Some aspects concerning non-Hamiltonians systems

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Abstract. The Helmholtz conditions and the canonical form of a non-Hamiltonian system are considered. We show that using in a suitable manner a Hamiltonian and an open domain, a canonical form can be given for the non-integrable Hamiltonian system defined by the Euler equation of a rigid body.

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1 Non-Hamiltonian systems

Let (M,Ω) be a symplectic manifold. It follows a one-to-one map between 1-forms and vector fields $\omega \to X_\omega$ and $X \to \omega_X$, given by $\omega = i_X\Omega$. A vector field X is *Hamiltonian* if the differentiable 1-form ω_X is closed. Every exact 1-form $\omega = df$, $f \in \Omega^0(M) = \mathcal{F}(M)$ is closed; a Hamiltonian vector field that comes from an exact form is called an *exact Hamiltonian.* The dynamic system of a Hamiltonian vector field is usually called a *Hamiltonian system.* It has the local form

$$\frac{dx^i}{dt} = X^i, \quad \frac{dY^i}{dt} = Y^i.$$

The vector field is Hamiltonian iff it fulfills the conditions

$$\frac{\partial X^i}{\partial y^j} - \frac{\partial X^i}{\partial y^j}, = 0; \ \frac{\partial X^i}{\partial x^j} + \frac{\partial Y^j}{\partial x^i} = 0; \ \frac{\partial Y^i}{\partial x^j} - \frac{\partial Y^i}{\partial x^j} = 0, \tag{1.1}$$

called *Helmholtz conditions.* These are just the conditions that the form $\omega = -\sum Y^i dx^i + \sum X^i dy^i$ be closed. According to Poincaré Lemma, there is a locally function H on a domain of coordinates $U \subset M$ such that

$$X^i = \frac{\partial H}{\partial y^i}, \quad Y^i = -\frac{\partial H}{\partial x^i}.$$

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A *non-Hamiltonian* system is a system that the Helmholtz conditions are not fulfilled, i.e. one of conditions (1.1) is not fulfilled.

An well-known example of a symplectic manifold is the cotangent space T^*M of a manifold M. For every manifold M, T^*M has a canonical symplectic structure. If (q^i, p_i) are local coordinates on T^*M, then $\Omega = dx^i \wedge dp_i$ $(= d(p_i dx^i))$ is a (global) symplectic form on T^*M. In this case the Hamiltonian system has the form

$$\frac{dx^i}{dt} = X^i, \quad \frac{dp_i}{dt} = Y_i,$$

where $X^i = -\dfrac{\partial H}{dp_i}$ and $Y_j = \dfrac{\partial H}{\partial x^i}$, for a local function H. The change rules of the local functions $\{X^i, Y_j\}$ are

$$\begin{cases} X^{i'} = \dfrac{\partial x^{i'}}{\partial x^i} X^j, \\ Z_{i'} = \dfrac{\partial x^i}{\partial x^{i'}} Z_i + \dfrac{\partial^2 x^u}{\partial x^{i'} \partial x^{u'}} p_u \dfrac{\partial x^{u'}}{\partial x^v} X^v \end{cases}.$$

The Helmholtz conditions have the form

$$\frac{\partial X^i}{\partial p_j} - \frac{\partial X^j}{\partial p_i} = \frac{\partial X^i}{\partial x^j} + \frac{\partial Y_j}{\partial p_i} = \frac{\partial Y_i}{\partial x^j} - \frac{\partial Y_j}{\partial x^i} = 0.$$

Let $X \in \mathcal{X}(T^*M)$ having the local form $X = X^i \dfrac{\partial}{\partial x^i} + Z_j \dfrac{\partial}{\partial p_j}$ and corresponding to a local domain $U \subset T^*M$. Let us consider also the local function $\Omega_U = \dfrac{\partial X^i}{\partial x^j} + \dfrac{\partial Y_j}{\partial p_i}$.

Proposition 1.1. *If $X \in \mathcal{X}(T^*M)$, then the local functions φ_U, where U cover T^*M, glue together to a global function $\varphi^X \in \mathcal{F}(T^*M)$.*

Notice that a vertical vector field $Y \in \Gamma(XT^*M)$ has the local form $Y = Y_i(x^j, p_k)\dfrac{\partial}{\partial p_i}$ and there is a natural vector bundle map $T^*T^*M \xrightarrow{I^*} V^*T^*M$, dual to the inclusion $VT^*M \xrightarrow{I} TT^*M$. For every function $f \in \mathcal{F}(T^*M)$ we define the *vertical differential* of f as $d_v f(X) = X(f)$. Analogously, one can define the vertical differential of a vertical form, i.e. $\omega \in \Omega^k(VT^*M)$, $k \geq 0$. An $\omega \in \Gamma(V^*T^*M) = \Omega^1(VT^*M)$ is *v-closed* if $d_v\omega = 0$ and *v-exact* if $\omega = d_v f$. One have $d_v^2 = d_v \circ d_v = 0$, thus every ω exact is closed.

We say that a non-Hamiltonian system defined by a vector field $X \in \mathcal{X}(T^*)$ is *top closed (exact)* if the vertical form $I^*(\omega_X)$ is v-closed (or v-exact respectively). So, if ω_X is v-exact, then there is a function $H \in \mathcal{F}(T^*M)$, i.e. a *Hamiltonian* on M such that the $d_v f = I^*\omega_X$. Using local coordinates, if $X = X^i \dfrac{\partial}{\partial x^i} + Z_j \dfrac{\partial}{\partial p_j}$, then $X^i = \dfrac{\partial H}{\partial p_i}$. In this case, if we denote by $Y_j = Z_j + \dfrac{\partial f}{\partial x^i}$, then $X = \dfrac{\partial}{\partial x^i}\dfrac{\partial H}{\partial p_i} + \left(Y_j - \dfrac{\partial H}{\partial x^j}\right)\dfrac{\partial}{\partial p_j}$, thus φ^X has the local form

$$\varphi^X = \frac{\partial Y_i}{\partial p_i}, \tag{1.2}$$

called in [4] the *velocity of the entropy density (v.e.d.)* of the vector field X. If $\varphi^X \leq 0$, but not vanishing, the system is called a *dissipative system.*

Notice that the case when $Y_j = 0$, the system is Hamiltonian.

The *total time derivative* along the trajectories of the vector field X is

$$\frac{d}{dt} = \frac{\partial}{\partial t} + X^i \frac{\partial}{\partial x^i} + Z_i \frac{\partial}{\partial p_i}.$$

Thus if $f \in \mathcal{F}(T^*M)$, then $\dfrac{df}{dt} = X(f)$.

The *entropy* ρ of the system is defined by the differential equation

$$\frac{d\rho}{dt} = -\varphi^X \cdot \rho, \tag{1.3}$$

called the *Liouville equation.*

If ρ comes from a real function on T^*M, then ρ is called the *distribution* of the entropy.

A top exact non-Hamiltonian system, $I^*\omega_X = d_v H$, is called in [4] *canonical* if the entropy ρ depends only on the Hamiltonian H. It is also proved [4, Proposition 2] that to every top exact non-Hamiltonian system one can associate locally, in a canonical way, a canonical non-Hamiltonian system that corresponds to the same Hamiltonian H.

Let us consider a top-exact Hamiltonian system, given by a vector field $X \in \mathcal{X}(T^*M)$, having the form $X = \dfrac{\partial H}{\partial p_i}\dfrac{\partial}{\partial x^i} + \left(Y_j - \dfrac{\partial H}{\partial x^j}\right)\dfrac{\partial}{\partial p_j}$. The total derivative of H is

$$\frac{dH}{dt} = X(H) = Y_j \frac{\partial H}{\partial p^j}. \tag{1.4}$$

Proposition 1.2. *(Tarasov [4, Proposition 1]) If for a non-dissipative system the non-potential forces Y_i satisfy the condition*

$$g(H) Y_i \frac{\partial H}{\partial p_i} = \frac{\partial Y_i}{\partial p_i}, \tag{1.5}$$

then the non-Hamiltonian system is in a canonical form with the distribution function

$$\rho(x^i, p_i) = C \exp[-G(H)],$$

where G is a primitive of the real function g, i.e. $G' = g$.

Proof. Using formula (1.4), it follows that $\dfrac{\partial Y_i}{\partial p_i} = g(H)\dfrac{dH}{dt} = \dfrac{d}{dt}G(H)$. Using relations (1.2) and (1.3), we have $\dfrac{d\rho}{dt} = -\dfrac{dG}{dt}\rho$, thus the result follows. □

Proposition 1.3. *If $\{\varphi_j(x^i, p_k)\}$ fulfills the conditions $\dfrac{\partial \varphi_i}{\partial p_i} = 0$ and $g(H)\varphi_i \dfrac{\partial H}{\partial p_i} \leq 0$, but not vanishing, then the non-Hamiltonian system obtained from*

$$Y_j = \varphi_j(x^i, p_k) \cdot \exp(G(H(x^i, p_j)))$$

is a non-dissipative one and the relation (1.5) holds.

For example:
1) φ comes from a 1-form $\varphi = \varphi_j(x^i)dx^j$ on M;
2) $\varphi = p_j \varphi_i^j(x^i)dx^i$, where $\varphi_i^i = 0$ (i.e. null trace).
An other non-trivial example is given in the next section.

2 The Euler equation and the canonical form

In the dynamics of a rigid-body, one use the Hamiltonian

$$H = \frac{1}{2}\left(\frac{\Pi_1^2}{I_1} + \frac{\Pi_2^2}{I_2} + \frac{\Pi_3^2}{I_3}\right), \tag{2.1}$$

where $(\Pi_i)_{i=\overline{1,3}}$ are *angular momenta* and $(I_i)_{i=\overline{1,3}}$ are some constants, the *principal moments of inertia* of the rigid body. Denoting $\alpha_1 = \dfrac{I_2 - I_3}{I_2 I_3}$, $\alpha_2 = \dfrac{I_2 - I_1}{I_3 I_1}$ and $\alpha_3 = \dfrac{I_1 - I_2}{I_1 I_2}$, the Euler's equations are:

$$\begin{aligned} \dot{\Pi}_1 &= \alpha_1 \Pi_2 \Pi_3, \\ \dot{\Pi}_2 &= \alpha_2 \Pi_3 \Pi_1, \\ \dot{\Pi}_3 &= \alpha_3 \Pi_1 \Pi_2. \end{aligned} \tag{2.2}$$

The study of Euler equation, as well as deep mechanical interpretations, can be found in the monograph [2]. We emphasis here only the aspect given by the non-integrable Hamiltonian system defined by the Euler equation.

Denoting by $(q^i)_{i=\overline{1,3}}$ the coordinates of the euclidean space such that the coordinates $(\Pi_i)_{i=\overline{1,3}}$ are coordinates in the dual and by $f_1 = \alpha_1 \Pi_2 \Pi_3$, $f_2 = \alpha_2 \Pi_3 \Pi_1$, $f_3 = \alpha_3 \Pi_1 \Pi_2$, we can write the equations:

$$\begin{cases} \dot{q}^i = \dfrac{\partial H}{\partial \Pi_i} \left(= \dfrac{\Pi_i}{I_i} \overset{not.}{=} \Omega_i \right), \\ \dot{\Pi}_i = -\dfrac{\partial H}{\partial q^i} + f_i \, (= f_i) \, . \end{cases}$$

These are the equations of a non-integrable Hamiltonian system, since $(f_i) \neq 0_3$. We have

$$(f_j^i) = \begin{pmatrix} 0 & \alpha_1 \Pi_3 & \alpha_1 \Pi_2 \\ \alpha_2 \Pi_3 & 0 & \alpha_2 \Pi_1 \\ \alpha_3 \Pi_2 & \alpha_3 \Pi_1 & 0 \end{pmatrix}.$$

The characteristic polynomial of this tensor is $P_f(\lambda) = -\lambda^3 - \Delta\lambda + \delta$, where $\Delta = \Pi_1^2 \alpha_2 \alpha_3 + \Pi_3^2 \alpha_2 \alpha_1 + \Pi_2^2 \alpha_1 \alpha_3$ and $\delta = \alpha_1 \alpha_2 \alpha_1 \Pi_1 \Pi_2 \Pi_3$. The entropy vanish, thus it is not a dissipative system. It is easy to see that

$$\sum_{i=1}^{3} f_i \frac{\partial H}{\partial \Pi_i} = 0, \tag{2.3}$$

thus $\dfrac{dH}{dt} = 0$ (i.e. H is constant along the trajectories of the dynamical system). The same conclusions hold taking instead of the Hamiltonian H given by the formula (2.1), the euclidean Hamiltonian

$$H = \frac{1}{2}\left(\Pi_1^2 + \Pi_2^2 + \Pi_3^2\right). \tag{2.4}$$

The equality (2.3) implies that Propositions 1.3 and 1.2 can not be used for Hamiltonians (2.1) and (2.4).

Let us consider instead the Hamiltonian

$$H = \frac{1}{2}\left(\alpha_1\Pi_1^2 + \alpha_2\Pi_2^2 + \alpha_3\Pi_3^2\right). \tag{2.5}$$

We have $\sum\limits_{i=1}^{3} f_i \frac{\partial H}{\partial \Pi_i} = \left(\alpha_1^2 + \alpha_2^2 + \alpha_3^2\right)\Pi_1\Pi_2\Pi_3$. Let us consider an open domain $D \subset I\!R^3$ such that $\Pi_1\Pi_2\Pi_3 \geq 0$ or $\Pi_1\Pi_2\Pi_3 \leq 0$ for $(\Pi_1, \Pi_2, \Pi_3) \in D$, a real function g of constant sign on D that fulfills the condition $g(H)\Pi_1\Pi_2\Pi_3 \leq 0$ and G be a primitive of g. Let us consider the functions $Y_i = f_i \cdot \exp(G(H(\Pi_j))$ and the non-Hamiltonian system $X = \sum\limits_{i=1}^{3}(\frac{\partial H}{\partial \Pi_i}\frac{\partial}{\partial q^i} + Y_i \frac{\partial}{\partial \Pi_i})$ on D. We can use Propositions 1.3 and 1.2 in order to obtain that the following assertion is true.

Proposition 2.1. *The vector field X defines a non-Hamiltonian system that has a suitable canonical form on D.*

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