Adjoint multidimensional acausal systems

Valeriu Prepeliţă

Abstract. Multidimensional (qD, $q \geq 2$) time-varying acausal systems are presented. These systems represent the continuous-time counterparts of the Attasi's 2D discrete-time time-invariant models. The state space representation is given and the formula of the input-output map is obtained. Two types of adjoint systems are introduced and the relationship between the input-output maps of the acausal systems and of their adjoints is emphasized by means of four suitable inner products.

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1 Introduction

Different state space models of two-dimensional 2D systems were proposed by Roesser [12], Fornasini and Marchesini [4], Attasi [3] and others. Their study has known an important development in the last three decades due to their significant applications in various areas as image processing, seismology, geophysics or computer tomography. The above mentioned papers and the subsequent ones have studied the causal systems, i.e. systems whose states and outputs are determined by the inputs and the initial states.

Linear one-dimensional (1D) acausal systems, i.e. systems with boundary conditions were introduced by A.J. Krener [9], [10] and were developed by I. Gohberg and M.A. Kaashoek in a series of papers [5], [6], [7], [8], motivated by the problem of boundary value regulation or by the analysis of Wiener-Hopf integral equation. M.B. Adams, A.S. Willsky and B.C. Levy [1], [2] have obtained important results in the linear estimation of stochastic processes governed by acausal systems.

In this paper we extend these results to multidimensional (qD, $q \geq 2$) acausal systems, by introducing a class of systems which represents the continuous-time time-varying counterpart of Attasi's discrete-time 2D model.

In Section 2 the state-space representation of the considered qD acausal systems is given, including well-posed boundary conditions. The formulas of the state and of

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the input-output map of the qD acausal systems are obtained, by means of a suitable variation-of-parameters formula.

Section 3 introduces the adjoint of a qD acausal system and the input-output map of the adjoint systems is derived. Two inner products are defined and they are used to obtain the relationship between the input-output operators of the qD acausal systems and their adjoints.

Section 4 is devoted to the study of a reduced adjoint system and a similar relationship is emphasized.

2 Time-varying qD systems with well-posed boundary conditions

We shall use the following notations: $[a,b] := \prod_{i=1}^{q}[a_i,b_i]$ where $q \in \mathbf{N}^*$, $a = (a_1,\ldots,a_q) \in \mathbf{R}^q$, $b = (b_1,\ldots,b_q) \in \mathbf{R}^q$, $a_i < b_i$; $t := (t_1,\ldots,t_q)$, $x(t) := x(t_1,\ldots,t_q)$; for $m \in \mathbf{N}^*$, $\bar{m} := \{1,2,\ldots,m\}$ and for $\tau = \{i_1,\ldots,i_l\} \subseteq \bar{q}$, $|\tau| := l$, $\tilde{\tau} := \bar{m} \setminus \tau$, $\frac{\partial}{\partial\tau} = \frac{\partial^l}{\partial t_{i_1}\ldots\partial t_{i_l}}$; if $\tau = \bar{q}$, $\frac{\partial}{\partial\tau} = \frac{\partial}{\partial t}$; $\tau \subset \bar{q}$ means τ included in $\bar{q}$ and $\tau \neq \bar{q}$.

For $i \in \bar{q}$, $\tilde{i} := \bar{q} \setminus \{i\}$ and $\tilde{\tilde{i}} = \{i+1,\ldots,q\}$. For $\tau \subset \bar{q}$, $x(k_\tau, l_{\tilde{\tau}})$ denotes the function $x(t)$ with the arguments $t_i = k_i$ if $i \in \tau$ and $t_i = l_i$ if $i \in \tilde{\tau}$.

We consider the linear spaces $X = \mathbf{R}^n$, $U = \mathbf{R}^m$ and $Y = \mathbf{R}^n$, called respectively the *state*, *input* and *output spaces*.

Definition 2.1. A *multidimensional* (qD) *time-varying acausal system* is a set $\Sigma = (\{A_i | i \in \bar{q}\},\ B, C, D, N_1, N_2, M_1, M_2)$ where $A_i(t_i) \in \mathbf{R}^{n\times n}$ are commutative matrices $\forall t_i \in [a_i,b_i]$, $\forall i \in \bar{q}$, $B(t) \in \mathbf{R}^{n\times n}$, $C(t) \in \mathbf{R}^{p\times n}$, $D(t) \in \mathbf{R}^{p\times m}$, $\forall t \in [a,b]$, $N_1, N_2, M_1, M_2 \in \mathbf{R}^{n\times n}$; the matrices A_i, B, C, D have continuous entries and the matrix $Q = \begin{bmatrix} N_1 & N_2 \\ M_1 & M_2 \end{bmatrix}$ is nonsingular. The drift matrices A_i are assumed to be integrable on $[a_i,b_i]$, $\forall i \in \bar{q}$, the input matrix B and the output matrix C are square integrable and the external matrix D is measurable and essentially bounded on $[a,b]$.

The system Σ has the state space representation

$$\frac{\partial}{\partial t}x(t) = \sum_{\tau\subset\bar{q}}(-1)^{q-|\tau|-1}\left(\prod_{i\in\tilde{\tau}}A_i(t_i)\right)\frac{\partial}{\partial\tau}x(t) + B(t)u(t) \tag{2.1}$$

$$y(t) = C(t)x(t) + D(t)u(t) \tag{2.2}$$

$$N_1x(a) + N_2x(b) = v \tag{2.3}$$

$$z = M_1x(a) + M_2x(b) \tag{2.4}$$

where the vectors $x(t) \in X$, $u(t) \in U$, $y(t) \in Y$ $\forall t \in [a,b]$, $v, z \in \mathbf{R}^n$. The vectors $x(\cdot) \in L^2([a,b],X)$, $u(\cdot) \in L^2([a,b],U)$, $y \in L^2([a,b],Y)$, v and z are called respectively *state, input function, output function, input vector* and *output vector.* The number n is called the *dimension* of the system Σ and it is denoted by dimΣ.

Since the matrices $A_i(t_i)$ commute, it results by Peano-Baker formula that their fundamental matrices $\Phi_i(t_i, s_i)$ commute too.

Definition 2.2. The vector $x^0 \in \mathbf{R}^n$ is called an *initial state* of the system Σ if

$$x(t_\tau, a_{\tilde{\tau}}) = \prod_{i\in\tau} \Phi_i(t_i, a_i)x^0 \tag{2.5}$$

for any $\tau \subset \bar{q}$; equalities (2.5) are called the *initial conditions* of Σ.

Let us consider the qD PDE

$$\frac{\partial}{\partial t}x(t) = \sum_{\tau\in\bar{q}}(-1)^{q-|\tau|-1}\left(\prod_{i\in\tilde{\tau}} A_i(t_i)\right)\frac{\partial}{\partial\tau}x(t) + f(t). \tag{2.6}$$

Adapting the proof in [11, Proposition 2.3] we obtain a multidimensional variation - of - parameters formula given by the following

Theorem 2.3. *The solution of the initial value problem* (2.6), (2.5) *is* (*with* $s = (s_1, \ldots, s_q)$)

$$x(t) = \left(\prod_{i=1}^{q} \Phi_i(t_i, a_i)\right)x^0 + \int_{a_1}^{t_1}\cdots\int_{a_q}^{t_q}\left(\prod_{i=1}^{q}\Phi_i(t_i, s_i)\right)f(s)ds_1\ldots ds_q. \tag{2.7}$$

The boundary condition (2.3) is said to be *well-posed* if the homogeneous problem corresponding to (2.1) and (2.3) (i.e. with $u \equiv 0$ and $v = 0$) with the initial conditions (2.5) has the unique solution $x = 0$; by replacing x^0 by $x(a)$ and t by b in (2.6), this is equivalent to the condition that the equation $\left(N_1 + N_2\prod_{i=1}^{q}\Phi_i(b_i, a_i)\right)x(a) = 0$ has the unique solution $x(a) = 0$. Therefore (2.3) is well-posed iff the matrix $F = N_1 + N_2\prod_{i=1}^{q}\Phi_i(b_i, a_i)$ is nonsingular.

Definition 2.4. The *canonical boundary value operator* of the acausal qD system Σ is the matrix

$$P = P_\Sigma = F^{-1}N_2\prod_{i=1}^{q}\Phi_i(b_i, a_i). \tag{2.8}$$

Theorem 2.5. *The state of the acausal system* Σ *determined by the input* $(u, v) \in L^2([a, b], U) \times \mathbf{R}^n$ *is*

$$\begin{aligned} x(t) = {} & \left(\prod_{i=1}^{q}\Phi_i(t_i, a_i)\right)F^{-1}v - \\ & -\int_{a_1}^{b_1}\cdots\int_{a_q}^{b_q}\left(\prod_{i=1}^{q}\Phi_i(t_i, a_i)\right)P\left(\prod_{i=1}^{q}\Phi_i(a_i, s_i)\right)B(s)u(s)ds_1\ldots ds_q + \\ & +\int_{a_1}^{t_1}\cdots\int_{a_q}^{t_q}\left(\prod_{i=1}^{q}\Phi_i(t_i, s_i)\right)B(s)u(s)ds_1\ldots ds_q. \end{aligned} \tag{2.9}$$

Proof. We replace $f(s)$ by $B(s)u(s)$ in (2.7) and we get the formula of the state of the (causal) system (2.1), (2.5)

$$x(t) = \left(\prod_{i=1}^{q} \Phi_i(t_i, a_i)\right) x^0 + \int_{a_1}^{t_1} \dots \int_{a_q}^{t_q} \left(\prod_{i=1}^{q} \Phi_i(t_i, s_i)\right) B(s)u(s)ds_1 \dots ds_q \quad (2.10)$$

hence, for $t = b$ we obtain

$$x(b) = \left(\prod_{i=1}^{q} \Phi_i(b_i, a_i)\right) x^0 + \int_{a_1}^{b_1} \dots \int_{a_q}^{b_q} \left(\prod_{i=1}^{q} \Phi_i(b_i, s_i)\right) B(s)u(s)ds_1 \dots ds_q. \quad (2.11)$$

By replacing $x(a)$ with x^0 and $x(b)$ (2.11) in the boundary condition 2.3 we get

$$N_1 + N_2 \left(\prod_{i=1}^{q} \Phi_i(b_i, a_i)\right) x^0 + N_2 \int_{a_1}^{b_1} \dots \int_{a_q}^{b_q} \left(\prod_{i=1}^{q} \Phi_i(b_i, s_i)\right) B(s)u(s)ds_1 \dots ds_q = v,$$

hence

$$x^0 = F^{-1}v - F^{-1}N_2 \int_{a_1}^{b_1} \dots \int_{a_q}^{b_q} \left(\prod_{i=1}^{q} \Phi_i(b_i, s_i)\right) B(s)u(s)ds_1 \dots ds_q. \quad (2.12)$$

By replacing x^0 (2.12) in (2.10) and taking into account the semigroup property $\Phi_i(t_i, s_i) = \Phi_i(b_i, a_i)\Phi_i(a_i, s_i)$ one obtains the state of the acausal system

$$\begin{aligned} x(t) &= \left(\prod_{i=1}^{q} \Phi_i(t_i, a_i)\right) F^{-1}v - \\ &- \left(\prod_{i=1}^{q} \Phi_i(t_i, a_i)\right) F^{-1}N_2 \int_{a_1}^{b_1} \dots \int_{a_q}^{b_q} \left(\prod_{i=1}^{q} \Phi_i(b_i, a_i)\right) \cdot \\ &\cdot \left(\prod_{i=1}^{q} \Phi_i(a_i, s_i)\right) B(s)u(s)ds_1 \dots ds_q + \\ &+ \int_{a_1}^{t_1} \dots \int_{a_q}^{t_q} \left(\prod_{i=1}^{q} \Phi_i(t_i, s_i)\right) B(s)u(s)ds_1 \dots ds_q \end{aligned}$$

which gives (2.9).

For $t = (t_1, \dots t_q)$, $t_i > a_i$, $\forall i \in \bar{q}$ we shall use the notations: $[a, t] := \prod_{i=1}^{q} [a_i, t_i]$ and $\widetilde{[a, t]} = [a, b] \setminus [a, t])$; $\int_a^b := \int_{a_1}^{b_1} \dots \int_{a_q}^{b_q}$, $\int_{[a,t]} = \int_a^t := \int_{a_1}^{t_1} \dots \int_{a_q}^{t_q}$ and a similar meaning has $\int_{\widetilde{[a,t]}}$; $ds := ds_1 \dots ds_q$.

Theorem 2.6. *The input-output map of the* qD *acausal system* Σ *is the operator* $T : L^2([a, b], U) \times \mathbf{R}^n \to L^2([a, b], Y) \times \mathbf{R}^n$, $T(u(\cdot), v) = (y(\cdot), z)$, *where*

$$\begin{aligned}
y(t) &= C(t)\left(\prod_{i=1}^{q}\Phi_i(t_i,a_i)\right)F^{-1}v- \\
&- \int_a^b C(t)\left(\prod_{i=1}^{q}\Phi_i(t_i,a_i)\right)P\left(\prod_{i=1}^{q}\Phi_i(a_i,s_i)\right)B(s)u(s)ds+ \\
&+ \int_a^t C(t)\left(\prod_{i=1}^{q}\Phi_i(t_i,s_i)\right)B(s)u(s)ds + D(t)u(t)
\end{aligned} \tag{2.13}$$

and

$$\begin{aligned}
z &= \left(M_1 + M_2\prod_{i=1}^{q}\Phi_i(b_i,a_i)\right)F^{-1}v- \\
&- \left(M_1 + M_2\prod_{i=1}^{q}\Phi_i(b_i,a_i)\right)P\int_a^b\left(\prod_{i=1}^{q}\Phi_i(a_i,s_i)\right)B(s)u(s)ds+ \\
&+ M_2\int_a^b\left(\prod_{i=1}^{q}\Phi_i(b_i,s_i)\right)B(s)u(s)ds.
\end{aligned} \tag{2.14}$$

Proof. The expression (2.13) of $y(t)$ results by replacing the sate $x(t)$ (2.9) in the output equation (2.2). Formula (2.14) can be obtained by replacing $x(a) = x^0$ (2.12) and $x(b)$ given by (2.11) in (2.4) and by using the semigroup property; we obtain

$$\begin{aligned}
&z = M_1x(a) + M_2x(b) = \\
&= M_1F^{-1}v - M_1F^{-1}N_2\int_{a_1}^{b_1}\dots\int_{a_q}^{b_q}\left(\prod_{i=1}^{q}\Phi_i(b_i,s_i)\right)B(s)u(s)ds+ \\
&+M_2\left(\prod_{i=1}^{q}\Phi_i(b_i,a_i)\right)F^{-1}v- \\
&-M_2\int_{a_1}^{b_1}\dots\int_{a_q}^{b_q}\left(\prod_{i=1}^{q}\Phi_i(b_i,a_i)\right)P\left(\prod_{i=1}^{q}\Phi_i(a_i,s_i)\right)B(s)u(s)ds+ \\
&+M_2\int_{a_1}^{b_1}\dots\int_{a_q}^{b_q}\left(\prod_{i=1}^{q}\Phi_i(b_i,s_i)\right)B(s)u(s)ds = \left(M_1 + M_2\left(\prod_{i=1}^{q}\Phi_i(b_i,a_i)\right)\right)F^{-1}v- \\
&-M_1F^{-1}N_2\left(\prod_{i=1}^{q}\Phi_i(b_i,a_i)\right)\int_{a_1}^{b_1}\dots\int_{a_q}^{b_q}\left(\prod_{i=1}^{q}\Phi_i(a_i,s_i)\right)B(s)u(s)ds+ \\
&+M_2\left(\prod_{i=1}^{q}\Phi_i(b_i,a_i)\right)P\int_{a_1}^{b_1}\dots\int_{a_q}^{b_q}\left(\prod_{i=1}^{q}\Phi_i(a_i,s_i)\right)B(s)u(s)ds+ \\
&+M_2\int_{a_1}^{b_1}\dots\int_{a_q}^{b_q}\left(\prod_{i=1}^{q}\Phi_i(b_i,s_i)\right)B(s)u(s)ds.
\end{aligned}$$

3. Adjoint systems

Let us consider the qD acausal system $\Sigma = (\{A_i | i \in \bar{q}\}, B, C, D, N, N_2, M_1, M_2)$ (2.1)-(2.4) with well-posed boundary condition (2.3) and the canonical boundary value operator P (2.8). Assume that the matrix $Q = \begin{bmatrix} N_1 & N_2 \\ M_1 & M_2 \end{bmatrix}$ is nonsingular and partition its inverse as $Q^{-1} = \begin{bmatrix} Q_1 & Q_2 \\ Q_3 & Q_4 \end{bmatrix}$.

From $Q^{-1}Q = I_{2n}$ and $QQ^{-1} = I_{2n}$ we obtain:

$$Q_1 N_1 + Q_2 M_1 = Q_3 N_2 + Q_4 M_2 = I_n, \tag{3.1}$$

$$Q_1 N_2 + Q_2 M_2 = Q_3 N_1 + Q_4 M_1 = O_n, \tag{3.2}$$

$$N_1 Q_1 + N_2 Q_3 = M_1 Q_2 + M_2 Q_4 = I_n, \tag{3.3}$$

$$N_1 Q_2 + N_2 Q_4 = M_1 Q_1 + M_2 Q_3 = O_n. \tag{3.4}$$

Lemma 3.1. *The matrix* $\hat{F} = -Q_2 + \left(\prod_{i=1}^{q} \Phi_i(a_i, b_i)\right) Q_4$ *is nonsingular.*

Proof. Let us assume that $\hat{F}$ is singular. Then $\exists v \in \mathbf{R}^n$, $v \neq 0$ such that

$$-Q_2 v + \left(\prod_{i=1}^{q} \Phi_i(a_i, b_i)\right) Q_4 v = 0. \tag{3.5}$$

Since Q^{-1} is nonsingular, rank $\begin{bmatrix} Q_2 \\ Q_4 \end{bmatrix} = n$, hence at least one of the products $Q_2 v$ or $Q_4 v$ is nonzero. If $Q_4 v \neq 0$ we premultiply (3.5) by N_1 and we get by (3.4) the following equalities:

$$0 = -N_1 Q_2 v + N_1 \left(\prod_{i=1}^{q} \Phi_i(a_i, b_i)\right) Q_4 v = N_2 Q_4 v + N_1 \left(\prod_{i=1}^{q} \Phi_i(a_i, b_i)\right) Q_4 v =$$

$$= \left(N_1 + N_2 \left(\prod_{i=1}^{q} \Phi_i(a_i, b_i)\right)\right) \left(\prod_{i=1}^{q} \Phi_i(a_i, b_i)\right) Q_4 v$$

and since $\left(\prod_{i=1}^{q} \Phi_i(a_i, b_i)\right)$ is nonsingular we have $w := \left(\prod_{i=1}^{q} \Phi_i(a_i, b_i)\right) Q_4 v \neq 0$; it results that the matrix $F = N_1 + N_2 \left(\prod_{i=1}^{q} \Phi_i(a_i, b_i)\right)$ is singular, contradiction of the assumption that Σ is with well-posed boundary condition. The case $Q_2 v = 0$ can similarly be treated.

Let us denote by $\hat{P}$ the *adjoint operator*

$$\hat{P} = \left(\prod_{i=1}^{q} \Phi_i(a_i, b_i)\right) Q_4 \hat{F}^{-1}. \tag{3.6}$$

Lemma 3.2. *The following equalities hold*:

$$\left(M_1 + M_2\left(\prod_{i=1}^{q}\Phi_i(b_i,a_i)\right)\right)F^{-1} = \hat{F}^{-1}\left(Q_1 - \left(\prod_{i=1}^{q}\Phi_i(a_i,b_i)\right)\right)Q_3, \tag{3.7}$$

$$F^{-1} = \left(\prod_{i=1}^{q}\Phi_i(a_i,b_i)\right)Q_3 + \hat{P}\left(Q_1 - \left(\prod_{i=1}^{q}\Phi_i(a_i,b_i)\right)Q_3\right), \tag{3.8}$$

$$\hat{F}^{-1} = M_2\left(\prod_{i=1}^{q}\Phi_i(b_i,a_i)\right) - \left(M_1 + M_2\left(\prod_{i=1}^{q}\Phi_i(a_i,b_i)\right)\right)P, \tag{3.9}$$

$$P + \hat{P} = I_n. \tag{3.10}$$

Proof. Taking into account the expressions of F and $\hat{F}$, (3.7) is equivalent to

$$\left(-Q_2 + \left(\prod_{i=1}^{q}\Phi_i(a_i,b_i)\right)Q_4\right)\left(M_1 + M_2\prod_{i=1}^{q}\Phi_i(b_i,a_i)\right) =$$
$$= \left(Q_1 - \left(\prod_{i=1}^{q}\Phi_i(a_i,b_i)\right)Q_3\right)\left(N_1 + N_2\prod_{i=1}^{q}\Phi_i(b_i,a_i)\right)$$

and this equality results by applying (3.1) and (3.2).

By (3.6) and (3.7), the right hand member of (3.8) can be written as

$$\left(\prod_{i=1}^{q}\Phi_i(a_i,b_i)\right)Q_3 + \left(\prod_{i=1}^{q}\Phi_i(a_i,b_i)\right)Q_4\left(M_1 + M_2\left(\prod_{i=1}^{q}\Phi_i(b_i,a_i)\right)\right)F^{-1}.$$

But by (3.2) and (3.1) $Q_4M_1 = -Q_3N_1$ and $Q_4M_2 = I - Q_3N_2$, hence taking into account the definition of F it becomes:

$$\left(\prod_{i=1}^{q}\Phi_i(a_i,b_i)\right)\left(Q_3F - Q_3\left(N_1 + N_2\prod_{i=1}^{q}\Phi_i(b_i,a_i)\right) + \prod_{i=1}^{q}\Phi_i(b_i,a_i)\right)F^{-1} =$$
$$= \left(\prod_{i=1}^{q}\Phi_i(a_i,b_i)\right)\left(\prod_{i=1}^{q}\Phi_i(b_i,a_i)\right)F^{-1} = F^{-1}.$$

Similarly we can prove (3.9).

Finally, by (3.8), (3.2), (3.1), (3.6) and again by (3.1) we get

$$P + \hat{P} = F^{-1} N_2 \prod_{i=1}^{q} \Phi_i(b_i, a_i) + \hat{P} =$$
$$= \left(\prod_{i=1}^{q} \Phi_i(a_i, b_i) \right) Q_3 N_2 \prod_{i=1}^{q} \Phi_i(b_i, a_i) + \hat{P} \left(Q_1 N_2 \prod_{i=1}^{q} \Phi_i(b_i, a_i) - \right.$$
$$\left. - \left(\prod_{i=1}^{q} \Phi_i(a_i, b_i) \right) Q_3 N_2 \prod_{i=1}^{q} \Phi_i(b_i, a_i) \right) + \hat{P} =$$
$$= \left(\prod_{i=1}^{q} \Phi_i(a_i, b_i) \right) Q_3 N_2 \prod_{i=1}^{q} \Phi_i(b_i, a_i) +$$
$$+ \hat{P} \left(-Q_2 + \left(\prod_{i=1}^{q} \Phi_i(a_i, b_i) \right) Q_4 \ M_2 \prod_{i=1}^{q} \Phi_i(b_i, a_i) - \right.$$
$$\left. - \left(\prod_{i=1}^{q} \Phi_i(a_i, b_i) \right) \prod_{i=1}^{q} \Phi_i(b_i, a_i) \right) + \hat{P} =$$
$$= \prod_{i=1}^{q} \Phi_i(a_i, b_i)(Q_3 N_2 + Q_4 M_2) \prod_{i=1}^{q} \Phi_i(b_i, a_i) - \hat{P} + \hat{P} =$$
$$= \left(\prod_{i=1}^{q} \Phi_i(a_i, b_i) \right) \prod_{i=1}^{q} \Phi_i(b_i, a_i) = I.$$

Definition 3.3. The system $\hat{\Sigma} = (\{\hat{A}_i | i \in \bar{q}\}, \hat{B}, \hat{C}, \hat{D}, \hat{N}_1, \hat{N}_2, \hat{M}_1, \hat{M}_2)$ with $\hat{A}_i = -A_i^*$, $\hat{B} = -C^*$, $\hat{C} = B^*$, $\hat{D} = D^*$, $\hat{N}_1 = -Q_2^*$, $\hat{N}_2 = Q_4^*$, $\hat{M}_1 = Q_1^*$ and $\hat{M}_2 = -Q_3^*$ is called the *adjoint of the system* Σ.

Let us denote the input and the output of $\hat{\Sigma}$ by $(\hat{u}(\cdot), \hat{v})$ and $(\hat{y}(\cdot), \hat{z})$ respectively and by $\hat{T}$ the input-output map of $\hat{\Sigma}$, i.e. $\hat{T}(\hat{u}, \hat{v}) = (\hat{y}, \hat{z})$.

Theorem 3.4. *The inputs and the outputs of the system* Σ *and of its adjoint* $\hat{\Sigma}$ *verify the relation*

$$\begin{aligned} &\hat{z}^* v + \int_{a_1}^{b_1} \dots \int_{a_q}^{b_q} \hat{y}(t_1, \dots, t_q)^* u(t_1, \dots, t_q) dt_1 \dots dt_q = \\ &= \hat{v}^* z + \int_{a_1}^{b_1} \dots \int_{a_q}^{b_q} \hat{u}(t_1, \dots, t_q)^* y(t_1, \dots, t_q) dt_1 \dots dt_q \end{aligned} \tag{3.11}$$

Proof. We apply (2.12) and (2.13) to the adjoint system $\hat{\Sigma}$. since the fundamental matrix of the matrix $-A^*$ is related to the matrix A by $\Phi_{-A^*}(t, s) = \Phi_A(s, t)^*$, we obtain the output

$$\begin{aligned}\hat{y}(t)^* &= \hat{v}^*\hat{F}^{-1}\left(\prod_{i=1}^{q}\Phi_i(a_i,t_i)\right)B(t)+\\ &+ \int_a^b \hat{u}(s)^*C(s)\left(\prod_{i=1}^{q}\Phi_i(s_i,a_i)\right)\hat{P}\left(\prod_{i=1}^{q}\Phi_i(a_i,t_i)\right)B(t)ds-\\ &- \int_a^t \hat{u}(s)^*C(s)\left(\prod_{i=1}^{q}\Phi_i(s_i,t_i)\right)B(t)ds+\hat{u}(t)^*D(t),\end{aligned} \tag{3.12}$$

$$\begin{aligned}\hat{z}^* &= \hat{v}^*\hat{F}^{-1}\left(Q_1-\left(\prod_{i=1}^{q}\Phi_i(a_i,b_i)\right)Q_3\right)+\\ &+ \int_a^b \hat{u}(s)^*C(s)\left(\prod_{i=1}^{q}\Phi_i(s_i,a_i)\right)\hat{P}\left(Q_1-\left(\prod_{i=1}^{q}\Phi_i(a_i,b_i)\right)Q_3\right)ds+\\ &+ \int_a^t \hat{u}(s)^*C(s)\left(\prod_{i=1}^{q}\Phi_i(s_i,b_i)\right)Q_3ds.\end{aligned} \tag{3.13}$$

By (2.12) and (2.13) the two members of (3.11) $\hat{z}^*v+\int_a^b \hat{y}(t)^*u(t)dt$ and $\hat{v}^*z+\int_a^b \hat{u}(t)^*y(t)dt$ are equal to the sums of seven terms S_1-S_7 and S_8-S_{14} respectively, where

$$S_1=\hat{v}^*\hat{F}^{-1}\left(Q_1-\left(\prod_{i=1}^{q}\Phi_i(a_i,b_i)\right)Q_3\right)v,$$

$$S_2=\int_a^b \hat{u}(s)^*C(s)\left(\prod_{i=1}^{q}\Phi_i(s_i,a_i)\right)\hat{P}\left(Q_1-\left(\prod_{i=1}^{q}\Phi_i(a_i,b_i)\right)Q_3\right)v\,ds,$$

$$S_3=\int_a^b \hat{u}(s)^*C(s)\left(\prod_{i=1}^{q}\Phi_i(s_i,b_i)\right)Q_3v\,ds,$$

$$S_4=\hat{v}^*\hat{F}^{-1}\int_a^b\left(\prod_{i=1}^{q}\Phi_i(a_i,t_i)\right)B(t)u(t)dt,$$

$$S_5=\int_a^b\left(\int_a^b \hat{u}(s)^*C(s)\left(\prod_{i=1}^{q}\Phi_i(s_i,a_i)\right)\hat{P}\left(\prod_{i=1}^{q}\Phi_i(a_i,t_i)\right)B(t)u(t)ds\right)dt,$$

$$S_6=-\int_a^b\left(\int_a^t \hat{u}(s)^*C(s)\left(\prod_{i=1}^{q}\Phi_i(s_i,t_i)\right)B(t)u(t)ds\right)dt,$$

$$S_7=\int_a^b \hat{u}(t)^*D(t)u(t)dt,\quad S_8=\hat{v}^*\left(M_1+M_2\prod_{i=1}^{q}\Phi_i(b_i,a_i)\right)F^{-1}v,$$

$$S_9 = -\hat{v}^* \left(M_1 + M_2 \prod_{i=1}^{q} \Phi_i(b_i, a_i) \right) P \int_a^b \left(\prod_{i=1}^{q} \Phi_i(a_i, s_i) \right) B(s)u(s)ds,$$

$$S_{10} = \hat{v}^* M_2 \int_a^b \left(\prod_{i=1}^{q} \Phi_i(b_i, s_i) \right) B(s)u(s)ds,$$

$$S_{11} = \int_a^b \hat{u}(t)^* C(t) \left(\prod_{i=1}^{q} \Phi_i(t_i, a_i) \right) F^{-1} v \, dt,$$

$$S_{12} = -\int_a^b \hat{u}(t)^* \left(\int_a^b C(t) \left(\prod_{i=1}^{q} \Phi_i(t_i, a_i) \right) P \left(\prod_{i=1}^{q} \Phi_i(a_i, s_i) \right) B(s)u(s)ds \right) dt,$$

$$S_{13} = \int_a^b \hat{u}(t)^* \left(\int_a^t C(t) \left(\prod_{i=1}^{q} \Phi_i(t_i, s_i) \right) B(s)u(s)ds \right) dt,$$

$$S_{14} = \int_a^b \hat{u}(t)^* D(t)u(t)dt.$$

Obviously $S_7 = S_{14}$ and $S_1 = S_8$ by (3.7). By (3.9) we have

$$\begin{aligned} S_9 + S_{10} &= \hat{v}^* \left(M_2 \prod_{i=1}^{q} \Phi_i(b_i, a_i) - \right. \\ &- \left. \left(M_1 + M_2 \prod_{i=1}^{q} \Phi_i(b_i, a_i) \right) P \right) \int_a^b \left(\prod_{i=1}^{q} \Phi_i(a_i, s_i) \right) B(s)u(s)ds = \\ &= \hat{v}^* \hat{F}^{-1} \int_a^b \left(\prod_{i=1}^{q} \Phi_i(a_i, s_i) \right) B(s)u(s)ds, \end{aligned}$$

hence $S_9 + S_{10} = S_4$. Similarly (3.8) implies $S_2 + S_3 = S_{11}$. By (3.10),

$$\begin{aligned} S_5 + S_6 &= \int_a^b \left(\int_a^b \hat{u}(s)^* C(s) \left(\prod_{i=1}^{q} \Phi_i(s_i, t_i) \right) B(t)u(t)ds \right) dt - \\ &- \int_a^b \left(\int_a^b \hat{u}(s)^* C(s) \left(\prod_{i=1}^{q} \Phi_i(s_i, a_i) \right) P \left(\prod_{i=1}^{q} \Phi_i(a_i, t_i) \right) B(t)u(t)ds \right) dt - \\ &- \int_a^b \left(\int_a^b \hat{u}(s)^* C(s) \left(\prod_{i=1}^{q} \Phi_i(s_i, t_i) \right) B(t)u(t)ds \right) dt. \end{aligned}$$

The second integral is S_{12} and, by changing the variables t and s and the order of integration in S_{13}, it results that $S_5 + S_6 = S_{12} + S_{13}$, hence the two members of (3.11) are equal.

Let us denote by $\langle \cdot, \cdot \rangle_1$ and $\langle \cdot, \cdot \rangle_2$ the inner products on $L^2([a, b], U) \times \mathbf{R}^n$ and $L^2([a, b], Y) \times \mathbf{R}^n$ respectively, where

$$\langle (\hat{y}, \hat{z}), (u, v) \rangle_1 = \hat{z}^* v + \int_{a_1}^{b_1} \dots \int_{a_2}^{b_2} \hat{y}(t_1, \dots, t_q)^* u(t_1, \dots, t_q) dt_1 \dots dt_q$$

and $\langle\cdot,\cdot\rangle_2$ has a similar definition. Then Theorem 3.4 can be restated as follows:

Corollary 3.5. *For any* qD *acausal system with well-posed boundary condition* Σ *and its adjoint* $\hat{\Sigma}$, *the following equality holds*:

$$\langle \hat{T}(\hat{u},\hat{v}),(u,v)\rangle_1 = \langle (\hat{u},\hat{v}),T(u,v)\rangle_2.$$

4. Reduced adjoint systems

Let us consider the system Σ (2.1), (2.2) with the well-posed boundary condition (2.3) (that is without the output equation (2.4)).

Definition 4.1. The (qD) system ($\hat{\Sigma}_r$) having the representation

$$\frac{\partial}{\partial t}\hat{x}(t) = \sum_{\tau\subset\bar{q}}(-1)^q\left(\prod_{i\in\tilde{\tau}}A_i(t_i)^*\right)\frac{\partial}{\partial\tau}x(t) - C(t)^*\hat{u}(t) \tag{4.1}$$

$$\hat{y}(t) = B(t)^*\hat{x}(t) + D(t)^*\hat{u}(t) \tag{4.2}$$

$$\hat{x}(a) = N_1^*\lambda, \quad \hat{x}(b) = -N_2^*\lambda \tag{4.3}$$

is called the *reduced adjoint of the system* Σ.

Proposition 4.2. The state of the reduced adjoint system $\hat{\Sigma}_r$ determined by the control $\hat{u}$ is

$$\begin{aligned}\hat{x}(t)^* \;&=\; \int_{\widetilde{[a,t]}}\hat{u}(s)^*C(s)\left(\prod_{i=1}^q\Phi_i(s_i,t_i)\right)ds- \\ &-\; \int_a^b \hat{u}(s)^*C(s)\left(\prod_{i=1}^q\Phi_i(s_i,a_i)\right)P\left(\prod_{i=1}^q\Phi_i(a_i,t_i)\right)ds.\end{aligned} \tag{4.4}$$

Proof. Equation (4.1) is similar to (2.1) with $-C(t)^*$ instead of $B(t)$ and $-A_i(t_i)^*$ instead of $A_i(t_i)$. Since $\Phi_{-A_i^*}(t_i,s_i) = \Phi_{A_i}(s_i,t_i)^*$, by (2.9) we obtain

$$\hat{x}(t)^* = \hat{x}(a)^*\prod_{i=1}^q\Phi_i(a_i,t_i) - \int_a^t \hat{u}(s)^*C(s)\left(\prod_{i=1}^q\Phi_i(s_i,t_i)\right)ds. \tag{4.5}$$

We replace t by b in (4.5) and (4.3) gives:

$$-\lambda^*N_2 = \hat{x}(b)^* = \lambda^*N_1\prod_{i=1}^q\Phi_i(a_i,b_i) - \int_a^b \hat{u}(s)^*C(s)\left(\prod_{i=1}^q\Phi_i(s_i,b_i)\right)ds. \tag{4.6}$$

We postmultiply (4.6) by $\prod_{i=1}^q\Phi_i(b_i,a_i)$ and we obtain

$$\lambda^*\left(N_1 + N_2\prod_{i=1}^q\Phi_i(a_i,b_i)\right) = \int_a^b \hat{u}(s)^*C(s)\left(\prod_{i=1}^q\Phi_i(s_i,a_i)\right)ds,$$

hence

$$\lambda^* = \int_a^b \hat{u}(s)^* C(s) \left(\prod_{i=1}^{q} \Phi_i(s_i, a_i) \right) F^{-1} ds. \tag{4.7}$$

Therefore

$$\hat{x}(b)^* = -\lambda^* N_2 = -\int_a^b \hat{u}(s)^* C(s) \left(\prod_{i=1}^{q} \Phi_i(s_i, a_i) \right) F^{-1} N_2 ds.$$

This is equal to $\hat{x}(b)^*$ given by (4.5) with $t = b$, hence

$$\begin{aligned} \hat{x}(a)^* \prod_{i=1}^{q} \Phi_i(a_i, b_i) - \int_a^b \hat{u}(s)^* C(s) \left(\prod_{i=1}^{q} \Phi_i(s_i, b_i) \right) ds = \\ = -\int_a^b \hat{u}(s)^* C(s) \left(\prod_{i=1}^{q} \Phi_i(s_i, a_i) \right) F^{-1} N_2 ds. \end{aligned} \tag{4.8}$$

We postmultiply (4.8) by $\prod_{i=1}^{q} \Phi_i(b_i, a_i)$ and we obtain

$$\hat{x}(a)^* = \int_a^b \hat{u}(s)^* C(s) \left(\prod_{i=1}^{q} \Phi_i(s_i, a_i) \right) ds - \int_a^b \hat{u}(s)^* C(s) \left(\prod_{i=1}^{q} \Phi_i(s_i, a_i) \right) P ds,$$

hence, by replacing $\hat{x}(a)^*$ in (4.5) we get

$$\begin{aligned} \hat{x}(t)^* &= \int_a^b \hat{u}(s)^* C(s) \left(\prod_{i=1}^{q} \Phi_i(s_i, t_i) \right) ds - \\ &- \int_a^b \hat{u}(s)^* C(s) \left(\prod_{i=1}^{q} \Phi_i(s_i, a_i) \right) P \left(\prod_{i=1}^{q} \Phi_i(a_i, t_i) \right) ds - \\ &- \int_a^t \hat{u}(s)^* C(s) \left(\prod_{i=1}^{q} \Phi_i(s_i, t_i) \right) ds \end{aligned}$$

i.e. (4.4).

Theorem 4.3. *The input-output map of the reduced adjoint system* $\hat{\Sigma}_r$ *is the operator* $\hat{T}_r : L^2([a,b], Y) \to L^2([a,b], U) \times \mathbf{R}^n$, $\hat{T}_r(\hat{u}(\cdot)) = (\hat{y}(\cdot), \lambda)$ *where*

$$\begin{aligned} \hat{y}(t)^* &= \int_{\widetilde{[a,t]}} \hat{u}(s)^* C(s) \left(\prod_{i=1}^{q} \Phi_i(s_i, t_i) \right) B(t) ds - \\ &- \int_a^b \hat{u}(s)^* C(s) \left(\prod_{i=1}^{q} \Phi_i(s_i, a_i) \right) P \left(\prod_{i=1}^{q} \Phi_i(a_i, t_i) \right) B(t) ds + \\ &+ \hat{u}(t)^* D(t) \end{aligned} \tag{4.9}$$

and

$$\lambda^* = \int_a^b \hat{u}(s)^* C(s) \left(\prod_{i=1}^q \Phi_i(s_i, a_i) \right) F^{-1} ds.$$

Proof. We obtain (4.9) by replacing the state $\hat{x}(t)$ (4.4) in the output equation (4.2). The expression of λ^* was obtained in (4.7).

Theorem 4.4. *The inputs and the outputs of the system* Σ *and of its reduced adjoint* $\hat{\Sigma}_r$ *verify the relation*

$$\begin{aligned} &\int_{a_1}^{b_1} \dots \int_{a_q}^{b_q} \hat{y}(t_1, \dots, t_q)^* u(t_1, \dots, t_q) dt_1, \dots dt_q + \lambda^* v = \\ &= \int_{a_1}^{b_1} \dots \int_{a_q}^{b_q} \hat{u}(t_1, \dots, t_q)^* y(t_1, \dots, t_q) dt_1 \dots dt_q \end{aligned} \tag{4.10}$$

Proof. By (4.9), (4.7) and (2.12) we can write the members of (4.10) as $\int_a^b \hat{y}(t)^* u(t) dt + \lambda^* v = S_1 + S_2 + S_3 + S_4$ and $\int_a^b \hat{u}(t)^* y(t) dt = S_5 + S_6 + S_7 + S_8$ where

$$\begin{aligned} S_1 &= \int_a^b \hat{u}(s)^* C(s) \left(\prod_{i=1}^q \Phi_i(s_i, a_i) \right) F^{-1} v \, ds, \\ S_2 &= \int_a^b \left(\int_{\widetilde{[a,t]}} \hat{u}(s)^* C(s) \left(\prod_{i=1}^q \Phi_i(s_i, t_i) \right) B(t) u(t) ds \right) dt, \\ S_3 &= -\int_a^b \left(\int_a^b \hat{u}(s)^* C(s) \left(\prod_{i=1}^q \Phi_i(s_i, a_i) \right) P \left(\prod_{i=1}^q \Phi_i(a_i, t_i) \right) B(t) u(t) ds \right) dt, \\ S_4 &= \int_a^b \hat{u}(t)^* D(t) u(t) dt, \\ S_5 &= \int_a^b \hat{u}(t)^* C(t) \left(\prod_{i=1}^q \Phi_i(t_i, a_i) \right) F^{-1} v \, dt, \\ S_6 &= -\int_a^b \left(\int_a^b \hat{u}(t)^* C(t) \left(\prod_{i=1}^q \Phi_i(t_i, a_i) \right) P \left(\prod_{i=1}^q \Phi_i(a_i, s_i) \right) B(s) u(s) ds \right) dt, \\ S_7 &= \int_a^b \left(\int_a^t C(t) \left(\prod_{i=1}^q \Phi_i(t_i, s_i) \right) B(s) u(s) ds \right) dt \\ S_8 &= \int_a^b \hat{u}(t)^* D(t) u(t) dt. \end{aligned}$$

Obviously, $S_1 = S_5$, $S_3 = S_6$, and $S_4 = S_8$. By changing the order of integration in S_2, we obtain

$$S_2 = \int_a^b \left(\int_a^t \hat{u}(s)^* C(s) \left(\prod_{i=1}^q \Phi_i(s_i, t_i) \right) B(t) u(t) dt \right) ds$$

and by changing the roles of s and t it results that $S_2 = S_7$, which proves (4.10).

Now, we denote by $\langle \cdot, \cdot \rangle_3$ and $\langle \cdot, \cdot \rangle_4$ the inner products on $L^2([a,b],U) \times \mathbf{R}^n$ and $L^2([a,b],Y)$ respectively, defined by

$$\langle (u,v), (\hat{y}, \lambda) \rangle_3 = \int_a^b \hat{y}(t)^* u(t) dt + \lambda^* v$$

and

$$\langle y, \hat{u} \rangle_4 = \int_a^b \hat{u}(t)^* y(t) dt.$$

Then Theorem 4.4 can be restated as follows:

Corollary 4.5. *For any* qD *acausal system with well-posed boundary condition and for its reduced adjoint* $\hat{\Sigma}_r$*, the following equality holds*:

$$\langle \hat{T}_r(\hat{u}), (u,v) \rangle_3 = \langle \hat{u}, T(u,v) \rangle_4.$$

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Valeriu Prepeliţă
Department of Mathematics and Informatics I,
Faculty of Applied Sciences,
University Politehnica of Bucharest,
313 Splaiul Independentei st., 060042, Bucharest, Romania.
E-mail: vprepelita@mathem.pub.ro