

Non-relativistic bound states for modified Woods-Saxon potential

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Abstract. In this investigation we solve the radial part of schrödinger equation for the mixed Woods-Saxon and coulomb potentials in the non-relativistic case. This potential can be use to study the behavior of protons in nucleus by neglecting spin interactions. By modification of this differential equation to become comparable with the associated Jacobi differential equation, we obtain the bound states of nucleons in nucleus. The results obtained using this simple approach is exactly conformed with the supersymmetry approach in quantum mechanics.

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Key words: Non-relativistic Schrödinger equation, Woods-Saxon potential, Coulombian repulsive potential, bound states.

1 Introduction

The existence of an strong nuclear force in atomic nuclei revealed by the exceptional role of the nuclear magic number provides the foundation of the nuclear shell model. This strong force is believed to be approximated most closely by a Wood-saxon potential [14] either from analyzing the radial dependence of the nuclear central force or by deriving it from a microscopic two body force acted in neutron proton scattering [3]. Since the woods-saxon potential can not be solved analytically, one often adopts the harmonic oscillator potential or the square well, in particular the former, in nuclear shell model for both spherical [9] and deformed nuclei as a good approximation. As an initial approximation, the harmonic type potential used to construct the nuclear interaction hamiltonian. However it is necessary to improve the asymptotic behavior of harmonic oscillator wave function by performed a local scaling transformation [11]. To obtain strongly bound level of nucleus it is necessary to use a proper potential. The spherical Woods-Saxon potential that was used as a major part of nuclear shell model, was successful to deduce the nuclear energy levels [6]. Also it was used as central part for the interaction of neutron with heavy nucleus [8]. With the help of the axially-deformed Woods-Saxon potential along with the spin-orbit interaction potential , we

may construct the structure of single-particle shell model [7]. The Woods-Saxon potential was used as a part of optical model in elastic scattering of some ions with heavy target in low range of energies [2]. Generally, the Woods-Saxon potential and its various modified shapes was successful to describe the metallic clusters [4]. Recently the relativistic Dirac equation has been solved using two component spinors for Woods-Saxon potential in a special case [1, 13]. The nuclear strong force between nucleons is charge independent. Therefore it is independent of whether nucleon is neutron or proton. Also it is strongly spin dependent that is ignored in this study as an starting point. The average potential for interaction between nucleons is called Woods-Saxon potential. But in addition to the strong force between nucleons, there is a repulsive force between positively charged protons in nucleus. So in absence of spin interaction the entire potential in nucleus are sum of the Woods-Saxon and the Coulombian repulsive potential. The time independent Schrödinger equation for this central potential, can be solved analytically. In this paper we used a new approach to solve the radial Schrödinger equation for this potential. In advance we compared the resultant differential equation with well known associated Jacobi differential equation to obtain the bound states of nucleons in nucleus. Of course, this approach is a approximated method, because we ignore the spins and isospins of nucleons in nucleus. The results obtained here is directly match with the supersymmetry approaches.

2 Solution of the modified Woods-Saxon potential

The Modified Woods-Saxon potential is included the Woods-saxon potential and Coulombian repulsive potential that it given by,

$$(2.1) \quad V(r) = \frac{K}{r} - \frac{V_0}{1 + e^{\frac{r-R_0}{a}}},$$

here K is a electrical constant, V_0 is the nuclear potential depth, R_0 is the width of the potential, and a is the surface thickness. The parameter a is usually adjusted to the experimental value of nuclear interaction barriers. By inserting the conversions $r - R_0 \equiv r$ and $\frac{1}{a} \equiv 2\omega$ the potential $V(r)$ is reformed as,

$$(2.2) \quad V(r) = \frac{K}{r} - \frac{V_0}{1 + e^{2\omega r}}.$$

The general case of radial part of Schrödinger equation for this potential can be written as,

$$(2.3) \quad -\frac{\hbar^2}{2m} \left(\frac{d^2}{dr^2} + \frac{2}{r} \frac{d}{dr} - \frac{L^2}{\hbar^2 r^2} \right) \psi_{nl}(r) + \left(\frac{K}{r} - \frac{V_0}{1 + e^{2\omega r}} \right) \psi_{nl}(r) = E_{nl} \psi_{nl}(r).$$

In order to obtain the eigenvalues E_{nl} and corresponding eigenfunction $\psi_{nl}(r)$, we have to modify this differential equation to be comparable with the standard associated Jacobi differential equation. So extended the Wood-Saxon potential to obtain variable r as,

$$(2.4) \quad r = \frac{1}{2\omega} (e^{2\omega r} - 1),$$

this approximation valid for $r \rightarrow 0$ in the nuclear size region. By considering wave function as,

$$(2.5) \quad \psi_{nl}(r) = \frac{1}{r} \varphi_{nl}(r),$$

and substituting these relations in the Schrödinger equation we have,

$$(2.6) \quad -\frac{\hbar^2}{2m} \left(\frac{d^2}{dr^2} - \frac{4\omega^2 L^2}{\hbar^2 (e^{2\omega r} - 1)^2} \right) \varphi_{nl}(r) + \left(\frac{2K\omega}{e^{2\omega r} - 1} - \frac{V_0}{1 + e^{2\omega r}} \right) \varphi_{nl}(r) = E_{nl} \varphi_{nl}(r).$$

to make calculation simpler we define following new variables, $\varepsilon^2 = \left| \frac{2mE_{nl}}{\hbar^2\omega^2} \right|$, $\nu^2 = \frac{mV_0}{\hbar^2\omega^2}$, $\tau^2 = \frac{2mK}{\omega\hbar^2}$ and a new variable as $x = \tanh(\omega r)$ the equation (6) is reformed as follows:

$$(2.7) \quad (1-x^2) \frac{d^2}{dx^2} \varphi_{nl}(x) - 2x \frac{d}{dx} \varphi_{nl}(x) + \left(\frac{\varepsilon^2}{1-x^2} + \frac{\nu^2}{1+x} - \frac{\tau^2}{x(1+x)} - \frac{l(l+1)}{x^2} \frac{1-x}{1+x} \right) \varphi_{nl}(x) = 0.$$

Now we have to modify this equation with following associated Jacobi differential equation. For the real parameters $\alpha, \beta > -1$, this differential equation in the interval $x \in (-1, 1)$ is introduced as follows [10]:

$$(2.8) \quad (1-x^2) \frac{d^2}{dx^2} P_{nl}^{(\alpha, \beta)}(x) + (\beta - \alpha - (\alpha + \beta + 2)x) \frac{d}{dx} P_{nl}^{(\alpha, \beta)}(x) + \left(n(\alpha + \beta + n + 1) - \frac{l(\alpha + \beta + l + (\alpha - \beta)x)}{1-x^2} \right) P_{nl}^{(\alpha, \beta)}(x) = 0,$$

where the indices n and l are non-negative integers with $0 \leq l \leq n < \infty$. The associated Jacobi function $P_{nl}^{(\alpha, \beta)}(x)$ as the solutions of the differential equation (25) have the following Rodrigues representation [12]:

$$(2.9) \quad P_{nl}^{(\alpha, \beta)}(x) = \frac{a_{nl}(\alpha, \beta)}{(1-x)^{\alpha+\frac{l}{2}}(1+x)^{\beta+\frac{l}{2}}} \left(\frac{d}{dx} \right)^{n-l} ((1-x)^{\alpha+n}(1+x)^{\beta+n}),$$

here $a_{nl}(\alpha, \beta)$ is the normalization coefficient. So it is required that to define $\varphi_{nl}(x)$ as,

$$(2.10) \quad \varphi_{nl}(x) = U(x)W(x),$$

by substituting this relation in equation (7) one obtain the following differential equation,

$$\begin{aligned}
(2.11) \quad & (1-x^2)U''(x) + \left(2(1-x^2)\frac{W'(x)}{W(x)} - 2x\right)U'(x) + \\
& \left((1-x^2)\frac{W''(x)}{W(x)}\right)U(x) \\
& + \left(\frac{\varepsilon^2}{1-x^2} + \frac{\nu^2}{1+x^2} - \frac{\tau^2}{x(1-x)} - \frac{l(l+1)}{x^2}\frac{1-x}{1+x}\right)U(x) = 0.
\end{aligned}$$

In the first step, by comparing equations (8) and (11) one may obtain the following differential equation for $W(x)$,

$$(2.12) \quad 2(1-x^2)\frac{W'(x)}{W(x)} = (\beta - \alpha) - (\beta + \alpha)x,$$

so after integrating we have,

$$(2.13) \quad W(x) = (1-x)^{\frac{\alpha}{2}}(1+x)^{\frac{\beta}{2}}.$$

Meanwhile the further comparison of these equations lead us to get the following conditions:

$$\begin{aligned}
(2.14) \quad & \left(\frac{\beta^2}{4} - \frac{\beta}{2}\right) + \left(\frac{\alpha^2}{4} - \frac{\alpha}{2}\right) - \frac{\alpha\beta}{2} + \varepsilon^2 + \nu^2 + \tau^2 - l(l+1) \\
& = n(\alpha + \beta + n + 1) - l(\alpha + \beta + l), \\
& -2\left(\frac{\beta^2}{4} - \frac{\beta}{2}\right) + \left(\frac{\alpha^2}{4} - \frac{\alpha}{2}\right) + \alpha - \beta - \nu^2 = -l(\alpha - \beta), \\
& \left(\frac{\beta^2}{4} - \frac{\beta}{2}\right) + \left(\frac{\alpha^2}{4} - \frac{\alpha}{2}\right) + \frac{\alpha\beta}{2} + \alpha + \beta = -n(\alpha + \beta + n + 1),
\end{aligned}$$

by solving above equations, the parameter ε^2 which is corresponding to energy spectrum can be obtained as,

$$(2.15) \quad \varepsilon^2 = -((\alpha + l)^2 - l(l+1)) - \tau^2,$$

here the parameter α is given by,

$$(2.16) \quad \alpha = \frac{-1}{2n + \beta - l + 1} \left(\beta(\beta + l + 1) + 2n(\beta + n + 1) + \frac{mV_0}{\hbar^2\omega^2} \right).$$

The negative energy levels are determined from relation between E_{nl} and ε^2 as,

$$(2.17) \quad E_{nl} = -\frac{\hbar^2\omega^2}{2m} ((\alpha + l)^2 - l(l+1)) - K\omega.$$

Without considering the centrifugal angular momentum and Colombian repulsive potential the negative energy spectrum is written as [5],

$$(2.18) \quad E_{nl} = -\frac{\hbar^2\omega^2}{2m}(\alpha + l)^2.$$

For $l = 0$ we have,

$$(2.19) \quad E_{nl} = -\frac{\hbar^2 \omega^2 \alpha^2}{2m}.$$

This relation for negative energy spectrum exactly match with the results of supersymmetry approach [5].

Finally the corresponding wave function for this energy is given by,

$$(2.20) \quad \psi_{nl}(r) = \frac{1}{r} (1 - \tanh \omega r)^{\frac{\alpha}{2}} (1 + \tanh \omega r)^{\frac{\beta}{2}} P_{nl}^{(\alpha, \beta)}(r).$$

3 Conclusion

In addition to the nuclear strong force between nucleons, there is a repulsive force between protons in nucleus. In this study the non-relativistic radial Schrödinger equation is solved for modified Woods-Saxon potential that is sum of the Wood-saxon potential and the coulombian repulsive potential. By modifying this equation to form the associated Jacobi differential equation, we simply obtained the negative energy spectrum for bound states of nucleons and corresponding wave functions. The results can extend for generalized nuclear potential which correspond to modify nucleus in the case of relativistic theory. This study is in progress. Also this simple method can be applied for other complicated potentials.

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