive integer n. It follows that the *independent*, global, smooth vector fields Y, TY, $\cdots$, T^nY are members of $CC(M)$ for any such n and hence that $CC(M)$ is infinite-dimensional. It can also be checked that, since Y annihilates $\Re$, the curvature class at each $p \in M$ is either C, D or O because the curvature map f has, at each $p \in M$, a range space of dimension at most three, consisting of simple bivectors whose orthogonal complements contain $Y(p)$. Thus the conditions of part (i) of the corollary are violated since the curvature class is not A. [It is remarked here, however, that it is possible to choose an example where $CC(M)$ is infinite-dimensional without any non-trivial covariantly constant vector fields being admitted by M [9]].

Next, consider a space-time (M, g) which is everywhere of Petrov type **N** and with M simply connected. The theory of the Petrov types reveals a unique, global, 1-dimensional, null, smooth distribution $\mathcal{D}$ on M (the *principal null direction* of C) whose local spanning (null) vector fields annihilate C at each $p \in M$. Since M is simply connected, $\mathcal{D}$ can be spanned by a *global* smooth null vector field Z on M and Z annihilates C at each $p \in M$ [13, 9]. Now choose this example (as one can; see [4]) so that Z is covariantly constant and an argument similar to that of the last paragraph shows that $WC(M)$ is infinite-dimensional (and, incidentally, $CC(M)$ is infinite-dimensional also). [It is remarked here that it is possible to find an example where $WC(M)$ is infinite- dimensional without M admitting any non-trivial covariantly constant vector fields [11].]

In the first example, two paragraphs above, it can be arranged that the conditions of part (ii) of the corollary are satisfied and so $WC(M)$ is finite-dimensional (and, in fact, contained in the infinite-dimensional algebra $CC(M)$). One may also construct an example to achieve $WC(M) \subset CC(M)$ (and $WC(M) \neq CC(M)$) and with each either finite-dimensional or infinite-dimensional. Similarly, in the second example, in the previous paragraph, a conformal scaling of the metric (which leaves C unchanged) can be used to ensure that the conditions of part (i) of the corollary are satisfied. Thus $W(M)$ is infinite-dimensional and $CC(M)$ is finite dimensional. (and $CC(M) \subset WC(M)$). In fact other examples can be used to show that one may achieve $CC(M) \subset WC(M)$ (with $CC(M) \neq WC(M)$) and, in addition, with $CC(M)$ and $WC(M)$ each finite-dimensional or each infinite-dimensional.

To illustrate these constructions, some examples will be given. First, consider the following metric g on the manifold $\mathbb{R}^4$,

$$ds^2 = e^{ux} du dv + dx^2 + dy^2 \tag{2.1}$$

This metric was originally introduced in a 3-dimensional context in [1] and applied to space-times in [5]. It has Petrov type **III** everywhere on M but admits the covariantly constant vector field $\partial/\partial y$. Hence, from theorem 1 and the corollary, $WC(M) \subset CC(M)$ with $WC(M)$ finite dimensional and $CC(M)$ infinite-dimensional. (The inclusion sign here follows from the fact that, for this metric, $C(M) = H(M)$ [5]).

Next consider the metric on some appropriate open subset of $\mathbb{R}^4$ given by

$$ds^2 = f(z,t)dt^2 + k(t,z)dx^2 + u(x)\exp(\frac{y^2}{2})(dx^2 + dy^2) \tag{2.2}$$

where f, k, and u are smooth functions on M with u positive and the product fk negative. This metric was given in [12] where it was shown to admit a finite-dimensional Lie algebra $CC(M)$ which *properly contains* $A(M)$ and hence $H(M)$. However, M may be restricted in such a way that the Petrov type is never **O** or **N** on M. It can be shown that $C(M) = H(M)$ (see e.g.[2]) and so $WC(M) = H(M) \subset CC(M)$ and with $WC(M) \neq CC(M)$.

Finally, it is possible that neither of $WC(M)$ and $CC(M)$ is contained in the other. To see this consider the following metric on some open, connected and simply connected subset M of $\mathbb{R}^4$ and given by

$$ds^2 = -dt^2 + dx^2 + e^{2x}h^2(y,z)(dy^2 + dz^2) \tag{2.3}$$

where h is a nowhere zero function on M. This metric is being considered in a different context in [3]. The manifold M here may be restricted so that (M,g) is Petrov type **D** and so, from the corollary, part (ii), $WC(M) = C(M)$ and so $WC(M)$ is finite-dimensional. However, the global vector field X given by $X = e^{x+t}(1,1,0,0)$ is conformal and, in fact, is not a member of $CC(M)$. Hence $WC(M)$ is not contained in $CC(M)$. But the global vector field $\partial/\partial t$ is covariantly constant and so, from previous remarks, $CC(M)$ is infinite-dimensional. Hence $CC(M)$ is not contained in $WC(M)$.

The following result is also of some interest.

Theorem 2 *Let (M,g) be a space-time satisfying the following "null fluid" conditions.*

(i) *(M,g) is of Petrov type* **N**.

(ii) *For each $p \in M$ there exists a coordinate neighbourhood U of p in which the Ricci tensor may be written as $R_{ab} = \mu l_a l_b$ for some real valued function μ and null vector field l on U and with l spanning the principal null direction $\mathcal{D}$ of C on U.*

(iii) *The curvature rank is not equal to one (or zero) at any $p \in M$.*

Then $CC(M) \subset WC(M)$ with equality if and only if $WC(M) \subset RC(M)$.

Proof It is first remarked that condition (ii), together with the Petrov type **N** condition in (i), imply that the curvature rank is *at most two* at each $p \in M$ and hence, with (iii), that the curvature rank is everywhere equal to two.

The proof runs, briefly, as follows. The Petrov type **N** condition (i) means that each $p \in M$ admits a coordinate neighbourhood U and a null vector field l on U spanning the null distribution $\mathcal{D}$ of C so that, on U, C satisfies $C^a{}_{bcd}l^d = 0$, whilst (ii) implies that the Ricci scalar $R = 0$. Then the condition (ii) can, from (1.2), be checked to be equivalent to the condition that $E^a{}_{bcd}l^d = 0$ on U. Thus, from (1), $R^a{}_{bcd}l^d = 0$ and then (iii) shows that l is the unique solution (up to a scaling function) of this last equation [9]. Now let $X \in CC(M)$. This last result, together with the uniqueness associated with it, shows that, on U [9]

$$\mathcal{L}_X g_{ab} = \phi g_{ab} + \lambda l_a l_b \qquad (\Rightarrow \mathcal{L}_X g^{ab} = -\phi g^{ab} - \lambda l^a l^b) \tag{2.4}$$

for smooth functions ϕ, λ, on M. The condition $\mathcal{L}_X \Re = 0$ then implies $\mathcal{L}_X Ricc = 0$. Thus $\mathcal{L}_X R^a{}_b = \mathcal{L}_X(g^{ac}R_{cb}) = -\phi R^a{}_b$ and so $\mathcal{L}_X E^a{}_{bcd} = 0$ and finally, from (1.2), $\mathcal{L}_X C = 0$ on each such neighbourhood U. Thus $X \in WC(M)$. The final part of the theorem is proved similarly if one bears in mind the result that $CC(M) \subset RC(M)$ in (1.4) and the fact that l is the unique solution (up to a scaling function) of the equation $C^a{}_{bcd} l^d = 0$. This last result leads to (2.4) and the proof easily follows. □

It is remarked that the rank condition (iii) in this theorem cannot be relaxed since an example exists which satisfies (i) and (ii) and for which the curvature rank is equal to one and where the conclusion of the theorem is false. To see this, briefly, consider the following metric on some simply connected open subset of $\mathbb{R}^4$ given in coordinates u, v, x and y by

$$ds^2 = 2dudv + f(u,x)dx^2 + dy^2 \tag{2.5}$$

where f is a positive function on M. Then $l \equiv \partial/\partial v$ and $Y \equiv \partial/\partial y$ are covariantly constant vector fields on M. For this metric one may always restrict M, if necessary, so that the Petrov type is **N**, with repeated principal null direction everywhere spanned by l and the curvature class nowhere O on M. Then the curvature class is **D** (curvature rank one) everywhere on M and the Ricci tensor is of the null fluid type, $R_{ab} = \alpha l_a l_b$, for some nowhere zero function α on M. Now let $X \in WC(M)$ so that (2.4) holds. Then, as is known (see for example [9]), $\mathcal{L}_X l^a = \alpha l^a$ for some function α. It can then be checked that $\mathcal{L}_X R_{ab} = \kappa R_{ab}$ and that

$$\mathcal{L}_X R^a{}_{bcd} = \mathcal{L}_X E^a{}_{bcd} = \kappa E^a{}_{bcd} \tag{2.6}$$

Now take the Lie derivative, with respect to X, of the equation $R^a{}_{bcd} Y_a = 0$ and, using (1.2), contract with Y^c to get $\kappa = 0$. Thus $X \in CC(M)$ (and then $X \in RC(M)$). So $WC(M) \subset CC(M)$. Further, the vector field $\gamma(y)l$, for some smooth non-constant function γ, is easily checked to be in $CC(M)$ but not in $WC(M)$. This gives an example of a situation referred to above (since, because l is covariantly constant, both $CC(M)$ and $WC(M)$ are infinite-dimensional) and shows that the rank condition in theorem 2 is important since conditions (i) and (ii), but *not* (iii), in that theorem are satisfied. In fact, the example in (2.5) can be generalised. One need only choose a space-time (M, g) of holonomy type R_3 (see [7, 9])and with M simply connected and chosen, as it can be, so that the curvature tensor is nowhere zero on M. Then (M, g) is of Petrov type **N** everywhere and the above results can be recovered.

3 Projective structure

Let M be a space-time manifold with g and g' Lorentz metrics for M whose respective Levi-Civita connections ∇ and ∇' have the property that, for each $p \in M$ and each $u \in T_pM$, there exists a path in M which passes through p and whose tangent vector at p is u and which is an unparametrised geodesic for ∇ and ∇'. Then ∇ and ∇'

(or g and g') are called *projectively related*. It is clear that the projective algebras $P(M)$ and $P'(M)$ of (M,g) and (M,g'), respectively, are identical. Also the Weyl projective tensors of these space-times, given in (1.3), are identical [19] and so their Weyl projective algebras $W(M)$ and $W'(M)$ are identical. Further, if $X \in P(M)$, a consideration of the local flows associated with X then shows that $X \in W(M)$. Thus $P(M) \subset W(M)$.

Some information on the algebra $P(M)$ can be found in [8] where, in particular, it was shown that if (M,g) is a *vacuum* space-time (that is, its Ricci tensor is identically zero on M) then $X \in P(M) \Rightarrow X \in A(M)$ and so no "proper" projective vector fields are admitted by vacuum space-times. An extension of this result in [10] says that if (M,g) and (M,g') are projectively related and if (M,g) is vacuum then $\nabla = \nabla'$ and so (M,g') is also a vacuum space-time and, with a very special case excepted, $g' = cg$ for some positive real number c. Thus the metric is determined up to units of measurement. This last result is interesting in connection with the principle of equivalence in general relativity theory.

A study of projective structure for space-times might then consider the following four conditions and investigate any relations and implications between them. So for two space-times, (M,g) and (M,g'), and using an obvious notation, consider the following statements;

(i) Their Weyl projective tensors W and W' are equal,
(ii) Their connections ∇ and ∇' are projectively related,
(iii) Their Weyl projective algebras $W(M)$ and $W'(M)$ are equal,
(iv) Their projective algebras $P(M)$ and $P'(M)$ are equal.

It turns out that no two of these statements are equivalent. Comments made earlier, together with obvious observations, reveal that (ii)$\Rightarrow$ (i), (ii)$\Rightarrow$ (iii), (ii)$\Rightarrow$(iv) and that (i)$\Rightarrow$ (iii). All other possible implications between distinct members in the collection of the above statements are false. A complete list of counterexamples can easily be found and will be given elsewhere. For example, consider, on the same manifold, two Schwarzschild metrics with mass constants m and $2m$ for some positive real number m and restrict the manifold in the usual Schwarzschild coordinates by $r > 4m$. Since each of these metrics is a *vacuum* metric admitting no projective vector fields apart from their (identical) 4-dimensional Lie algebra of Killing vector fields (see e.g. [8, 18]), it follows that $P(M) = P'(M) = K(M)$. Further, since they are vacuum metrics, it follows from (1.3) that $W = \Re$ and $W' = \Re'$ and since their curvature ranks can then be shown equal to six at each $p \in M$, it follows from the corollary, and in an obvious notation, that $W(M) = CC(M) = K(M)$ and that $W'(M) = CC'(M) = K(M)$. So $W(M) = W'(M) = P(M) = P'(M) = K(M)$. However, $\Re \neq \Re'$ and so $W \neq W'$. But these space-times are vacuum space-times with $\nabla \neq \nabla'$ and so, by a result given earlier in this section, they are not projectively related. Hence, the following implications are false; (iii)$\Rightarrow$(i), (iii)$\Rightarrow$ (ii), (iv)$\Rightarrow$ (i) and (iv)$\Rightarrow$ (ii). More information will be given elsewhere.

4 Local symmetries

The above discussion of symmetries considered only *global* (symmetry) vector fields. However, the physical interpretation of symmetry in general relativity arises from local observations in space-time. Thus a few remarks will be made here regarding local symmetries. Let $p \in M$, let V be an open neighbourhood of p and let X be a Killing vector field defined on V. Then X is called a *local Killing vector field* (with domain V) on M. One asks about the possibility of extending such a vector field X to the whole of M. In general, this is not possible (the "rotational" local Killing vector field on the flat cylinder being such an example). In relativistic cosmology, for example, one might ask the question; if *all* astronomers discover the usual Robertson-Walker symmetries of isotropy and homogeneity in some neighbourhood U of their observatory, are the six independent local Killing vector fields they discover simply restrictions to their neighbourhood of a 6-dimensional Lie algebra $K(M)$ of global Killing vector fields on the whole of space-time? If such an extension is possible for each such astronomer in the universe, then each of their local Killing vector fields is extendible to the whole of M. Clearly, a necessary condition for this result is that each of the local Killing algebras (the set of all Killing vector fields which are defined on some open neighbourhood V of M (and then restricted to V) and denoted by $K(V)$), is of the same dimension. But the above example of the cylinder shows that more is required. The following result shows that the only extra barrier to global extension is the possible non-simply connectedness of M. It is taken from [14, 6]. Let A denote the collection of all global and local Killing vector fields on M and, for $p \in M$, let A^*_p denote the vector space of germs of Killing vector fields defined on some open neighbourhood of p (that is, the vector space of equivalence classes of Killing vector fields under the equivalence relation of equality on some open neighbourhood of p). Nomizu showed that there then exists an open neighbourhood U of p (a *special* neighbourhood of p) such that A^*_p and $K(U)$ are naturally isomorphic. In this sense, no "smaller" open neighbourhood W of p will give a greater dimension for this local Lie algebra than U. Now suppose that dim A^*_p is the same for each $p \in M$ and that M is *simply connected*. Then (and in this statement, "Killing" may be replaced by "homothetic" or "affine" or "conformal" or "projective") if X is a local Killing vector field on U it may be globally extended to a Killing vector field on M.

If one attempts to investigate any extension of this result to the algebras $CC(M)$, $WC(M)$, $RC(M)$ or $W(M)$, problems arise. These problems can each be clearly seen by considering one easily adaptable case. Consider the metric g on some connected open subset M of $\mathbb{R}^4$ given by

$$ds^2 = -dt^2 + g_{\alpha\beta}dx^\alpha dy^\beta \tag{4.1}$$

where Greek letters take the values 1, 2, 3 and the $g_{\alpha\beta}$ are functions only of the x^α. [Then $T \equiv \partial/\partial t$ is a covariantly constant vector field on M.] Now for $p \in M$ and any open neighbourhood U of p, one may always choose an open neighbourhood V of p such that $V \subset U$, together with a smooth real-valued function $f(t)$ on V which can not be smoothly extended beyond V. Then one constructs the smooth vector field $X = f(t)T$, which, by the covariant constancy of (the restriction of) T is then a curvature collineation on V, but where X is not extendible smoothly beyond V. (For example choose V to be a coordinate "box" neighbourhood with $-\epsilon < t < \epsilon$, for $0 < \epsilon \in \mathbb{R}$, and choose $f(t) = \tan(\frac{\pi t}{2\epsilon})$). In this sense, there are no such "special neighbourhoods" as in the finite-dimensional cases because for each open neighbourhood of p there is always a "smaller" one containing a non-extendible curvature collineation.

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