

About almost symplectic conjugations

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Abstract. The study of almost symplectic structures on differentiable manifolds with even dimension is the starting point of a lot of applications to Physics and Hamiltonian Mechanics. We present several results obtained within the theory of the conjugated almost symplectic structures on the total space of a vector bundle.

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1 Introduction

The papers written by the Romanian mathematicians (see the references) have suggested to the authors several ideas, which motivate and inspire the present and further study on the topic. To begin with, consider a vector bundle $\xi = (E, \pi, M)$, with m -dimensional fiber and a C^∞ -differentiable, n -dimensional, paracompact structure $M = (M, [A], \mathbb{R}^n)$, where $\dim E = n + m$ and $E_{n+m} = (E, [A], \mathbb{R}^{n+m})$. Let us consider an almost symplectic structure (E, ω) , which has its restriction to the vertical bundle nondegenerate.

In [4] R. Miron and M. Anastasiei prove that we can introduce a well-defined nonlinear connection, N , given by

$$(1.1) \quad \omega(hX, vY) = 0.$$

We further consider $N = \overset{\omega}{N}$, which corresponds to an horizontal distribution, and $H = \overset{\omega}{H}$ such that

$$T_u E = H_u E \oplus V_u E, \quad \forall u \in E.$$

Then we have

$$(1.2) \quad \omega = h\omega + v\omega,$$

where

$$(h\omega)(X, Y) = \omega(hX, hY); \quad (v\omega)(X, Y) = \omega(vX, vY).$$

In [7] the author elaborates the theory of spaces with linear g -conjugated connections. In the recent book [6], a systematic theory of spaces with almost symplectic, ω -conjugated structures is developed. We start by presenting the theory of the ω -conjugation in the case $\xi = (E, \pi, M)$.

2 Bundled ω -conjugations

Assuming that a nonlinear connection is given $N = \overset{\omega}{N}$ which provides the splitting $T_u E = H_u E \oplus V_u E$ ($E = TM$), an important role in Physics and in Hamiltonian Mechanics have the linear, special connections D which have the geometric property that the parallel transport preserves both the horizontal (H) and the vertical (V) distributions. In the general theory, these are called N -linear connections (or d -linear connections [1]).

Based on this geometrical property, we have defined the special operators $\overset{(12)}{D}_X^h \omega$, $\overset{(12)}{D}_X^v \omega$, $\overset{(21)}{D}_X^h \omega$, $\overset{(21)}{D}_X^v \omega$ ([6]) and studied their general properties, where we have respectively denoted by $\overset{(1)}{D}_X^h$, $\overset{(1)}{D}_X^v$, $\overset{(2)}{D}_X^h$, and $\overset{(2)}{D}_X^v$ the h -covariant derivative, and the v -covariant derivative.

We will further generalize the results obtained in [3].

Definition 2.1. Let $\xi = (E, \pi, M)$, (E, ω) be an almost symplectic structure and let $N = \overset{\omega}{N}$ and $\overset{(1)}{D}$, $\overset{(2)}{D}$ be two N -linear connections on E . Then we define $\overset{(12)}{D}$ by:

$$(2.1) \quad \overset{(12)}{D}_X^h h\omega(hY, hZ) = \overset{(1)}{D}_X^h h\omega(hY, hZ) - (h\omega)(hY, h\overset{(21)}{\tau}(hX, hZ))$$

$$(2.2) \quad \overset{(12)}{D}_X^h v\omega(vY, vZ) = \overset{(1)}{D}_X^h v\omega(vY, vZ) - (v\omega)(vY, v\overset{(21)}{\tau}(hX, vZ))$$

$$(2.3) \quad \overset{(12)}{D}_X^v h\omega(hY, hZ) = \overset{(1)}{D}_X^v h\omega(hY, hZ) - (h\omega)(hY, h\overset{(21)}{\tau}(vX, hZ))$$

$$(2.4) \quad \overset{(12)}{D}_X^v v\omega(vY, vZ) = \overset{(1)}{D}_X^v v\omega(vY, vZ) - (v\omega)(vY, v\overset{(21)}{\tau}(vX, vZ)),$$

$X, Y, Z \in \mathcal{X}(E)$, and we have the corresponding $\overset{(21)}{D}$ object, obtained by interchanging (1) with (2) in (2.1)-(2.4) where

$$\overset{(21)}{\tau}(X, Z) = \overset{(2)}{D}_X Z - \overset{(1)}{D}_X Z.$$

Proposition 2.1. In the general case we have:

$$(2.5) \quad \overset{(12)}{D}_X \omega(YZ) \neq \pm \overset{(12)}{D}_X \omega(ZY), \quad \overset{(12)}{D}_X \omega \neq \pm \overset{(21)}{D}_X \omega$$

P. Stavre has studied the special cases in which:

$${}^{(12)}(D_X\omega)(YZ) = \pm {}^{(12)}(D_X\omega)(ZY)$$

and

$${}^{(12)}D_X\omega = \pm {}^{(21)}D_X\omega,$$

the mean connection ${}^{(m)}D$ of the connection ${}^{(1)}D, {}^{(2)}D$.

An important role, in the general theory, have the ω -compatible connections D ($D\omega = 0$) [7]. Analogously, we consider the case in which ${}^{(12)}D$ is ω -compatible and we obtain a special geometrical interpretation of the condition ${}^{(12)}D\omega = 0$. We obtain:

Proposition 2.2. ${}^{(12)}D$ is ω -compatible if and only if:

$$(2.6) \quad {}^{(12)}D_X^h\omega = 0; \quad {}^{(12)}D_X^h v\omega = 0$$

$$(2.7) \quad {}^{(12)}D_X^v h\omega = 0; \quad {}^{(12)}D_X^v v\omega = 0.$$

Proposition 2.3. The geometric objects ${}^{(12)}D_X^h h\omega, {}^{(12)}D_X^h v\omega, {}^{(12)}D_X^v h\omega, {}^{(12)}D_X^v v\omega$ are d -tensor fields of type:

$$\begin{pmatrix} 0 & 0 \\ 3 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 \\ 1 & 2 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 \\ 2 & 1 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 \\ 0 & 3 \end{pmatrix}.$$

Proposition 2.4. If ${}^{(12)}D$ is ω -compatible then ${}^{(21)}D$ will be ω -compatible and conversely.

As well, we obtain:

Proposition 2.5. If ${}^{(12)}D$ is ω -compatible, then ${}^{(1)}D$ or ${}^{(2)}D$ is not ω -compatible.

Proposition 2.6. If ${}^{(12)}D$ is ω -compatible then ${}^{(2)}D$ is well-determined by ${}^{(1)}D$ and conversely.

Proof. If ${}^{(12)}D\omega = 0$, then (2.1)-(2.4) yield

$$(2.8) \quad (h\omega)(hY, {}^{(2)}D_X^h hZ) = (h\omega)(hY, {}^{(1)}D_X^h hZ) + ({}^{(1)}D_X^h h\omega)(hY, hZ)$$

$$(2.9) \quad (v\omega)(vY, {}^{(2)}D_X^h vZ) = (v\omega)(vY, {}^{(1)}D_X^h vZ) + ({}^{(1)}D_X^h v\omega)(vY, vZ)$$

$$(2.10) \quad (h\omega)(hY, {}^{(2)}D_X^v hZ) = (h\omega)(hY, {}^{(1)}D_X^v hZ) + ({}^{(1)}D_X^v h\omega)(hY, hZ)$$

$$(2.11) \quad (v\omega)(vY, D_X^{(2)}vZ) = (v\omega)(vY, D_X^{(1)}vZ) + (D_X^{(1)}v\omega)(vY, vZ)$$

Since $h\omega, v\omega$ are nondegenerate, it results that $D^{(2)h}, D^{(2)v}$ are well-determined by $D^{(1)h}, D^{(1)v}$, and we have:

$$h^{(21)}\tau(X, vZ) = 0, \quad v^{(21)}\tau(X, hZ) = 0, \quad \forall X, Z \in \mathcal{X}(E),$$

because $D^{(1)}, D^{(2)}$ are N -linear connections. In the same way we will obtain $D^{(1)h}, D^{(1)v}$ from $D^{(2)h}, D^{(2)v}$ if we interchange the indices (1) with (2) in (2.8)-(2.11).

Proposition 2.7. *The mean connection $D^{(\omega)}$ of the connection $D^{(1)}, D^{(2)}$ is ω -compatible if $D^{(12)}$ is ω -compatible.*

Proof. If $D^{(12)}\omega = 0$, then $D^{(21)}\omega = 0$ and conversely. From (2.1)-(2.8), taking into account that

$$(h\omega)(hY, D_X^{(2)}hZ) = (h\omega)(hY, D_X^{(1)}hZ)$$

we infer

$$(2.12) \quad (h\omega)(hY, D_X^{(m)h}hZ) = (h\omega)(hY, D_X^{(1)h}hZ) + \frac{1}{2}(D_X^{(1)h}h\omega)(hY, hZ)$$

In the same way we obtain

$$(2.13) \quad (v\omega)(vY, D_X^{(m)v}vZ) = (v\omega)(vY, D_X^{(1)v}vZ) + \frac{1}{2}(D_X^{(1)v}v\omega)(vY, vZ)$$

$$(2.14) \quad (h\omega)(hY, D_X^{(m)v}hZ) = (h\omega)(hY, D_X^{(1)v}hZ) + \frac{1}{2}(D_X^{(1)v}h\omega)(hY, hZ)$$

$$(2.15) \quad (v\omega)(vY, D_X^{(m)h}vZ) = (v\omega)(vY, D_X^{(1)h}vZ) + \frac{1}{2}(D_X^{(1)h}v\omega)(vY, vZ)$$

On the other hand, we have

$$(2.16) \quad D_X^{(m)}Z = \frac{1}{2}(D_X^{(1)}Z + D_X^{(2)}Z)$$

$$(2.17) \quad vD_X^{(m)}hZ = \frac{1}{2}(vD_X^{(1)}hZ + vD_X^{(2)}hZ) = 0$$

$$(2.18) \quad hD_X^{(m)}vZ = \frac{1}{2}(hD_X^{(1)}vZ + hD_X^{(2)}vZ) = 0, \quad \forall X, Z \in \mathcal{X}(E).$$

Therefore, $\overset{(m)}{D}$ is an N -linear connection and it is well-determined by $\overset{(m)}{D}_X^h, \overset{(m)}{D}_X^v$ given by (2.12), (2.13), (2.14) and (2.15).

As mentioned before, a linear ω -compatible connection plays an important role in Hamiltonian Mechanics. If $\overset{(12)}{D}\omega = 0$, then $\overset{(1)}{D}, \overset{(2)}{D}$ will not be ω -compatible. But we associate to them $\overset{(m)}{D}$, which is ω -compatible, since we have

$$(2.19) \quad (\overset{(m)}{D}_X^h h\omega)(Y, Z) = \frac{1}{2} [(\overset{(1)}{D}_X^h h\omega)(hY, hZ) + (\overset{(2)}{D}_X^h h\omega)(hY, hZ)] = 0 \quad \text{etc.}$$

Definition 2.2. Let us consider $\overset{(1)}{D}, \overset{(2)}{D}$, two one-dimensional, ω -conjugated distributions on E , in $u \in E$ i.e.,

$$(2.20) \quad \omega_u(Y, Z) = 0, \quad \forall Y \in \overset{(1)}{D}_u, Z \in \overset{(2)}{D}_u.$$

If subjected to parallel transport, $\overset{(1)}{D}$ and $\overset{(2)}{D}$ remain ω -conjugated, then we will say that $\overset{(1)}{D}$ and $\overset{(2)}{D}$ are ω -conjugated, and we will write $\overset{(1)}{D} \overset{\omega}{\sim} \overset{(2)}{D}$. From the definition, by denoting $f = \omega(Y, Z)$, based on

$$(2.21) \quad df = (\overset{(1)}{D}\omega)(Y, Z) - \omega(Y, \overset{(21)}{\tau}(XZ)) + \omega(\overset{(1)}{D}Y, Z) + \omega(Y, \overset{(2)}{D}Z)$$

it results that

$$(2.22) \quad df = \omega(\overset{(1)}{D}Y, Z) + \omega(Y, \overset{(2)}{D}Z)$$

if $\overset{(12)}{D}$ is ω compatible. Under the parallel transport of $\overset{(1)}{D}$ and $\overset{(2)}{D}$ with respect to $\overset{(1)}{D}$ and $\overset{(2)}{D}$, we have

$$\overset{(1)}{D}_X Y \in \overset{(1)}{D}, \quad \overset{(2)}{D}_X Y \in \overset{(2)}{D}$$

(or the reverse situation). From the condition of conservation of ω conjugation of $\overset{(1)}{D}$

and $\overset{(2)}{D}$ under parallel transport and from (2.22), it results:

Proposition 2.8. *If $\overset{(12)}{D}$ is ω compatible then $\overset{(1)}{D} \overset{\omega}{\sim} \overset{(2)}{D}$.*

More generally, from (2.21) we have

Proposition 2.9. *The necessary and sufficient condition that the ω conjugation of any two distributions $\overset{(1)}{D}, \overset{(2)}{D}$ be conserved, under the parallel transport with respect to $\overset{(1)}{D}, \overset{(2)}{D}$, is*

$$(2.23) \quad (D_X^{(12)h} h\omega)(hY, hZ) = (h\alpha)(X)(h\omega)(hY, hZ)$$

$$(2.24) \quad (D_X^{(12)h} v\omega)(vY, vZ) = (h\beta)(X)(v\omega)(vY, vZ)$$

$$(2.25) \quad (D_X^{(12)v} h\omega)(hY, hZ) = (v\gamma)(X)(h\omega)(hY, hZ)$$

$$(2.26) \quad (D_X^{(12)v} v\omega)(vY, vZ) = (v\sigma)(X)(v\omega)(vY, vZ)$$

$\forall X, Y, Z \in \mathcal{X}(E)$, where $h\alpha, h\beta$ are arbitrary d -covectors of type $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ and $v\gamma, v\sigma$ of type $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$.

From (2.23)-(2.26) we obtain

Proposition 2.10. *If $D \stackrel{(1)}{\sim} \omega \stackrel{(2)}{\sim} D$, then we have $D \stackrel{(2)}{\sim} \omega \stackrel{(1)}{\sim} D$*

$$(D_X^{(21)h} h\omega)(hY, hZ) = (h\alpha)(X)(h\omega)(hY, hZ)$$

$$(D_X^{(21)h} v\omega)(vY, vZ) = (h\beta)(X)(v\omega)(vY, vZ)$$

$$(D_X^{(21)v} h\omega)(hY, hZ) = (v\gamma)(X)(h\omega)(hY, hZ)$$

$$(D_X^{(21)v} v\omega)(vY, vZ) = (v\sigma)(X)(v\omega)(vY, vZ)$$

$\forall X, Y, Z \in \mathcal{X}(E)$, and therefore $D \stackrel{(12)}{\omega} = D \stackrel{(21)}{\omega}$.

Hence we have obtained a generalization of our previous results.

Proposition 2.11. *In the general case, if $D \stackrel{(1)}{\sim} \omega \stackrel{(2)}{\sim} D$ by fixing D , then D is not any more determined by D . There exists a set $\{D\}$ of ω -conjugated with D connections, given by:*

$$(2.27) \quad (h\omega)(hY, D_X^{(2)h} hZ) = (h\omega)(hY, D_X^{(1)h} hZ) + (D_X^{(1)h} h\omega)(hY, hZ) - (h\alpha)(X)(h\omega)(hY, hZ)$$

$$(2.28) \quad (v\omega)(vY, D_X^{(2)h} vZ) = (v\omega)(vY, D_X^{(1)h} vZ) + (D_X^{(1)h} v\omega)(vY, vZ) - (h\beta)(X)(v\omega)(vY, vZ)$$

$$(2.29) \quad (h\omega)(hY, \overset{(2)}{D}_X^v hZ) = (h\omega)(hY, \overset{(1)}{D}_X^v hZ) + (\overset{(1)}{D}_X^v h\omega)(hY, hZ) \\ - (v\gamma)(X)(h\omega)(hY, hZ)$$

$$(2.30) \quad (v\omega)(vY, \overset{(2)}{D}_X^v vZ) = (v\omega)(vY, \overset{(1)}{D}_X^v vZ) + (\overset{(1)}{D}_X^v v\omega)(vY, vZ) \\ - (v\sigma)(X)(v\omega)(vY, vZ)$$

$\forall X, Y, Z \in \mathcal{X}(E)$.

Taking into account the tensorial d -components which characterize the torsion of one N -linear connection D ,

$$(2.31) \quad hT(hX, hZ) = D_X^h hZ - D_Z^h hX - h[hX, hZ]$$

$$(2.32) \quad vT(hX, hZ) = \Omega(hX, hZ),$$

where $\Omega(hX, hY) = -v[hX, hY]$ is the curvature of the nonlinear connection N ,

$$(2.33) \quad (hT)(hX, vY) = -D_Y^v hX - h[hX, vY]$$

$$(2.34) \quad (vT)(hX, vY) = -D_X^h vY - v[hX, vY]$$

$$(2.35) \quad (hT)(vX, vY) = 0,$$

since the vertical distribution is integrable, we have

$$(2.36) \quad (vT)(vX, vY) = D_X^v vY - D_Y^v vX - v[vX, vY].$$

Then we obtain:

Proposition 2.12. *If $D \overset{(1)}{\sim} \overset{(2)}{D}$ and*

$$(2.37) \quad hT^{(1)}(hX, hZ) = hT^{(2)}(hX, hZ)$$

$$(2.38) \quad hT^{(1)}(hX, vZ) = hT^{(2)}(hX, vZ)$$

$$(2.39) \quad vT^{(1)}(hX, vZ) = vT^{(2)}(hX, vZ)$$

$$(2.40) \quad vT^{(1)}(vX, vZ) = vT^{(2)}(vX, vZ),$$

then $\overset{(2)}{D}$ is well-determined by $\overset{(1)}{D}$ and conversely.

Thus we have obtained a generalization of Proposition 2.6.

In nearby future, we plan to study the relations between the six essential d -tensor fields of curvature: $R(hX, hY)hZ$, $R(hX, hY)vZ$, $R(hX, vY)hZ$, $R(hX, vY)vZ$, $R(vX, vY)hZ$, $R(vX, vY)vZ$ (not taking into account the antisymmetry), for $D \stackrel{(1)}{\sim} \omega \stackrel{(2)}{\sim} D$. As well, an important role in the Hamiltonian Mechanics play the symplectic structures (in which ω is locally integrable, i.e. $d\omega = 0$). Taking into account that $\omega = h\omega + v\omega$, $D \stackrel{(1)}{\sim} \omega \stackrel{(2)}{\sim} D$, the authors are considering this matter for future research, which will provide a connection with the pseudo-Riemannian g -conjugations, applied to General Relativity.

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