Deformation of three real scalar fields with defect structure solution

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Abstract. In this paper, we consider three coupled scalar fields with super-potential $W(\phi_1, \phi_2, \phi_3)$. We deform the corresponding super-potential and obtain the defect solutions. Then, we compare the deformed and non-deformed solutions with considering amount of parameter r. Therefore, we draw the graph of super-potential and fields in terms of x and observe that the graphs for deformed and non-deformed cases are changed by the scale.

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1 Introduction

As we know the defect structures applied in different branches of physics, such as domain wall, kinks, vortices, monopoles, condensed matter and string theory, and in higher - dimension (D,1) space- time they are solved by real scalar fields. The defects in physics can be support by fields, for example the single real scalar field support just single defect as kink-like and the double sin-Gordon model may support two different defect. On the other hand, models containing two or more real scalar fields give rise to at least two other classes of systems those that support defect that engender internal structure and those that support junctions of defects. Also two and three scalar fields describe the regular hexagonal network, Higgs model [10, 9, 6, 4, 13] and bent brane in five dimensions [11]. Most of work has been done in two scalar fields system, such as brane scenario, inflation in cosmology and quintom model of dark energy. As we know the deformation method for the single and two coupled scalar fields are discussed by Bazeia. But we are going to consider three coupled scalar fields in (1,1) dimensions in flat space-time, where metric is $\eta_{\mu\nu} = (1, -1)$ [10, 9, 6, 4]. And also the corresponding equations of motion reduce to first - order differential equation, the static solutions such as topological solutions for stable states has been discussed as Bogomol'nyi-Prasad-Sommerfield (BPS) in Refs.[5, 7].

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In this paper, we use deformation procedure in defect solutions. This deformation play important role to the energy of systems. The cosmological parameters such as energy density, pressure and equation of states can be controlled by the deformation method on the fields .
These all give us motivation to discuss the deformation procedure, so the outline of paper as follows; In the next section three coupled scalar fields with example are discussed. The analysis of the solution without deformation are shown. The deformation of procedure for the three coupled scalar field with diagram are discussed in Sec.3. Also we compare two diagram to each other and show that the variation of fields with respect to coordinates are changed by scale. These never change the physical phenomena such as energy spectrum.

2 Three coupled scalar fields system

We are going to introduce the Lagrangian's density by three coupled scalar fields,

$$L = \frac{1}{2}\partial_\mu\phi_1\partial^\mu\phi_1 + \frac{1}{2}\partial_\mu\phi_2\partial^\mu\phi_2 + \frac{1}{2}\partial_\mu\phi_3\partial^\mu\phi_3 - U(\phi_1,\phi_2,\phi_3), \tag{2.1}$$

where $U = U(\phi_1,\phi_2,\phi_3)$ is the potential. By using the Euler-Lagrange equations, one can obtain equations of motion,

$$\begin{aligned} \frac{\partial^2\phi_1}{\partial x^2} &= \frac{dU}{d\phi_1} \\ \frac{\partial^2\phi_2}{\partial x^2} &= \frac{dU}{d\phi_2} \\ \frac{\partial^2\phi_3}{\partial x^2} &= \frac{dU}{d\phi_3}. \end{aligned} \tag{2.2}$$

In order to obtain solution for the equation (2.2), we define super-potential function $W = W(\phi_1,\phi_2,\phi_3)$ such that one may write the potential in terms of super potential,

$$U(\phi_1,\phi_2,\phi_3) = \frac{1}{2}W_{\phi_1}^2 + \frac{1}{2}W_{\phi_2}^2 + \frac{1}{2}W_{\phi_3}^2, \tag{2.3}$$

where

$$W_{\phi_i} = \frac{\partial W}{\partial \phi_i}, \qquad i = 1,2,3. \tag{2.4}$$

The energy spectrum associated with these configurations could be written as [8, 2],

$$E = \frac{1}{2}\int_{-\infty}^{+\infty} dx \sum_{i=1}^{3}\left[(\frac{d\phi_i}{dx})^2 + W_{\phi_i}^2\right], \tag{2.5}$$

we can rewrite also it as a following,

(2.6) $$E = E_{BPS} + \frac{1}{2}\int_{-\infty}^{+\infty} dx \sum_{i=1}^{3}\left[\frac{d\phi_i}{dx} - W_{\phi_i}\right]^2.$$

In here we set the BPS energy,

(2.7) $$E_{BPS} = |\Delta W| = |W(\phi_i(\infty)) - W(\phi_i(-\infty))|.$$

This procedure shows that the energy is minimized to,

(2.8) $$E = E_{BPS}.$$

Also we can obtain the first - order equation as,

(2.9) $$\frac{d\phi_i}{dx} = W_{\phi_i}, \qquad i = 1, 2, 3.$$

Now we are going to consider special example of three coupled scalar fields which describe the regular hexagonal network. By using the first - order deferential equations, we solve this system. In that case we introduce the corresponding super-potential,

(2.10) $$W(\phi_1, \phi_2, \phi_3) = \phi_1 - \frac{\phi_1^3}{3} - r\phi_1(\phi_2^2 + \phi_3^2),$$

and use equations (2.9) and (2.10), we have the following equation,

$$\begin{aligned}\frac{d\phi_1}{dx} &= 1 - \phi_1^2 - r(\phi_2^2 + \phi_3^2)\\ \frac{d\phi_2}{dx} &= -2r\phi_1\phi_2\\ \frac{d\phi_3}{dx} &= -2r\phi_1\phi_3\end{aligned}$$

(2.11) .

As we know the exact solution for the above equations not clear, so, we use the following elliptical orbit procedure [11, 8, 3],

(2.12) $$\phi_1^2 + \frac{\phi_2^2}{\frac{1}{r} - 2} + \frac{\phi_3^2}{\frac{1}{r} - 2} = 1,$$

finally we will arrive at,

$$\begin{aligned}\phi_1(x) &= \pm\tanh(2rx)\\ \phi_2(x) &= \pm\sqrt{\frac{1}{r} - 2}\cos(\theta)sech(2rx)\\ \phi_3(x) &= \pm\sqrt{\frac{1}{r} - 2}\sin(\theta)sech(2rx)\end{aligned}$$

(2.13) ,

where θ is the arbitrary phase.
We note that there are several orbit for solving of equation (2.11). But here we consider the condition $0 < r < \frac{1}{2}$ and orbit equation (2.12).

By putting equation (2.13) in (2.10) and (2.3) the corresponding super-potential and potential in terms of x are respectively,

$$W(x) = \frac{2}{3} tanh(2rx)[1 + (3r - 1)sech^2(2rx)], \tag{2.14}$$

$$U(x) = 2rsech^2(2rx)[(3r - 1)sech^2(2rx) + 1 - 2r]. \tag{2.15}$$

3 Deformation procedure for three scalar fields

We are going to apply the deformation procedure for three couple scalar fields. As we know the deformation method for two and single field discussed by Ref.s. [8, 1, 12]. First we transform three initial scalar fields $\phi_1(x)$, $\phi_2(x)$ and $\phi_3(x)$ to form of scalar fields $\chi_1(x)$, $\chi_2(x)$ and $\chi_3(x)$ respectively. In order to deform three scalar fields we introduce the following deformed function,

$$\phi_i(x) = f_i(\chi_i), \qquad i = 1, 2, 3, \tag{3.1}$$

where non-deformed function $\phi_i(x)$ and deformed function $\chi_i(x)$ are differentiable and invertible. We can write inverse function as $\chi_i(x) = f_i^{-1}(\phi_i)$ for $i = 1, 2, 3$. For simplicity the deformed functions are just functional of single field. The essential condition for the deformation functions are given by,

$$\frac{df_i(\chi_i)}{d\chi_i} = constant, \qquad i = 1, 2, 3. \tag{3.2}$$

This equation shows us that the variation of non-deformed functions equivalent to the deformed functions. Finally the deformed superpotential $\mathcal{W}(\chi_1, \chi_2, \chi_3)$ from eqs. (2.4) and (2.9) will be as,

$$\mathcal{W} = \int \frac{W_{\phi_i}}{\frac{df_i(\chi_i)}{d\chi_i}} d\chi_i. \tag{3.3}$$

With the help of essential condition and eq. (2.3) we have,

$$\mathcal{U} = \frac{U}{(\frac{df_i(\chi_i)}{d\chi_i})^2}, \tag{3.4}$$

where $\mathcal{U}$ deformed potential in terms of deformed superpotential is,

$$\mathcal{U} = \sum_{i=1}^{3} \frac{1}{2} \mathcal{W}_{\chi_i}^2. \tag{3.5}$$

The energy of deformed defects can be written by following expression,

$$\mathcal{E}_{BPS} = \Delta\mathcal{W}|_{-\infty}^{+\infty}, \tag{3.6}$$

where $\mathcal{E}$ is deformed BPS energy .

Now we apply deformation method for super-potential (2.10). To starting we introduce the deformed function χ_1,

$$\phi_1 = f_1(\chi_1) = \tan(\chi_1), \tag{3.7}$$

and from eq. (2.9) χ_1 is given by,

$$\chi_1 = arctan(\phi_1). \tag{3.8}$$

By using the following expression

$$\frac{\phi_3}{\phi_2} = c = \tan\theta$$

the orbit equation (2.12) will be as,

$$\phi_1^2 + \frac{(1+c^2)}{\frac{1}{r}-2}\phi_2^2 = 1. \tag{3.9}$$

In order to obtain the deformation fields χ_2 and χ_3, we use eq. (2.15), so we have,

$$\chi_2 = \int \frac{d\chi_1}{d\phi_1} d\phi_2, \qquad \chi_3 = \int \frac{d\chi_1}{d\phi_1} d\phi_3. \tag{3.10}$$

Here we use equations (3.9), (3.10) and (2.13) the corresponding deformed fields χ_2 and χ_3 are respectively,

$$\chi_2 = \sqrt{\frac{1}{2r} - 1} cos(\theta) arctanh(\frac{1}{\sqrt{2}} sech(2rx)), \tag{3.11}$$

$$\chi_3 = \sqrt{\frac{1}{2r} - 1}\, \sin(\theta) arctanh(\frac{1}{\sqrt{2}} sech(2rx)), \tag{3.12}$$

also the deformation functions will be following,

$$\phi_2 = f_2(\chi_2) = \sqrt{2(\frac{1}{r} - 2)}\, \cos(\theta) tanh(\frac{sec(\theta)}{\sqrt{\frac{1}{2r} - 1}}\chi_2), \tag{3.13}$$

$$\phi_3 = f_3(\chi_3) = \sqrt{2(\frac{1}{r} - 2)} sin(\theta) tanh(\frac{\csc(\theta)}{\sqrt{\frac{1}{2r} - 1}}\chi_3). \tag{3.14}$$

The deformed super-potential and deformed potential from equations (3.3) and (3.4)are given by,

$$\mathcal{W} = rtanh(4rx), \tag{3.15}$$

and

$$\mathcal{U} = \frac{2rsech^2(2rx)[(3r-1)sech^2(2rx)+1-2r]}{(1+tanh^2(2rx))^2}. \tag{3.16}$$

Finally we can say the topological solutions of non-deformed and deformed are compared together by drawing their graphs. In that case, we see that figs. (1) and (2) as follows,

Fig. (1) shows us that plots of non-deformed fields ϕ_1 and deformed χ_1 were drawn as the kink. Plots of non-deformed fields ϕ_2 and ϕ_3 and deformed fields χ_2 and χ_3 are lump. We see that the graphs of three fields for non-deformed and deformed are same and just by scale are different.

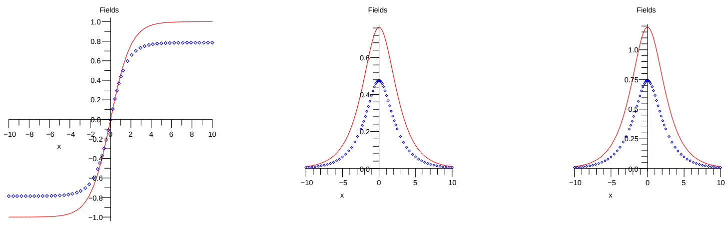

Figure 1. Left hand: Graphs of the fields $\phi_1(x)$ (line) and $\chi_1(x)$ (point). Middle: Graphs of the fields $\phi_2(x)$ (line) and $\chi_2(x)$ (point). Right hand: Graphs of the fields $\phi_3(x)$ (line) and $\chi_3(x)$ (point). All correspond to $r = 0.25$ and $\theta = 45$.

Fig. (2) shows plots of non-deformed and deformed superpotentials. Similarly, we see that the variation of two case are same, just by scale are different.

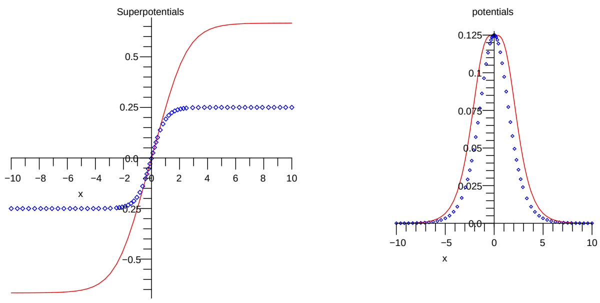

Figure 2. Left hand: Graphs of W (line) and $\mathcal{W}$ (point),Right hand: Graphs of U (line) and $\mathcal{U}$ (point), both plotted for $r = 0.25$.

4 Conclusion

In this paper, we introduced three coupled scalar fields ϕ_1, ϕ_2 and ϕ_3 and obtained topological solution for the super-potential by the orbit method. Next we obtained the deformation form of initial fields as χ_1, χ_2 and χ_3. Also, the topological solution for deformed field are obtained by the orbit method. The solution of deformed and non-deformed three coupled scalar fields lead us to compare two cases. So, we have shown that the variation of fields for two cases with respect to coordinates in fig.(1) and (2) are same and differ only by scale.

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