Affine invariant iterated function systems and the minimal simplex problem

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Abstract. The AIFS (Affine invariant Iterated Function System) is a slightly modified IFS defined by Barnsley [*Fractal Everywhere*, Academic Press, 1988]. Instead of the usual Descartes coordinates, the barycentric system has been used. This allows more handy manipulation with the attractors generated by the IFS. The open question is still determination of the smallest simplex that contains the attractor. Some ideas concerning this problem are presented and elaborated in this note.

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1 Introduction

A large class of fractal sets has been defined by means of iterated function systems (IFS). IFS are defined as follows ([1]): Let (X, d) be a complete metric space. If $\mathcal{H}(X)$ denotes the set of all nonempty compact subsets of X and $h(d)$ denotes the Hausdorff metric induced by d, then $(\mathcal{H}(X), h(d))$ is a complete metric space. Let $\{w_i \,|\, i = 1, 2, \ldots, n\}$ be a set of contractive mappings with Lipschitz factors s_i, $|s_i| < 1$, respectively. A complete metric space (X, d) together with the above mentioned set is called *(hyperbolic) Iterated Function System.* Its notation is $\{X;\, w_i, i = 1, 2, \ldots, n\}$ and its contractivity factor is $s = \max\{s_i, i = 1, 2, \ldots, n\}$. The transformation $W : \mathcal{H}(X) \to \mathcal{H}(X)$ (W is called *Hutchinson operator*), defined by

$$W(B) = \bigcup_{i=1}^{n} w_i(B), \quad \forall B \in \mathcal{H}(X)$$

is a contractive mapping on the complete metric space $(\mathcal{H}(X), h(d))$ with contractivity factor s. Its unique fixed point, $A \in \mathcal{H}(X)$, obeys

$$A = W(A) = \bigcup_{i=1}^{n} w_i(A)$$

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and is given by $A = \lim_{n\to\infty} W^{\circ n}(B)$ for any $B \in \mathcal{H}(X)$, where $W^{\circ 0}(B) = B$, $W^{\circ n}(B) = W^{\circ(n-1)}(W(B))$. The fixed point $A \in \mathcal{H}(X)$ is called the *attractor* of the IFS. It usually has fractional Hausdorff – Besicovitch dimension and in that case the attractors are fractal sets ([8]).

If the IFS code is given, the associated attractor can be constructed by some of the known algorithms ([1], [2]). The simplicity of the IFS based schemes makes them very popular in computer aided modeling. Such scheme contains much less information then the image of the fractal attractor. This is considered as one of its main advantages and therefore the IFS code is the basis of several image compression algorithms ([3]).

Besides being so attractive, the IFS code has some essential disadvantages. The last assertion refers to the problem of prediction of the shape and position of the attractor, the minimal convex body that contains the attractor, as well as the interactive change of once constructed attractor.

In this paper we will try to trace a way for overcoming these shortcomings. First, we will introduce the affine invariant IFS (AIFS), a system based on a control simplex, which increases the flexibility of fractal objects ([4], [7]). A connection between IFS and AIFS, will be given, offering a practical use of the AIFS. Some directions for finding the minimal body (simplex) that contains the attractor will also be considered.

2 Switching between IFS and AIFS

Let $(X, d) = (\mathbf{R}^m, d)$ where d is usual Euclidian metric and $m \geq 2$.

We denote by $\mathbf{E} = \{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_m\}$ the orthonormal basis of $\mathbf{R}^m$ ($\mathbf{e}_i = [\delta_{ij}]_{j=1}^m, i = 1, 2, \ldots, m$). The set $\mathbf{E}$ defines m points $\{\mathbf{P}_1, \mathbf{P}_2, \ldots, \mathbf{P}_m\}$, that are vertices of the $(m-1)$- simplex $\mathbf{P}_\triangle = \{\sum_{i=1}^n \rho_i \mathbf{P}_i : \sum \rho_i = 1, \rho_i \geq 0\}$. The notation $\mathbf{V}$ is used for affine hull of the points $\{\mathbf{P}_1, \mathbf{P}_2, \ldots, \mathbf{P}_m\}$, i.e. $\mathbf{V} = \mathrm{aff}(\mathbf{P}_1, \mathbf{P}_2, \ldots, \mathbf{P}_m) \subset \mathbf{R}^m$.

It is to be noticed that position of any point $\mathbf{X} \in \mathbf{V}$ can be described by Descartes coordinates with respect to the base $\mathbf{E}$ and with barycentric coordinates with respect to the simplex $\mathbf{P}_\triangle$.

Let $S_i = [s_{ij}]_{i,j=1}^m$ be a real nonsingular matrix, whose rows sum up to unit (sometimes called *row-stochastic matrix*), and let a linear transformation $\mathcal{L} : \mathbf{V} \to \mathbf{V}$ be defined by

$$\mathcal{L}\mathbf{P} = S^T \mathbf{P}. \tag{2.1}$$

This linear mapping is naturally connected with an affine transformation (projection of $\mathcal{L}$),

$$\mathcal{A}(\mathbf{y}) = A\mathbf{y} + \mathbf{b}, \quad \mathbf{y} \in \mathbf{V}_0 = \mathrm{span}\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_{m-1}\} \tag{2.2}$$

where A is an $(m-1)\times(m-1)$ real, nonsingular matrix and $\mathbf{b}$ is an $(m-1)$-dimensional real vector.

Lemma 2.1. *The matrix S is uniquely determined by the pair $(A, \mathbf{b})$ and vice versa. Moreover, S, A and $\mathbf{b}$ are connected with the relation*

$$S^T Q = \begin{bmatrix} A & \vdots & \mathbf{b} \\ \cdots & \cdots & \cdots \\ \mathbf{c}^T & \vdots & s_{mm} \end{bmatrix}, \tag{2.3}$$

where $\mathbf{c}$ *is an* $(m-1)$*–dimensional vector and*

$$Q = \begin{bmatrix} I_{m-1} & \vdots & 0 \\ \cdots & \cdots & \cdots \\ -1 & \vdots & 1 \end{bmatrix}, \quad I_{m-1} = [\delta_{ij}]_{i,j=1}^{m-1}. \tag{2.4}$$

Proof. The proof can be seen in [6].

Lemma 2.2. *Let* $\mathcal{L}$ *and* $\mathcal{A}$ *be the mappings of the metric spaces* $(\mathbf{V}, d)$ *and* $(\mathbf{V}_0, d_0)$*, respectively, given by* (2.1) *and* (2.2)*. Then, the contractiveness of the one induces the contractiveness of the other.*

Proof. (This proof is different then the one in [6]) Suppose that $\mathcal{A}$ is contraction, i.e. there exists a vector norm $||\cdot||$ in $\mathbf{V}_0$, such that

$$||\mathcal{A}(\mathbf{x}_0) - \mathcal{A}(\mathbf{y}_0)|| = ||A(\mathbf{x}_0) - A(\mathbf{y}_0)|| \le ||A||||\mathbf{x}_0 - \mathbf{y}_0|| < ||\mathbf{x}_0 - \mathbf{y}_0||,$$

for any $\mathbf{x}_0, \mathbf{y}_0 \in \mathbf{V}_0$. Let d_0 be a metric in $\mathbf{V}_0$ induced by the above mentioned norm. Define the metric in $\mathbf{V}$ by $d(\mathbf{x}, \mathbf{y}) = d_0(\mathbf{x}_0, \mathbf{y}_0) = ||\mathbf{x}_0 - \mathbf{y}_0||,\ \mathbf{x},\ \mathbf{y} \in \mathbf{V}$. Now, we have

$$\begin{aligned} ||\mathcal{L}(\mathbf{x}) - \mathcal{L}(\mathbf{y})|| &= d(\mathcal{L}(\mathbf{x})), \mathcal{L}(\mathbf{y})) = d_0(\mathcal{A}(\mathbf{x}_0), \mathcal{A}(\mathbf{y}_0)) \\ &= ||\mathcal{A}(\mathbf{x}_0) - \mathcal{A}(\mathbf{y}_0)|| < ||\mathbf{x}_0 - \mathbf{y}_0|| = ||\mathbf{x} - \mathbf{y}|| \end{aligned}$$

which means that $\mathcal{L}$ is a contraction. Almost on the same way, it can be shown that the contractiveness of $\mathcal{L}$ induces a contractiveness of $\mathcal{A}$. □

The affine invariant iterated function system can be defined by means of an $(m-1)$–dimensional simplex and two or more real square m–dimensional row-stohastic matrices.

Definition 2.1. Let $\mathbf{T} = [\mathbf{T}_1\, \mathbf{T}_2\, \ldots\, \mathbf{T}_m]^T$ define a nondegenerate simplex $\mathbf{T}_\triangle$ in $\mathbf{R}^{m-1}$. The system

$$\Omega(\mathbf{T}) = \{\mathbf{T}; S_1, S_2, \ldots, S_n\} \tag{2.5}$$

where S_k are real, nonsingular row-stochastic matrices, is called *Affine Invariant Iterated Function System (AIFS)* ([4]-[7]).

Theorem 2.1. *Let a hyperbolic IFS* w*,*

$$w = \{\mathbf{R}^{m-1}; w_1, w_2, \ldots, w_n\} \tag{2.6}$$

be given and let $F = \mathrm{att}(w)$*. Then, for any nondegenerate simplex* $\mathbf{T}_\triangle \subset \mathbf{R}^{m-1}$*, the attractor* $\mathrm{att}(\Omega(\mathbf{T}))$ *is affinly equivalent to* F*.*

Proof. Let w_i from IFS (2.6) be given by $w_i(\mathbf{y}) = A_i\mathbf{y} + \mathbf{b}_i$, and let $F \in \mathbf{R}^{m-1}$ be its attractor. Then, according to Lemma 1, n row-stochastic matrices $S_1, S_2, \dots, S_n$ are uniquely determined, and thereby n linear mappings $\mathcal{L}_1, \mathcal{L}_2, \dots, \mathcal{L}_n : \mathbf{R}^m \to \mathbf{R}^m$ such that $\mathcal{L}_i(\mathbf{P}) = S_i^T\mathbf{P}$. If we identify $\mathbf{R}^m$ with $\mathbf{V}$ and $\mathbf{R}^{m-1}$ with $\mathbf{V}_0$, this means that the projection of $\mathcal{L}_i(\mathbf{P})$ on $\mathbf{V}_0$ is $w_i(\mathbf{P}_0)$, where $\mathbf{P}_0 = \mathrm{proj}_{\mathbf{V}_0}\mathbf{P}$, and since $\mathbf{P}$ and $\mathbf{P}_0$ are affinly equivalent, $w_i(\mathbf{P}_0)$ and $\mathcal{L}_i(\mathbf{P})$ are also affinly equivalent. According to Lemma 2, $\mathcal{L}_1, \mathcal{L}_2, \dots, \mathcal{L}_n$ are contractions of $(\mathbf{V}, d)$, which means that AIFS $\Omega(\mathbf{P}) = \{\mathbf{P}; S_1, S_2, \dots, S_n\}$ is hyperbolic, and has $G \in \mathcal{H}(\mathbf{V})$ as an attractor. The same is with $\Omega(\mathbf{P}_0) = \{\mathbf{P}_0; S_1, S_2, \dots, S_n\}$ and its attractor is F. The Hutchinson operator for $\Omega(\mathbf{P})$ is $W(M) = \cup_{i=1}^n \mathcal{L}_i(M), \forall M \in \mathcal{H}(\mathbf{V})$, while for $\Omega(\mathbf{P}_0)$ it is $W_0(N) = \cup_{i=1}^n w_i(N)$, $\forall N \in \mathcal{H}(\mathbf{V}_0)$. For any set $M \in \mathcal{H}(\mathbf{V})$, $w_i(\mathrm{proj}_{\mathbf{V}_0} M) = \mathrm{proj}_{\mathbf{V}_0}(\mathcal{L}_i(M))$, thereby, $F = \mathrm{proj}_{\mathbf{V}_0} G$. For arbitrary $\mathbf{T}_\triangle$ the point set $\mathbf{T}$ is affinly equivalent with $\mathbf{P}_0$, i.e. there exists an affine mapping $\mathcal{A} : \mathbf{V}_0 \to \mathbf{V}_0$, such that $\mathbf{P}_0 = \mathcal{A}(\mathbf{T})$. The images $S_i(P_0)$ and $S_i(T)$ are connected in analogous way. So, the attractor of $\Omega(T) = \{T; S_1, S_2, \dots, S_n\}$ is $\mathcal{A}(F)$. □

Corollary 2.1. ([4, 5]) The AIFS code (2.5) has the affine invariant property, that is: for any affine transformation $\mathcal{A} : \mathbf{R}^{m-1} \to \mathbf{R}^{m-1}$,

$$\mathrm{att}\{\Omega(\mathcal{A}(\mathbf{T}))\} = A\{\mathrm{att}(\Omega(\mathbf{T}))\}.$$

3 AIFS as a design oriented tool

In [5] an algorithm for constructing $\mathrm{att}(\Omega(\mathbf{T}))$ is given. Here we will give two examples, wherefrom a practical use of the AIFS as design oriented tool can be seen.

Example 3.1. *Given IFS* $\{\mathbf{R}^2; w_1, w_2, w_3, w_4\}$, $w_i(\mathbf{x}) = A_i\mathbf{x} + b_i,\ i = 1, 2, 3, 4,$ *where*

$$w_1: \quad A_1 = \begin{bmatrix} -0.4 & 0 \\ 0 & -0.4 \end{bmatrix}, \quad \mathbf{b}_1 = \begin{bmatrix} 0.24 \\ 0.37 \end{bmatrix},$$

$$w_2: \quad A_2 = \begin{bmatrix} 0.50 & 0.00 \\ 0.00 & 0.50 \end{bmatrix}, \quad \mathbf{b}_2 = \begin{bmatrix} -1.37 \\ 0.25 \end{bmatrix},$$

$$w_3: \quad A_3 = \begin{bmatrix} 0.21 & 0.00 \\ 0.00 & 0.21 \end{bmatrix}, \quad \mathbf{b}_3 = \begin{bmatrix} 1.00 \\ 1.47 \end{bmatrix},$$

$$w_4: \quad A_4 = \begin{bmatrix} 0.50 & 0.00 \\ 0.00 & 0.50 \end{bmatrix}, \quad \mathbf{b}_4 = \begin{bmatrix} 0.76 \\ -1.16 \end{bmatrix}.$$

From (2.3), we have

$$S_1 = \begin{bmatrix} -0.16 & 0.37 & 0.79 \\ 0.24 & -0.03 & 0.79 \\ 0.24 & 0.37 & 0.39 \end{bmatrix}, \quad S_2 = \begin{bmatrix} -0.87 & 0.25 & 1.62 \\ -1.37 & 0.75 & 1.62 \\ -1.37 & 0.25 & 2.12 \end{bmatrix},$$

$$S_3 = \begin{bmatrix} 1.21 & 1.47 & -1.68 \\ 1.00 & 1.68 & -1.68 \\ -1.00 & 1.47 & -1.47 \end{bmatrix}, \quad S_4 = \begin{bmatrix} 1.26 & -1.16 & 0.90 \\ 0.76 & -0.66 & 0.90 \\ 0.76 & -1.16 & 1.40 \end{bmatrix}.$$

The attractor of this IFS (AIFS) is very similar to Sierpinski triangle. Knowing the AIFS we can interactively change the attractor (Figure 1).

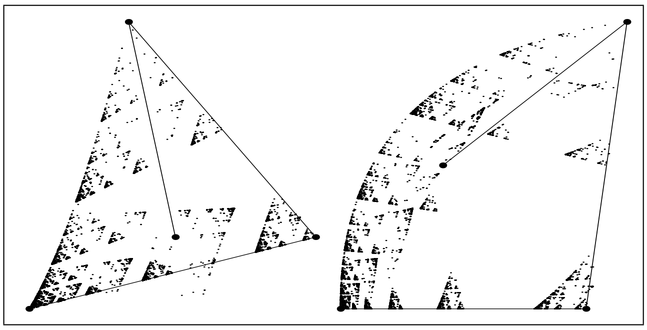

Figure 1.

Example 3.2. *Consider the AIFS* $\Omega(\mathbf{T}) = \{\mathbf{T}; S_1, S_2, S_3, S_4\}$, *where*

$$S_1 = \begin{bmatrix} -0.12 & 0.03 & 1.09 \\ -0.08 & -0.39 & 1.47 \\ -0.08 & 0.26 & 0.82 \end{bmatrix}, \quad S_2 = \begin{bmatrix} 0.68 & 3.50 & -3.18 \\ 0.07 & 3.81 & -2.88 \\ 0.07 & 3.50 & -2.57 \end{bmatrix},$$

$$S_3 = \begin{bmatrix} 1.39 & 0.09 & -0.48 \\ 1.03 & 0.87 & -0.90 \\ 0.74 & 0.39 & -0.13 \end{bmatrix}, \quad S_4 = \begin{bmatrix} 0.08 & 0.76 & 0.16 \\ -0.86 & 1.16 & 0.70 \\ -0.56 & 0.60 & 0.96 \end{bmatrix}.$$

The affine transformations $w_i,\ i = 1, 2, 3, 4$ can be found from (2.3).

$$w_1 : \quad A_1 = \begin{bmatrix} -0.04 & 0.00 \\ -0.23 & -0.65 \end{bmatrix}, \quad \mathbf{b}_1 = \begin{bmatrix} -0.08 \\ 0.26 \end{bmatrix},$$

$$w_2 : \quad A_2 = \begin{bmatrix} 0.61 & 0.00 \\ 0.00 & 0.31 \end{bmatrix}, \quad \mathbf{b}_2 = \begin{bmatrix} 0.07 \\ 3.50 \end{bmatrix},$$

$$w_3 : \quad A_3 = \begin{bmatrix} 0.65 & 0.29 \\ -0.30 & 0.48 \end{bmatrix}, \quad \mathbf{b}_3 = \begin{bmatrix} 0.74 \\ 0.39 \end{bmatrix},$$

$$w_4 : \quad A_4 = \begin{bmatrix} 0.64 & -0.30 \\ 0.16 & 0.56 \end{bmatrix}, \quad \mathbf{b}_4 = \begin{bmatrix} -0.56 \\ 0.60 \end{bmatrix}.$$

This IFS (AIFS) has a maple tree as an attractor (Figure 2).

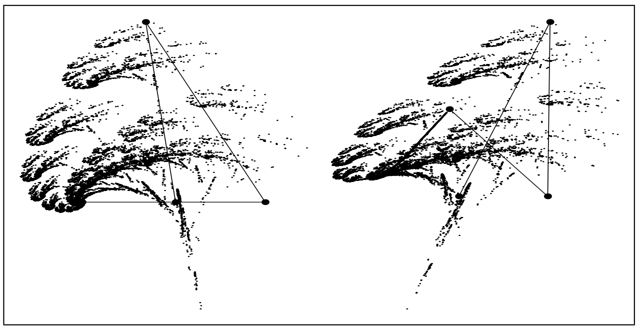

Figure 2.

4 Minimal simplex problem

The first step in predicting an attractor's shape is to find a polyhedral body that contains it. At the same time, this polyhedral body should be as simple as possible. At the first sight, it will be quite acceptable to have a simplex.

One of the differences between IFS code, and AIFS code is that the above task has a very good chance to be accomplished with AIFS code. Let us state the problem.

Minimal simplex problem. *Let the AIFS*

$$\Omega(\mathbf{T}) = \{\mathbf{T}; S_1, S_2, \ldots, S_n\},$$

be given.

Find a simplex $\mathbf{M}_\triangle$ *defined by* $\mathbf{M} = [\mathbf{M}_1\, \mathbf{M}_2\, \ldots\, \mathbf{M}_m]^T$ *from* $\mathbf{R}^{m-1}$, *that fulfils the following two conditions:*

1^o *It contains all points of the attractor, i.e.*

$$X \in att(\Omega(\mathbf{T})) \quad \Rightarrow \quad X \in \mathbf{M}_\triangle\,;$$

2^o *If* $\mathbf{N}_\triangle$, *defined by* $\mathbf{N} = [\mathbf{N}_1\, \mathbf{N}_2\, \ldots\, \mathbf{N}_m]^T$ *is another simplex that satisfies* 1^o, *then*

$$\mathrm{Vol}(\mathbf{M}_\triangle) \le \mathrm{Vol}(\mathbf{N}_\triangle)\,.$$

Having no success in gaining analytic solution, we step forward with the numerical one. In this sense, we made numerous numerical experiments, finding pre-attractors having about 10^5 points and constructing a simplex that contains almost every point of such pre-attractor. The simplex had a predetermined form, but we have transformed it affinely, so to approximately satisfied the above two conditions.

Here we present this solution in 2D case (for two dimensional attractor S) pointing out that it can be naturally generalized to m-dimensional space. This numerical solution we call *quasi-minimal 2D simplex* (*QMS*). The simplex in 2D is isosceles right-angled triangle having its two sides parallel to x and y axes.

Quasi-minimal 2D simplex algorithm

1. Using "random" algorithm [1], an attractor S consisting of N points is generated;
2. Find the minimal rectangle that contains S, $MinRect = [minX,\ maxX] \times [minY,\ maxY]$;
3. Using the affine transformation $u(\mathbf{x}) = A\mathbf{x} + \mathbf{b}$, where

$$A = \frac{1}{2}\begin{bmatrix} \dfrac{1}{maxX - minX} & 0 \\ 0 & \dfrac{1}{maxY - minY} \end{bmatrix}, \quad \mathbf{b} = -\frac{1}{2}\begin{bmatrix} \dfrac{minX}{maxX - minX} \\ \dfrac{minY}{maxY - minY} \end{bmatrix},$$

   transform $MinRect$ into the square $[0,\ 1/2] \times [0,\ 1/2]$;
4. Find d^* such that $d^* = \max d$ in line equation $L_d : x + y = d$, provided $\mathrm{dist}(u(S), L_d) = 0$;
5. The quasi-minimal 2D–simplex is $QMS = u^{-1}(T),\ T[(0,0),(d^*,0),(0,d^*)]$.

The advantage of having minimal simplex is illustrated in Figure 3. With minimal simplex containing the attractor we can at least change the attractors form affinely with predictable effect of this change. It is helpful and handy, especially in interactive designing process. We also can understand now why it is not so important to have exact minimal simplex. In fact, the "size" of the simplex is not of essential value so far it contains the attractor. This fact, that simplex contains the attractor, makes our operation of transforming this attractor a numerical process that is stable in comparison with changing some parameters directly in IFS that may be very unstable. There is a complete parallelism with free-form curve models like Bézier or spline that, having the convex hull property is numerically stable vs. unstable models like Lagrange interpolating model. The next two examples illustrate the usage of QMS.

Example 4.1. *The QMS for the attractor known as "seahorse", defined by the IFS*

$$w_1: \quad A_1 = \begin{bmatrix} 0.824074 & 0.281482 \\ -0.212346 & 0.864198 \end{bmatrix}, \quad \mathbf{b}_1 = \begin{bmatrix} -1.88229 \\ -0.110607 \end{bmatrix},$$

$$w_2: \quad A_2 = \begin{bmatrix} 0.088272 & 0.520988 \\ -0.463889 & -0.377778 \end{bmatrix}, \quad \mathbf{b}_2 = \begin{bmatrix} 0.78536 \\ 8.095795 \end{bmatrix}.$$

has $d^* = 0.188574$ (Figure 3).

Example 4.2. *The IFS for the fractal set known as "cascade" is given with:*

$$w_1: \quad A_1 = \begin{bmatrix} 0.35 & -0.13 \\ 0.13 & 0.35 \end{bmatrix}, \quad \mathbf{b}_1 = \begin{bmatrix} 0.37 \\ 0.68 \end{bmatrix},$$

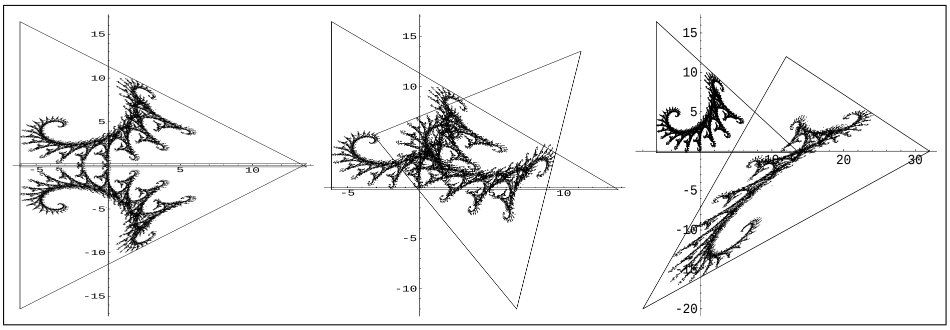

Figure 3. $d^* = 0.188574$ for the "seahorse" QMS

$$w_2: \quad A_2 = \begin{bmatrix} 0.32 & -0.19 \\ 0.19 & 0.32 \end{bmatrix}, \quad \mathbf{b}_2 = \begin{bmatrix} 2.23 \\ -0.67 \end{bmatrix},$$

$$w_3: \quad A_3 = \begin{bmatrix} -0.11 & 0.36 \\ -0.36 & -0.11 \end{bmatrix}, \quad \mathbf{b}_3 = \begin{bmatrix} -2.94 \\ 1.18 \end{bmatrix},$$

$$w_4: \quad A_4 = \begin{bmatrix} -0.09 & 0.37 \\ -0.37 & -0.09 \end{bmatrix}, \quad \mathbf{b}_4 = \begin{bmatrix} 1.84 \\ 2.33 \end{bmatrix},$$

$$w_5: \quad A_5 = \begin{bmatrix} -0.29 & -0.24 \\ 0.24 & -0.29 \end{bmatrix}, \quad \mathbf{b}_5 = \begin{bmatrix} 0.18 \\ -0.53 \end{bmatrix},$$

$$w_6: \quad A_6 = \begin{bmatrix} 0.36 & -0.09 \\ 0.09 & 0.36 \end{bmatrix}, \quad \mathbf{b}_6 = \begin{bmatrix} -1.40 \\ 2.66 \end{bmatrix},$$

$$w_7: \quad A_7 = \begin{bmatrix} 0.36 & -0.09 \\ 0.09 & 0.36 \end{bmatrix}, \quad \mathbf{b}_7 = \begin{bmatrix} 0.98 \\ 3.26 \end{bmatrix}.$$

Here, $d^* = 0.154936$. Some transformations of "cascade" are shown in Figure 4.

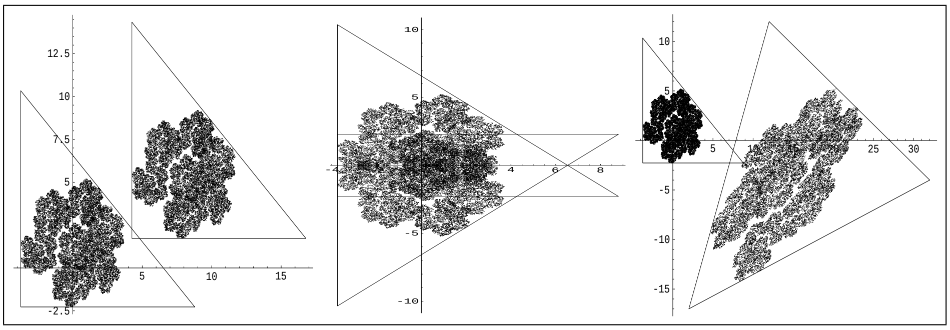

Figure 4. $d^* = 0.154936$ for the "cascade" QMS

Although the approximate solution, given by QMS algorithm, satisfies need to gain "convex hull" property, important for interactive modelling process, the analytical solution of the problem still remains an open problem. On the other hand, there is still room for further improvement of this numerical, approximate solution. Possible improvements might be in further minimizing of the QMS. Also, the higher dimensional cases will be treated in the next papers.

5 Conclusion

The AIFS model, introduced in [4] and [5] is a special kind of iterated function systems using only linear maps, but providing predictability and interactive change of the attractor. By this reason the AIFSs have big practical importance for Computer Aided Geometric Design. Besides the affine invariant property, we wanted to provide the AIFS with the convex hull property and came to the idea of the minimal simplex that contains the attractor. A "minimal simplex problem" is stated here and numerically solved. The analytical solution is still missing.

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