An intrinsic link between scalar and volume-valued Lagrangians

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Abstract. The aim of this paper is to provide a natural frame for affine Lagrangians and affine Hamiltonians, the focus being on some Hamiltonians applicable in classical fields and their generalizations. A unitary treatment of scalar and volume-valued Hamiltonians in a special class is obtained using some suitable lifting procedures.

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1 Introduction

A general setting concerning Lagrangians and Hamiltonians on affine bundles is given in [4] and [5]. The most known examples of affine bundle used in differential geometry are the higher order tangent space and the jet space of a fibered manifold. These two classical cases were recently studied in many papers. The higher order spaces are studied from the affine point of view in [4]. The jet spaces are studied in the context of multi-time Lagrangian and Hamiltonian geometry in [7] and in an affine setting in [1]-[3]. The purpose of our paper is to indicate a link between these two cases, and also to give a general setting for Lagrangians and Hamiltonians on affine bundles. An F-Hamiltonian (volume-valued) and an affine Hamiltonian (scalar valued) are defined as sections in certain affine bundles, both naturally lifting to $\tilde{F}$-Hamiltonians. Further investigations are given in [6], where considering a Hamilton-Jacobi variational principle for $\tilde{F}$-Hamiltonians, one obtain some Hamilton-Jacobi equations that extend the classical ones studied in [1]-[3].

2 Affine Lagrangians and Hamiltonians on affine spaces

Let A be an affine space modeled on the real (finite dimensional) vector space V. A *Lagrangian* on A is a differentiable function $L : A \to I\!R$. An *affine Hamiltonian* on

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A is a differentiable map (non-necessary linear) $h : V^* \to A^\dagger$ such that $\pi \circ h = 1_{V^*}$. Using local coordinates, the affine Hamiltonian is

$$(p_i) \xrightarrow{h} (p_i, h_0(p_i)). \tag{2.1}$$

If the coordinates change, then

$$h'_0(p_{i'}) = h_0(p_i) + p_i a^i. \tag{2.2}$$

For example, if $x_0(\alpha_i) \in A$, then $(p_i) \xrightarrow{h_{x_0}} \left(p_i, \alpha^i p_i\right)$ is an affine Hamiltonian. For more details concerning affine Lagrangians and Hamiltonians on affine spaces and affine bundles see [4] and [5].

If h_1 and h_2 are two affine Hamiltonians, then $h_1 - h_2$ induces a map $H : V^* \to \mathbb{R}$, called a *vectorial Hamiltonian*; we write $H = h_1 - h_2$, or $h_1 = H + h_2$. In particular, if h is an affine Hamiltonian and $x_0 \in A$, then $H_{x_0} = h - h_{x_0}$ is a vectorial Hamiltonian. Every vectorial Hamiltonian $H : V^* \to \mathbb{R}$ has this form, using the affine Hamiltonian $H + h_{x_0}$.

The *vertical Hessian* of a Lagrangian L (affine Hamiltonian h) is defined by $g_{ij}(y^k) = \dfrac{\partial^2 L}{\partial y^i \partial y^j}(y^k)$ (by $h^{ij}(p_k) = \dfrac{\partial^2 h_0}{\partial p_i \partial p_j}(p_k)$ respectively).

The *Legendre map* defined by a Lagrangian $L : A \to \mathbb{R}$ is $\mathcal{L} : A \to V^*$, $\mathcal{L}(y^i) = \dfrac{\partial L}{\partial y^i}(y^j)e^i$ and the *co-Legendre* map defined by an affine Hamiltonian $h : V^* \to A^\dagger$ of the form (2.1) is $\mathcal{H} : V^* \to A, \mathcal{L}(p_i) = \left(\dfrac{\partial h_0}{\partial p_i}(p_j)\right)$.

The Lagrangian L is *regular* (*hyperregular*) if the Legendre map is a local diffeomorphism (global diffeomorphism). Analogous one say that an affine Hamiltonian h is *regular* (*hyperregular*) if its co-Legendre map is a local diffeomorphism (global diffeomorphism). A Lagrangian (affine Hamiltonian) is *singular* if it is not regular. For example, the image of the co-Legendre map of an affine Hamiltonian of the form h_{x_0} is $\{x_0\}$ and its vertical Hessian is null (degenerate; an extreme case).

Then L (or h) is regular iff the vertical Hessian is non-degenerate in every point (as a bilinear form).

Let $L : A \to \mathbb{R}$ be a hyperregular Lagrangian. Then let us denote by $\mathcal{L}^{-1} : V^* \to A$ the inverse of the Legendre map; using coordinates, $\mathcal{L}^{-1}(p_i) = (\mathcal{L}^j(p_i))$. Then $h : V^* \to A^\dagger, h(p_i) = (p_i, h_0(p_i))$, $h_0(p_i) = p_j L^j(p_i) - L(\mathcal{L}^j(p_i))$, is an affine Hamiltonian.

Conversely, let $h : V^* \to A^\dagger$ be a hyperregular affine Hamiltonian and $\mathcal{H}^{-1} : A \to V^*$ the inverse of the co-Legendre map; using coordinates, $H^{-1}(y^i) = (\mathcal{H}_j(y^i))$. Then $L : A \to \mathbb{R}$, $L(y^i) = y^j \mathcal{H}_j(y^i) - h_0(\mathcal{H}_j(y^i))$, is an affine Lagrangian.

A surjective submersion $E \xrightarrow{\pi} M$ is usually called a *fibered manifold*; the manifold E is called the *total space*, M is the *base space* and π is the (*canonical*) *projection*. If the projection is understood, the fibered manifold is denoted by E. If $x \in M$, the submanifold $E_x = \pi^{-1}(x) \subset E$ is the fiber of π at x. In general, the fibers need not to be all homeomorphic; for example the fibered manifold $\pi_1 : E = \mathbb{R}^2 \backslash \{(0,0)\} = M, \pi_1(x,y) = y$ has not all the fibers connected.

A *fibered manifold map* (*fmm*) sends fibers in fibers: if $\pi : E \to M$ and $\pi' : E' \to M$, then $f : E \to E'$ is a *fmm* if $\pi = \pi' \circ f$; if $\pi : E \to M$ and $\pi' : E' \to M'$, then $f : E \to E'$ is an fmm if there is an induced $f_0 : M \to M'$ such that $f_0 \circ \pi = \pi' \circ f$.

The local coordinates on M and E *adapted* to the submersion π are

$$\text{(2.3)} \qquad \begin{cases} (x^i) \text{ on } M \text{ and} \\ (x^i, y^\alpha) \text{ on } E, \end{cases}$$

such that π has the local form $(x^i, y^\alpha) \to (x^i)$.

A case when all the fibers are homeomorphic is that of a *locally trivial fibration* $E \xrightarrow{\pi} M$ with the *fiber type* a manifold F. In this case there is an open cover of M with sets U such that for every U there is a locally diffeomorphism $\psi : \pi^{-1}(U) \to U \times F$. For example, a vector bundle is a locally trivial fibration with the fiber type a vector space and each of its fibers has an intrinsic structure of a vector bundle. In particular, the tangent and the contangent bundles of a manifold are vector bundles.

A locally trivial fibration $A \xrightarrow{\pi} M$ is an *affine bundle* if its fiber is modeled by a (real) affine space A_0 and the structural functions are affine transformations of A_0. A vector bundle is a particular case of an affine bundle. An affine bundle $\pi : A \to M$ gives rise to the vector bundles $\bar{\pi} : \bar{A} \to M$ (given by the director vector space in every point) and its dual vector bundle $\bar{\pi}' : \bar{A}^* \to M$, called the *dual vector bundle* of the given affine bundle and usually denoted by $\pi^* : A^* \to M$, or A^* for shortness.

Let $\pi_1 : F \to M$ be an affine bundle with the affine line $I\!R$ as typical fiber (i.e. with a one-dimensional fiber). The local coordinates on F change according to the rules

$$\text{(2.4)} \qquad \begin{cases} x^{i'} = x^{i'}(x^i) \\ y' = y\sigma(x^i) + \tau(x^i). \end{cases}$$

If $\sigma = 1$ and $\tau = 0$, then $\pi_1 : F \to F = M \times I\!R \to M$ is the projection on the first factor, thus it is the trivial vector bundle. If only $\sigma(x^i) = 1$ (for every local chart), then the affine bundle is associated with the trivial vector bundle $M \times I\!R \to M$; we say that the affine bundle F has *structural translations.*

Let $\pi : A \to M$ be an affine bundle and $\pi_1 : F \to M$ be an affine bundle with a one-dimensional fiber. The $\bar{F}$-dual of A is $L(\bar{A}, \bar{F})$, denoted by A^{*F}. The local coordinates on A^{*F} change according to the rules

$$\text{(2.5)} \qquad \begin{cases} x^{i'} = x^{i'}(x^i) \\ \sigma p_{\alpha'} a_\alpha^{\alpha'}(x^i) = p_\alpha. \end{cases}$$

Let us consider $\bar{F}_* \subset \bar{F}$, the fibered submanifold of the vector bundle $\pi_0 : \bar{F} \to M$, consisting in non-null vectors. Denote $\tilde{A} = A \times_M \bar{E}$. The natural projection $\bar{\pi} : \tilde{A} \to \bar{F}_*$ is the canonical projection of an affine bundle. Let us denote also by $\tilde{F} = F \times_M \bar{F}_*$ and by $\tilde{\pi}_0 : \tilde{F} \to \bar{F}$ the canonical projection.

Proposition 1. *The projection* $\tilde{\pi}_0 : \tilde{F} \to \bar{F}_*$ *is the canonical projection of an affine bundle with structural translation (i.e. the associated vector bundle is the trivial vector bundle* $M \times I\!R \to M$*).*

Proof. The local coordinates on F change according to the rules (2.4). Let us denote by $(x^i, \bar{y})$ the local coordinates on $\bar{F}_*$ and by $(x^i, \bar{y}, y)$ the local coordinates on $\tilde{F}$, such that $\tilde{\pi}_1$ has the local form $(x^i, \bar{y}, y) \to (x^i, \bar{y})$. The local coordinates $\bar{y}$ change according to the rule

$$\bar{y}' = \sigma(x^i)\bar{y}.$$

Since on F one have $y'' = \sigma' y\prime + \tau' = \sigma'(\sigma y + \tau) + \tau' = \sigma'\sigma y + (\sigma'\tau + \tau')$ and $y'' = \sigma'' y + \tau''$, it follows $\sigma'' = \sigma'\sigma$ and $\tau'' = \sigma'\tau + \tau'$. Also $\bar{y}'' = \sigma'\bar{y}' = \sigma''\bar{y}$. Consequently $\dfrac{\tau'}{\sigma'\bar{y}'} + \dfrac{\tau}{\sigma\bar{y}} = \dfrac{\tau'}{\sigma'\sigma\bar{y}} = \dfrac{\tau' + \sigma'\tau}{\sigma'\sigma\bar{y}} = \dfrac{\tau''}{\sigma''\bar{y}''}$. Also denoting $z = \dfrac{y}{\bar{y}}$ on $\tilde{F} = F \times_M \bar{F}_*$, one have $z'' = z' + \dfrac{\tau}{\sigma\bar{y}}$, thus the conclusion follows. □

If $\pi : E \to M$ is a fibered manifold, its first jet space $J^1\pi$ can be regarded as an affine bundle $J^1\pi \to E$. Using local coordinates (2.3), adapted to the submersion, the coordinates on $J^1\pi$ have the form $(x^i, y^\alpha, y^\alpha_i)$ and change according to the rules:

$$\begin{cases} x^{i'} = x^{i'}(x^i) \\ y^{\alpha'} = y^{\alpha'}(x^i, y^\alpha) \\ y^{\alpha'}_{i'}\dfrac{\partial x^{i'}}{\partial x^i} = y^\alpha_i \dfrac{\partial y^{\alpha'}}{\partial y^\alpha} + \dfrac{\partial y^{\alpha'}}{\partial x^i}. \end{cases} \tag{2.6}$$

If $s : M \to E$ is a section (it can be a local one), then it lifts to a section $s' : M \to J^1E$ of the fibered manifold $J^1E \to M$. Using local coordinates, if s has the local form $(x^i) \to (x^i, s^\alpha(x^i))$, then s' is $(x^i) \to (x^i, s^\alpha(x^i), \dfrac{\partial s^\alpha}{\partial x^i})$.

The manifold $J^1\pi^* = V^*E \otimes \pi^*TM$ is the total space of a vector bundle over E. The local coordinates on $J^1\pi^*$ have the form $(x^i, y^\alpha, p^i_\alpha)$. The change rule of (x^i) and (y^α) is given by relations (2.6), while

$$p^{i'}_{\alpha'}\frac{\partial y^{\alpha'}}{\partial y^\alpha} = p^i_\alpha \frac{\partial x^{i'}}{\partial x^i}. \tag{2.7}$$

If $E = M \times T$, where T is a manifold, then $x^{i'} = x^{i'}(x^i)$, $y^{\alpha'} = y^{\alpha'}(y^\alpha)$ and the coordinates (y^α_i) on $J^1\pi$ change in a tensor manner, thus $J^1\pi = VE \otimes \pi^*T^*M$ is a vector bundle and $J^1\pi^*$ is its dual vector bundle. This vector bundle is used in a systematic way in the study of multi-time Lagrangians and Hamiltonians (see [8] and the references therein). Another particular case, considered below, is when $\pi_1 : F \to M$ is an affine bundle with a one dimensional fiber. In this case the formulas (2.6) have the form:

$$\begin{cases} x^{i'} = x^{i'}(x^i) \\ y' = y\sigma(x^i) + \tau(x^i) \\ y_{i'}\dfrac{\partial x^{i'}}{\partial x^i} = y_i\sigma(x^i) + y\dfrac{\partial \sigma}{\partial x^i} + \dfrac{\partial \tau}{\partial x^i}. \end{cases} \tag{2.8}$$

If $\pi_1 : F \to M$ is a vector bundle, then $\tau = 0$.

Let us suppose that $\pi_1 : F \to M$ is an affine bundle with structural translations. If (x^i) and (x^i, y) are local coordinates on M and on F respectively, then the coordinates change according to the formula

$$x^{i'} = x^{i'}(x^i), \ y' = y + f(x^i).$$

The first jet bundle $J^1\pi_1$ has as coordinates (x^i, y, u_i), where the coordinates (u_i) change following the rule: $u'_i = u_i + \dfrac{\partial f}{\partial x^i}$. There is an affine bundle $\nu : F_1 \to M$

such that F_1 has as coordinates (x^i, u_i) and the affine bundle $J^1\pi_1$ is canonically isomorphic with the induced bundle $\pi_1^*\nu$; we write $J^1\pi_1 = \pi_1^*\nu$.

A section $s \in \Gamma(\pi_1)$ lifts naturally to a section $s' \in \Gamma(J^1F_1 \to M)$, given locally by $(x^i) \to (x^i, s(x^i), \dfrac{\partial s}{\partial x^i})$. It induces a section $s'' \in \Gamma(\nu)$ and implicitely an affine section $s^J \in \Gamma(J^1F \to F)$ having the local form $(x^i, y) \to (x^i, y, \dfrac{\partial s}{\partial x^i})$. The section s^J defines a null curvature connection on the bundle $\pi : E \to M$. If a connection on $\pi : E \to M$ is defined by a section $\xi \in \Gamma(J^1F \to F)$, $(x^i, y) \to (x^i, y, \xi_i(x^i, y^\alpha))$, its curvature is locally given by $R_{ij} = \dfrac{\partial \xi_i}{\partial x^j} - \dfrac{\partial \xi_j}{\partial x^i}$. The curvature vanishes iff locally ξ has the form $\xi = s^J$, i.e. it is a lift of a local section $s \in \Gamma(\pi_1)$.

We are going to prove that one can associate an affine bundle with a one dimensional fiber and structural translations with every affine bundle with a one dimensional fiber.

3 Lagrangians and Hamiltonians on affine bundles

Let $\pi : A \to M$ be an affine bundle and $\pi_1 : F \to M$ be an affine bundle with a one-dimensional fiber. An F-Lagrangian on E is a fibered manifold map $L : A \to F$ (i.e. $\pi_1 \circ L = \pi$). Since every affine map induces a linear map on the director vector space, then there is a canonical projection $\Pi : Aff(A, F^*) \to A^{*F}$. An F-Hamiltonian on E is a fibered manifold map $h : A^{*F} \to Aff(A, F^*)$ such that $\Pi \circ h = 1_{A^{*F}}$. For example, let us consider $F = M \times I\!R$ and $p_1 : M \times I\!R \to M$ be the projection on the first factor. The F-dual of A is just A^*. An F-Lagrangian has the form $L(e) = (\pi(e), L_0(e))$, where $L_0 : A \to I\!R$ is usually called a *Lagrangian.* An *F-Hamiltonian* on A has the form $h : A^* \to Aff(A, M \times I\!R)$. This case was considered in the study of affine Hamiltonians of higher order (see [3]). Another more elaborated example, using jet spaces, is given in [6].

Then an F-Lagrangian L has the local form $(x^i, y^\alpha) \xrightarrow{L} (x^i, L_0(x^i, y^\alpha))$ and the local functions L_0 change according to the rules given by (2.4):

$$L'_0(x^{i'}, y^{\alpha'}) = L_0(x^i, y^\alpha)\sigma(x^i) + \tau(x^i). \tag{3.1}$$

Since

$$\frac{\partial L_0}{\partial y^\alpha} = \sigma \frac{\partial L'_0}{\partial y^{\alpha'}} \frac{\partial y^{\alpha'}}{\partial y^\alpha} = \sigma \frac{\partial L'_0}{\partial y^{\alpha'}} a_\alpha^{\alpha'},$$

the formula $(x^i, y^\alpha) \to (x^i, \dfrac{\partial L_0}{\partial y^\alpha})$ defines a *Legendre map* $\mathcal{L} : A \to A^{*F}$ of L. The local form of a map $\Omega \in Aff(A, F^*)$ is $(y^\alpha) \xrightarrow{\Omega} \begin{pmatrix} y^\alpha p_\alpha & b \end{pmatrix}$; then $\Pi(\Omega)$ has the local form $(y^\alpha) \xrightarrow{\Pi(\Omega)} (y^\alpha p_\alpha)$.

There are also local forms of $\Pi : Aff(A, F^*) \to A^{*F}$ and of an F-Hamiltonian $h : A^{*F} \to Aff(A, F^*)$ given by $\begin{pmatrix} p_\alpha & p \end{pmatrix} \xrightarrow{\Pi} (p_\alpha)$ and by $(x^i, p_\alpha) \xrightarrow{h} (x^i, p_\alpha, h_0(x^i, p_\alpha))$, respectively. The change rules of local coordinates are:

$$\begin{pmatrix} p_\alpha & p \end{pmatrix} = \sigma \cdot \begin{pmatrix} p_{\alpha'} & p' \end{pmatrix} \begin{pmatrix} a_\alpha^{\alpha'} & a^{\alpha'} \\ 0 & 1 \end{pmatrix},$$

or $\begin{pmatrix} p_{\alpha'} & p' \end{pmatrix} = \sigma' \cdot \begin{pmatrix} p_\alpha & p \end{pmatrix} \begin{pmatrix} a^\alpha_{\alpha'} & a^\alpha \\ 0 & 1 \end{pmatrix}$, where $(a^\alpha_{\alpha'}) = (a^{\alpha'}_\alpha)^{-1}$, $\sigma = (\sigma')^{-1}$ and $a^\alpha = -a^{\alpha'} a^\alpha_{\alpha'}$. Thus $h'_0(x^{i'}, p_{\alpha'}) = \sigma^{-1}(x^i) \cdot (p_\alpha a^\alpha(x^i) + h_0(x^i, p_\alpha))$. Since

$$p_\alpha = \sigma p_{\alpha'} a^{\alpha'}_\alpha$$

$$\frac{\partial h'_0}{\partial p_{\alpha'}} = \frac{\partial h_0}{\partial p_\alpha} \frac{\partial y^{\alpha'}}{\partial y^\alpha} - a^{\alpha'} = \frac{\partial h_0}{\partial p_\alpha} a^{\alpha'}_\alpha - a^{\alpha'},$$

it follows that $(x^i, p_\alpha) \to (x^i, -\dfrac{\partial h_0}{\partial p_\alpha})$ defines a *co-Legendre map* $\mathcal{H}^* : A^{*F} \to A$ of h. A Lagrangian $L : A \to F$ is *regular* if its Legendre map is a local diffeomorphism; it is equivalent to say that the *vertical hessian*, given by the local matrix

$$\left(g_{\alpha\beta} = \frac{\partial^2 L}{\partial y^\alpha \partial y^\beta} \right)$$

is non-singular. The Lagrangian is *hyperregular* if its Legendre map is a (global) diffeomorphism.

If $L : A \to F$ is an F-Lagrangian, then $\tilde{L} : \tilde{A} \to \tilde{F}$ defined locally by $\tilde{L}(x^i, y^\alpha, \bar{y}) = \dfrac{L(x^i, y^\alpha)}{\bar{y}}$ is an $\tilde{F}$-Lagrangian on $\tilde{A}$. We say that $\tilde{L}$ is the *lift* of L from A to $\tilde{A}$. It is easy to see that the following statement is true.

Proposition 2. *The lift $\tilde{L}$ is regular (hyperregular) iff L is regular (hyperregular).*

Analogously, if $h : A^{*F} \to Aff(A, F)$ is an F-Hamiltonian, then one can consider an $\tilde{F}$-Hamiltonian $\tilde{h} : \tilde{A}^{*\tilde{F}} \to Aff(\tilde{A}, \tilde{F})$ defined by $\tilde{h}(x^i, \bar{y}, \tilde{p}_\alpha) = \dfrac{1}{\bar{y}} h(x^i, \dfrac{1}{\bar{y}} \tilde{p}_\alpha)$. We say that $\tilde{h}$ is the *lift* of h (from A^{*F} to $\tilde{A}$). It is easy to see that the following statement is also true.

Proposition 3. *The lift $\tilde{h}$ is regular (hyperregular) iff h is regular (hyperregular).*

There are natural maps $\Phi : A^* \times_M \bar{F} = \tilde{A}^* \to A^{*F}$ and $\Psi : Aff(A, I\!R) \times_M \bar{F} = Aff(\tilde{A}, I\!R) \to Aff(A, F)$ given in local coordinates by

$$\Phi : (x^i, \tilde{p}_\alpha, \bar{z}) \to (x^i, p_\alpha = \bar{z}^{-1} \tilde{p}_\alpha),$$
$$\Psi : (x^i, \tilde{p}_\alpha, \bar{z}, \tilde{p}) \to (x^i, p_\alpha = \bar{z}^{-1} \tilde{p}_\alpha, .\bar{z}^{-1} \tilde{p}).$$

One can see that considering the natural maps

$$\tilde{\Pi} : Aff(\tilde{A}, I\!R) \to \tilde{A}^*,$$
$$\Pi : Aff(A, F) \to A^{*F},$$

the following diagram

$$\begin{tikzcd} Aff(\tilde{A}, I\!R) \arrow[r, "\tilde{\Pi}"] \arrow[d, "\Psi"'] & \tilde{A}^* \arrow[d, "\Phi"] \\ Aff(A, F) \arrow[r, "\Pi"] & A^{*F} \end{tikzcd}$$

is commutative. If $\tilde{h}$ is the lift an F-Hamiltonian h, then the diagram

$$\begin{tikzcd}
Aff(\tilde{A}, \mathbb{R}) \arrow[d, "\Psi"'] & \tilde{A}^{*} \arrow[l, "\tilde{h}"'] \arrow[d, "\Phi"] \\
Aff(A, F) & A^{*F} \arrow[l, "h"']
\end{tikzcd}$$

is commutative.

Analogously, if $\bar{h} : A^{*} \to Aff(A, \mathbb{R})$ is an affine Hamiltonian, then one can consider an $\tilde{F}$-Hamiltonian $\tilde{h} : \tilde{A}^{*\tilde{F}} \to Aff(\tilde{A}, \tilde{F}^{*})$ defined by $\tilde{h}(x^{i}, \bar{y}, \tilde{p}_{\alpha}) = \bar{h}(x^{i}, \tilde{p}_{\alpha})$. We say that $\tilde{h}$ is the *lift* of $\bar{h}$ (from A^{*} to $\tilde{A}$).

We consider below some examples.

Let $\pi : E \to M$ be a fibered manifold (or a bundle). The vector bundle $\Lambda^{m}(TM) \to M$, $m = \dim M$, has a one-dimensional fiber; it has as sections the top forms (or volume densities) on M. It is easy to see that $\Lambda^{m}(TM)^{*} = \Lambda^{m}(T^{*}M)$. For our purpose we consider also the induced vector bundles with one dimensional fibers $\pi_{1} : F = \pi^{*}\Lambda^{m}(TM) \to E$, $\pi_{1}' : F^{*} = \pi^{*}\Lambda^{m}(T^{*}M) \to E$. In this particular case, our F*-Hamiltonian* on E is just a Hamiltonian considered in [1, 2, 3] as a section $h : J^{1}\pi^{*F} \to J^{1}\pi^{\dagger F}$ having the local form

$$(x^{i}, y^{\alpha}, p_{\alpha}^{i}) \to (x^{i}, y^{\alpha}, p_{\alpha}^{i}, h(x^{i}, y^{\alpha}, p_{\alpha}^{i})). \tag{3.2}$$

The local coordinates (p_{α}^{i}) and the local functions h change according to the rules

$$p_{\alpha'}^{i'} \frac{\partial y^{\alpha'}}{\partial y^{\alpha}} = \sigma^{-1} p_{\alpha}^{i} \frac{\partial x^{i'}}{\partial x^{i}}, \qquad h' = \sigma^{-1}\left(h + p_{\alpha}^{i} \frac{\partial y^{\alpha}}{\partial y^{\alpha'}} \frac{\partial y^{\alpha'}}{\partial x^{i}}\right).$$

An *affine Hamiltonian* on $J^{1}\pi^{*}$ is a section $\bar{h} : J^{1}\pi^{*} \to J^{1}\pi^{\dagger}$ and it has the local form

$$(x^{i}, y^{\alpha}, \tilde{p}_{\alpha}^{i}) \to (x^{i}, y^{\alpha}, \tilde{p}_{\alpha}^{i}, \bar{h}(x^{i}, y^{\alpha}, \tilde{p}_{\alpha}^{i})). \tag{3.3}$$

The local coordinates $(\tilde{p}_{\alpha}^{i})$ and the local functions $\bar{h}$ change according to the rules

$$\tilde{p}_{\alpha'}^{i'} \frac{\partial y^{\alpha'}}{\partial y^{\alpha}} = \tilde{p}_{\alpha}^{i} \frac{\partial x^{i'}}{\partial x^{i}}, \qquad \bar{h}' = \bar{h} + \tilde{p}_{\alpha}^{i} \frac{\partial y^{\alpha}}{\partial y^{\alpha'}} \frac{\partial y^{\alpha'}}{\partial x^{i}}.$$

We are going to put together F-Hamiltonians and affine Hamiltonians. In order to do this we consider $\tilde{F}$-Hamiltonians. In order to simplify notations and the exposition, we consider F_{*}^{*} instead of $\bar{F}_{*}$ previously. We denote by $\tilde{F} = F \times_{M} \bar{F}_{*}^{*}$ and we use the canonical projection $\tilde{\pi}_{0} : \tilde{F} \to \bar{F}_{*}^{*}$. Also, $\tilde{J} = J^{1}E \times_{M} \bar{F}_{*}^{*}$ and $\tilde{\pi} : \tilde{J} \to \tilde{E} = E \times_{M} \bar{F}_{*}^{*}$ (a canonical projection of a fibered manifold). An $\tilde{F}$*-Hamiltonian* on E is a section $\tilde{h} : \tilde{J}^{*} \to \tilde{J}^{\dagger}$ that has the local form

$$(x^{i}, y^{\alpha}, \bar{\omega}, \tilde{p}_{\alpha}^{i}) \to (x^{i}, y^{\alpha}, \bar{\omega}, \tilde{p}_{\alpha}^{i}, \tilde{h}(x^{i}, y^{\alpha}, \bar{\omega}, \tilde{p}_{\alpha}^{i})). \tag{3.4}$$

The local functions $\tilde{h}$ change according to the rules $\tilde{h}' = \tilde{h} + \tilde{p}_{\alpha}^{i} \dfrac{\partial y^{\alpha}}{\partial y^{\alpha'}} \dfrac{\partial y^{\alpha'}}{\partial x^{i}}$. As we have already seen, an F-Hamiltonian, as well as an affine Hamiltonian, lift both to an $\tilde{F}$-Hamiltonian. More specifically,

– if h is an F-Hamiltonian that has the local form (3.2), then its lift $\tilde{h}$ is an $\tilde{F}$-Hamiltonian that has the local form (3.4), with $\tilde{h}(x^{i}, y^{\alpha}, \bar{\omega}, \tilde{p}_{\alpha}^{i}) = \dfrac{1}{\bar{\omega}} h(x^{i}, y^{\alpha}, \bar{\omega}\tilde{p}_{\alpha}^{i})$;

– if $\bar{h}$ is an affine Hamiltonian that has the local form (3.3), then its lift $\tilde{h}$ is an $\tilde{F}$-Hamiltonian that has the local form (3.4), with $\tilde{h}(x^i, y^\alpha, \bar{\omega}, \tilde{p}^i_\alpha) = \bar{h}(x^i, y^\alpha, \tilde{p}^i_\alpha)$. An important tool in the study of F-Hamiltonians (Hamiltonians in the classical terminology) can be found in the multi-symplectic formalism developped in [1, 2, 3] (see also the bibliography therein). In [1] one define the action of an F-Hamiltonian h on sections on $E \to M$ and one deduce the equation of a critical section of this action, using a Hamilton-Jacobi principle. In [6] one defines an action for an $\tilde{F}$-Hamiltonian, in order to recover the same action for the lift of an F-Hamiltonian.

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