On tt*-bundles and related pluriharmonic maps

Lars Schäfer

Abstract. This article contains the extended notes of the author's talk given at the conference DGDS 2007. First we recall the notion of (generalized) (para-)pluriharmonic maps and of (para-)tt*-bundles and their metric/symplectic versions. Then we discuss the relation between (para-)tt*-bundles and (para-)pluriharmonic maps into the symmetric spaces $GL(r,\mathbb{R})/O(p,q)$ and $GL(r,\mathbb{R})/Sp(\mathbb{R}^r)$. In the last section we consider a general Ansatz to obtain solutions of (para-)tt*-bundles on the tangent bundle. We discuss solutions obtained from special (para-)Kähler and flat nearly (para-)Kähler manifolds. The associated (para-)pluriharmonic maps are analyzed.

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Contents

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1 Introduction

tt^*-geometry is a geometry of physical origin. Around 1990 physicists began to study topological field-theories and their moduli-spaces, in particular N=2 supersymmetric field-theories and discovered a special geometric structure called topological-antitopolgical fusion (see for example [3] and [8]). A definition of tt^*-geometry on abstract vector bundles was formulated in [13]. From the mathematical point of view this geometry can be considered as a generalization of variations of Hodge-structures (VHS) as it was done in [13]. The definition in terms of real differential geometry, which is used in this text was given in [5, 18]. A result of Dubrovin [8] associates to a tt^*-bundle with positive definite metric a pluriharmonic map to $GL(r)/O(r)$ where r is the dimension of the base-manifold and vice-versa to every such map a tt^*-geometry. This result was generalized by the author in different directions, as it is discussed later in this article. For example para-complex versions were introduced and a number of results from complex geometry were generalized to this setting.
An interesting class of solutions is harmonic bundles E which were first introduced by Simpson [24]. In his paper he related harmonic bundles to pluriharmonic maps into $GL(r,\mathbb{C})/U(r)$, where r is the (complex) rank of E. In the reference [19] we established the relation between the pluriharmonic maps associated to harmonic bundles and the pluriharmonic maps related to the tt*-bundle coming from the harmonic bundle.
In the present paper we discuss the results for complex and para-complex geometry in a parallel manner, which was neither done in the talk nor in the references. In addition we present the solutions on the tangent bundle in a general Ansatz and analyse these solutions in more details than in former work.

2 Para-complex differential geometry

The notion of para-complex geometry was first introduced in 1952 by P. Libermann [16]. A survey on this subject is [7]. One can obtain examples from para-hermitian symmetric spaces, see [2]. Such spaces provide examples of para-Kähler manifolds. Another name for para-Kähler manifolds is bi-Lagrangian manifolds (cf. [9] for a survey).
Let us now recall some notions and facts of para-complex (differential) geometry. The basic idea of para-complex geometry is to replace the complex structure J with $J^2 = -\mathbb{1}$ (on a finite dimensional vector space V) by the para-complex structure $\tau \in \mathrm{End}(V)$ satisfying $\tau^2 = \mathbb{1}$ such that the ± 1-eigenspaces have the same dimension. The pair (V,τ) is called para-complex vector space. An endomorphism-field τ, which is a point-wise para-complex structure is called an almost para-complex structure on a smooth manifold M. An almost para-complex structure τ is called para-complex structure on M if the eigendistributions $T^{\pm}M$ are integrable. In this case M is called a para-complex manifold. Similar to complex geometry, the obstruction to the integrability of the para-complex structure is given by a tensor, also called Nijenhuis tensor.
We consider the real algebra, which is generated by 1 and by the para-complex unit e subject to the relation $e^2 = 1$. It is called the para-complex numbers and is denoted by C. For all elements $z = x + ey \in C$ with $x, y \in \mathbb{R}$ one defines the para-complex conjugation as $\bar{\cdot} : C \to C, x + ey \mapsto x - ey$. The real part and imaginary part of z are defined as $\Re(z) := x,\ \Im(z) := y$. The free C-module C^n is a para-complex vector

space on which the para-complex structure is given by the multiplication with e and the para-complex conjugation of C extends to $\bar{\cdot} : C^n \to C^n$, $v \mapsto \bar{v}$.
Note, that $z\bar{z} = x^2 - y^2$. For this reason the algebra C is also called the hypercomplex numbers. The circle $\mathbb{S}^1 = \{z = x + iy \in \mathbb{C} \,|\, x^2 + y^2 = 1\}$ is replaced by the four hyperbolas $\{z = x + ey \in C \,|\, x^2 - y^2 = \pm 1\}$. We define $\tilde{\mathbb{S}}^1$ to be the hyperbola given by the one-parameter group $\{z(\theta) = \cosh(\theta) + e\sinh(\theta) \,|\, \theta \in \mathbb{R}\}$.
A para-complex vector space (V, τ) endowed with a pseudo-Euclidean metric g is called para-hermitian vector space, if g is τ-anti-invariant, i.e. $\tau^* g = -g$. The para-unitary group of V is defined as the group of automorphisms

$$U^\pi(V) := \mathrm{Aut}(V, \tau, g) := \{L \in GL(V) | [L, \tau] = 0 \text{ and } L^* g = g\}.$$

For $C^n = \mathbb{R}^n \oplus e\mathbb{R}^n$ the standard para-hermitian structure is defined by the above para-complex structure and the metric $g = \mathrm{diag}(\mathbb{1}, -\mathbb{1})$ (cf. Example 7 of [4]). The corresponding para-unitary group is given by (cf. Proposition 2 of [4]):

$$U^\pi(C^n) = \left\{ \begin{pmatrix} A & B \\ B & A \end{pmatrix} | \, A, B \in \mathrm{End}(\mathbb{R}^n), A^T A - B^T B = \mathbb{1}_n, \, A^T B - B^T A = 0 \right\}.$$

There exist two bi-gradings on the exterior algebra: The one is induced by the splitting in $T^{\pm}M$ and denoted by $\Lambda^k T^* M = \bigoplus_{k=p+q} \Lambda^{p+,q-} T^* M$ and induces an obvious bi-grading on exterior forms with values in a vector bundle E. The second is induced by the decomposition of the para-complexified tangent bundle $TM^C = TM \otimes_{\mathbb{R}} C$ into the subbundles $T_p^{1,0}M$ and $T_p^{0,1}M$ which are defined as the $\pm e$-eigenbundles of the para-complex linear extension of τ. This induces a bi-grading on the C-valued exterior forms noted $\Lambda^k T^* M^C = \bigoplus_{k=p+q} \Lambda^{p,q} T^* M$ and finally on the C-valued differential forms on M noted as $\Omega^k_C(M) = \bigoplus_{k=p+q} \Omega^{p,q}(M)$. In the case $(1,1)$ and $(1+, 1-)$ the two gradings induced by τ coincide, in the sense that $\Lambda^{1,1} T^* M = (\Lambda^{1+,1-} T^* M) \otimes C$.

Definition 1. An almost para-hermitian manifold (M, τ, g) is an almost para-complex manifold (M, τ) endowed with a pseudo-Riemannian metric g such that $\tau^* g = -g$. If τ is integrable, we call (M, τ, g) a para-hermitian manifold. The two-form $\omega := g(\tau\cdot, \cdot)$ is called the fundamental two-form of the almost para-hermitian manifold (M, τ, g). A para-hermitian manifold is called para-Kähler manifold if it ω is a closed form.

We remark that (almost) para-hermitian and para-Kähler manifolds M^{2n} are forced to have a metric of split signature (n, n).
In the following sections we want to consider the complex and the para-complex case at the same time. Therefore we introduce the following notation: For the ϵcomplex unit we use the symbol $\hat{i}$, i.e. $\hat{i} := e$, for $\epsilon = 1$, and $\hat{i} = i$, for $\epsilon = -1$. Further we define $\mathbb{C}_\epsilon$ by $\mathbb{C}_1 = C$ and $\mathbb{C}_{-1} = \mathbb{C}$. Our language is extended by the following ϵ-notation: If a word has a prefix ϵ with $\epsilon \in \{\pm 1\}$, i.e. is of the form ϵX, this expression is replaced by

$$\epsilon X := \begin{cases} \text{X, for } \epsilon = -1, \\ \text{para-X, for } \epsilon = 1. \end{cases}$$

The ϵunitary group is called

$$U^\epsilon(p,q) := \begin{cases} U^\pi(C^r), & \text{for } \epsilon = 1, \\ U(p,q), & \text{for } \epsilon = -1, \end{cases}$$

where in the complex case (p,q) for $r = p+q$ is the hermitian signature. In the rest of the text we denote the ϵcomplex structure by the symbol $J = J_\epsilon$, i.e. $J_{-1} = J$ and $J_1 = \tau$ omitting the ϵ.

3 Pluriharmonic maps

Let (M,g) and (N,h) be pseudo-Riemannian manifolds. Denote by ∇^g and ∇^h the Levi-Civita connections of g and h. Further consider a (smooth) map $f: M \to N$. The energy of f is defined as :

$$E(f) := \frac{1}{2}\int_M \|df\|^2_{g\otimes f^*h} vol_g,$$

where df is considered as a section of $T^*M \otimes f^*TN$. The map f is called harmonic, if $|E(f)| < \infty$ and f is an extremal value of $E(f)$ with respect to variations with compact support.

Proposition 1. *The* Euler-Lagrange equations *of the harmonic functional are*

$$tr_g \nabla df = 0.$$

Here ∇ is the connection on $T^*M \otimes f^*TN$ induced by ∇^g and ∇^h.

Since the harmonic functional is conformally invariant in dimension two the next definition does not depend on the choice of the metric in the conformal class of the (para-)complex structure J.

Definition 2. A smooth map $f: M \to N$ from an ϵcomplex manifold (M,J) into a pseudo-Riemannian manifold (N,h) is called ϵpluriharmonic if its restriction $f_{|\Sigma}$ to arbitrary ϵcomplex curves Σ is harmonic.

A map is ϵpluriharmonic if it satisfies the following differential equation

$$(\nabla df)^{1,1} = 0, \tag{3.1}$$

where ∇ is the connection on $T^*M \otimes f^*TN$ induced by a torsion-free ϵcomplex connection D on M, i.e. a torsion-free connection D satisfying $DJ = 0$ and by the Levi-Civita connection ∇^h of h.
More generally, any ϵcomplex connection D on M such that for the torsion it holds $(T^D)^{1,1} = 0$ leads to the same equation (3.1).
On an almost ϵcomplex manifold (M,J) there exists a connection D with torsion proportional to the Nijenhuis tensor, which has vanishing $(1,1)$-component (compare [15] for the complex and [21] for the para-complex case.).
Therefore we can define the notion of an ϵpluriharmonic map for maps in this context of non-integrable ϵcomplex structure by equation (3.1).

Definition 3. A map from an almost ϵcomplex manifold to a pseudo-Riemannian manifold (N, h) is called (generalized) ϵpluriharmonic if it satisfies the following differential equation

$$(\nabla df)^{1,1} = 0, \tag{3.2}$$

where ∇ is the connection on $T^*M \otimes f^*TN$ induced by an ϵcomplex connection D on M with vanishing $(1,1)$-torsion and by the Levi-Civita connection ∇^h.

4 tt*-geometry

For the origin of tt^*-geometries in topological field-theories we refer to [3] and the references within. The version on an abstract vector bundle in the language of complex geometry is introduced in [13]. The definition which we want to use in this text was formulated in [18, 5], para-complex versions were first considered in [21]. The structure of symplectic tt*-bundles appeared in a natural way in the analysis of solutions given by flat nearly ϵKähler manifolds [20, 22].
In order to formulate the next definition we use the notation

$$\cos_\epsilon(\theta) := \begin{cases} \cos(\theta), & \epsilon = -1, \\ \cosh(\theta), & \epsilon = 1, \end{cases} \text{ and } \sin_\epsilon(\theta) := \begin{cases} \sin(\theta), & \epsilon = -1, \\ \sinh(\theta), & \epsilon = 1. \end{cases}$$

Definition 4. An ϵtt*-bundle (E, D, S) over an (almost) ϵcomplex manifold (M, J) is a real vector bundle $E \to M$ endowed with a connection D and a section $S \in \Gamma(T^*M \otimes \operatorname{End} E)$, which satisfy the ϵtt*-equation

$$R^\theta = 0,$$

for all $\theta \in \mathbb{R}$, where R^θ is the curvature of the family of connections given by

$$D^\theta_X := D_X + \cos_\epsilon(\theta)\, S_X + \sin_\epsilon(\theta)\, S_{JX},\ X \in TM.$$

A metric/symplectic ϵtt*-bundle (E, D, S, β) is an ϵtt*-bundle (E, D, S) endowed with a D-parallel non-degenerate symmetric/skew-symmetric bilinear form β on the fibers of E such that S_X is β-symmetric for all $X \in TM$.
It is called unimodular if $\operatorname{tr} S_X = 0$ for all $X \in TM$.

The next proposition describes the ϵtt^*-equations in explicit equations on the data (D, S).

Proposition 2. *The ϵtt^*-equation $R^\theta = 0,\ \forall\, \theta \in \mathbb{R}$, is equivalent to the following system of equations:*

$$\begin{aligned} &d^D S = d^D S_J = 0, \ \textit{where} \\ &(d^D S)(X, Y) = D_X(S_Y) - D_Y(S_X) - S_{[X,Y]}, \\ &[S_X, S_Y] = -\epsilon[S_{JX}, S_{JY}], \quad \forall X, Y \in TM, \\ &R^D(X, Y) + [S_X, S_Y] = 0, \quad \forall X, Y \in TM. \end{aligned}$$

Proof: The explicit form of the tt*-equations in the complex case can be obtained by the decomposition of the curvature equation $R^\theta = 0$ into its Fourier-coefficients with respect to the parameter θ. For the computations we refer to [18]. In the para-complex case $R^\theta = 0$ has to be developed in terms of the functions sinh and cosh (cf. [21] for the details). □

Remark 1. If (E, D, S) is an ϵtt*-bundle, then (E, D, S^θ) with

$$S^\theta = \cos_\epsilon(\theta)S + \sin_\epsilon(\theta)S_J$$

defines an ϵtt*-bundle. The same observation applies to metric/symplectic ϵtt*-bundles.

5 tt*-geometry and pluriharmonic maps

The results discussed in this section generalize a construction of Dubrovin [8]. In his paper Dubrovin gave a correspondence between solutions of the tt*-equations on some real subbundle of $T^{1,0}M$ endowed with a metric of positive definite signature and pluriharmonic maps into $GL(n)/O(n)$ with $n = \dim_{\mathbb{C}} M$. A related construction for harmonic bundles was obtained by Simpson [24]. As can be seen in Theorem 1 and 2, these results have been developed by the author in different directions: as for example indefinite signature, para-complex geometry and symplectic pairings.
In this section we consider (M, J) to be a simply connected ϵcomplex manifold.

Theorem 1. *[8, 18, 21, 23] Let (E, D, S, β) be a metric/symplectic ϵtt*-bundle over an ϵcomplex manifold (M, J). For any $\theta \in \mathbb{R}$ there exists a D^θ-parallel frame $(e_1^\theta, \ldots, e_r^\theta)$ of E and the association*

$$x \mapsto f_\theta(x) := \big(\beta(e_i^\theta(x), e_j^\theta(x))\big)$$

defines an ϵpluriharmonic map

$$f_\theta : M \to \begin{cases} Sym_{p,q}(\mathbb{R}^r) \cong GL(r, \mathbb{R})/O(p, q), \\ Skew_{reg}(\mathbb{R}^r) \cong GL(r, \mathbb{R})/Sp(r, \mathbb{R}), \end{cases}$$

$$\text{if } \beta \text{ is } \begin{cases} \text{symmetric of signature } (p, q),\ p + q = r, \\ \text{skew-symmetric.} \end{cases}$$

Remark 2. Let us remark the following points:

(i) The target manifold N is a pseudo-Riemannian symmetric space with the metric induced by the (bi-invariant) trace form on $\mathfrak{gl}(r, \mathbb{R})$, i.e. $\langle A, B \rangle = \mathrm{tr}\,(A\,B)$.

(ii) In all cases the ϵpluriharmonic map $f = f_\theta$ has the following additional property:

(5.1) $$R^N(df\,T^{1,0}M, df\,T^{1,0}M) = 0,$$

where R^N is the curvature tensor of the symmetric target manifold N.

(iii) If the source manifold M is complex and the target manifold N is Riemannian then each map f is admissible, i.e. satisfies equation (5.1), since it holds

$$0 = \langle R^N(Z,W)\bar{W}, \bar{Z}\rangle = -\|[Z,W]\|^2 \Rightarrow [Z,W] = 0, \forall Z, W \in T^{1,0}N.$$

(iv) Suppose, that (E, D, S, β) is oriented and unimodular then f_θ takes values in the irreducible symmetric space

$$N = \begin{cases} SL(r,\mathbb{R})/SO(p,q), \\ SL(r,\mathbb{R})/Sp(r,\mathbb{R}) \end{cases}$$

in the case that β is

$$\begin{cases} \text{symmetric of signature } (p,q),\ p+q=r, \\ \text{skew-symmetric.} \end{cases}$$

(v) For manifolds M, which are not simply connected, one has to consider twisted ϵpluriharmonic maps.

To formulate the converse direction of the correspondence we need the following definition.

Definition 5. An ϵpluriharmonic map $f : M \to N$ is called admissible if it satisfies the condition (5.1).

Theorem 2. *[8, 18, 21, 23] An admissible ϵpluriharmonic map*

$$f : M \to N = \begin{cases} GL(r,\mathbb{R})/O(p,q), \\ GL(r,\mathbb{R})/Sp(r,\mathbb{R}) \end{cases}$$

induces a metric/symplectic ϵtt^*-bundle (E, D, S, β).
An admissible ϵpluriharmonic map

$$f : M \to N = \begin{cases} SL(r,\mathbb{R})/SO(p,q), \\ SL(r,\mathbb{R})/Sp(r,\mathbb{R}) \end{cases}$$

induces an oriented unimodular metric/symplectic ϵtt^*-bundle (E, D, S, β).

We want to emphasize the case of a complex source and a Riemannian target manifold:

Theorem 3. *Let (M, J) be a simply connected complex manifold and*

$$f : M \to N = GL(r,\mathbb{R})/O(r)$$

be a pluriharmonic map then f induces a metric tt^*-bundle (E, D, S, g).
Let (M, J) be a simply connected complex manifold and

$$f : M \to N = SL(r,\mathbb{R})/SO(r),$$

be a pluriharmonic map then f induces an oriented unimodular metric tt^*-bundle (E, D, S, g).

6 Solutions on the tangent bundle

6.1 A general Ansatz

In this section we recall the Ansatz of [20, 22], which was motivated by the study of ϵtt*-bundles on the tangent bundle, i.e. $E = TM$, related to *special ϵKähler manifolds* in [5, 21].
Let (M, J, ∇) be an almost ϵcomplex manifold endowed with a flat connection ∇. We consider the one-parameter family ∇^θ of connections, which is defined by

(6.1) $$\nabla^\theta_X Y = \exp(\theta J)\nabla_X(\exp(-\theta J)Y) \text{ for } X, Y \in \Gamma(TM),$$

where $\exp(\theta J) = \cos_\epsilon(\theta)Id + \sin_\epsilon(\theta)J$. Since ∇ is flat the family ∇^θ is also flat.
The natural question is, if this family is related to some solution $(E = TM, D, S)$ of the ϵtt*-equations. An answer (cf. Theorem 4) was given in [20, 22] under the assumption that D is an ϵcomplex connection, i.e. a connection such that $DJ = 0$. The existence of such a connection is ensured by the above cited ϵcomplex connection with torsion which is proportional to the Nijenhuis tensor.

Theorem 4.

(i) Let an almost ϵcomplex manifold (M, J) endowed with a flat connection ∇ and a decomposition of $\nabla = D + S$ in a connection D and a section S in $T^*M \otimes End(TM)$, such that J is D-parallel, i.e. $DJ = 0$, be given. If (TM, D, S) defines an ϵtt^*-bundle, such that $D^\theta = \nabla^{\alpha\theta}$ with factor $\alpha = \pm 2$, then D and S are uniquely determined by

(6.2) $$S = -\frac{1}{2}\epsilon J(\nabla J)$$

and $D = \nabla - S$.
Moreover, (TM, D, S) as above defines an ϵtt^*-bundle, such that $D^\theta = \nabla^{\alpha\theta}$ with factor $\alpha = \pm 2$, if and only if J satisfies

(6.3) $$(\nabla_{JX} J) \mp J(\nabla_X J) = 0$$

and D and S are given by $S = -\frac{1}{2}\epsilon J(\nabla J)$ and $D = \nabla - S$.

(ii) Let (M, J, g) be an almost ϵhermitian manifold endowed with a flat connection ∇, such that (∇, J) satisfies the condition (6.3) and the metric g is ∇-parallel. Define S, a section in $T^*M \otimes End(TM)$, by

(6.4) $$S := -\frac{1}{2}\epsilon J(\nabla J),$$

then $(TM, D = \nabla - S, S, \omega = \epsilon g(J\cdot, \cdot))$ defines a symplectic ϵtt^*-bundle.

(iii) Let (M, J, g) be an almost ϵhermitian manifold endowed with a flat connection ∇, such that (∇, J) satisfies the condition (6.3) and the two-form $\omega = \epsilon g(J\cdot, \cdot)$ is ∇-parallel. Define S, a section in $T^*M \otimes End(TM)$, by

(6.5) $$S := -\frac{1}{2}\epsilon J(\nabla J),$$

then $(TM, D = \nabla - S, S, g)$ defines a metric ϵtt^*-bundle.

Proof: The proof of part (i) can be found in [20, 22].
Let us prove (ii) and (iii): By part (i) the data (TM, D, S) defines an ϵtt^*-bundle.
(ii) It remains to check that $D\omega = 0$ and that S is ω-symmetric.
First, the information that g is ϵhermitian and $\nabla g = 0$ imply that $\nabla_X J$ is skew-symmetric with respect to g. By a short calculation it follows, that S is skew-symmetric with respect to g :

$$\begin{aligned}-2\epsilon\, g(S_X Y, Z) &= g(J(\nabla_X J)Y, Z) = -g((\nabla_X J)Y, JZ)\\ &= g(Y, (\nabla_X J)JZ) = -g(Z, J(\nabla_X J)Y) = 2\epsilon\, g(Y, S_X Z).\end{aligned}$$

Using $\omega = \epsilon g(J\cdot, \cdot)$ and $\{S_X, J\} = 0$ gives the ω-symmetry of S_X.
¿From $D = \nabla + \frac{1}{2}\epsilon J\nabla J$, one obtains

$$DJ = \nabla J + \frac{1}{2}\epsilon[J\nabla J, J] = 0.$$

This means $D\omega = 0$ if and only if $Dg = 0$. Since $\nabla g = 0$ and S is skew-symmetric with respect to g, it follows that g is parallel for $D = \nabla - S$.
Summarizing $(TM, D = \nabla - S, S, \omega)$ is a symplectic ϵtt^*-bundle.
(iii) It remains to check $Dg = 0$ and that S is g-symmetric.
First, we remark that $\omega(JX, Y) = -\omega(X, JY)$ as g is ϵhermitian. This yields using $\nabla\omega = 0$ the ω-skew-symmetry of $\nabla_X J$, which implies that $S_X = -\frac{1}{2}\epsilon J(\nabla J)$ is ω-skew-symmetric, since $J(\nabla_X J) = -(\nabla_X J)J$. Finally $\{S_X, J\} = 0$ shows the g-symmetry of S_X. Similary to (ii) one has $DJ = 0$. Therefore $Dg = 0$ is equivalent to $D\omega = 0$. From $\nabla\omega = 0$ and the ω-skew-symmetry of S it follows $D\omega = (\nabla - S)\omega = 0$. Thus we have shown that $(TM, D = \nabla - S, S, g)$ is a metric ϵtt^*-bundle. □

Remark 3. (i) If the torsion of ∇ vanishes then the condition (6.3) with the negative sign, i.e.

$$(\nabla_{JX} J)Y - J(\nabla_X J)Y = 0 \tag{6.6}$$

implies that J is an integrable almost ϵcomplex structure. This follows immediately from the formula for the Nijenhuis tensor in terms of a torsion-free connection ∇

$$N_J(X, Y) = (\nabla_{JX} J)Y - (\nabla_{JY} J)X - J(\nabla_X J)Y + J(\nabla_Y J)X \overset{(6.6)}{=} 0.$$

(ii) If the connection ∇ is the Levi-Civita connection $\nabla = \nabla^g$ of the almost ϵhermitian metric g then the condition (6.3) with +-sign, i.e.

$$(\nabla_{JX} J)Y + J(\nabla_X J)Y = 0 \tag{6.7}$$

describes a subclass of almost ϵhermitian structures of Gray-Hervella class $\mathcal{G}_1$ (cf. [12, 17]). As we observed in a discussion with P.-A. Nagy the subclass related to equation (6.7) is exactly the class of *nearly Kähler manifolds*. Solutions of this type are discussed in subsection 6.3. We observe, that the Nijenhuis tensor in this case is given by

$$N_J(X, Y) = -4J(\nabla_X J)Y.$$

(iii) Another possibility to solve the condition (6.3) is by a special connection ∇, i.e. $(\nabla_X J)Y = (\nabla_Y J)X$. Solutions of this type are for example given by *special ϵcomplex manifolds* and by *special ϵKähler manifolds*. This is discussed in more detail in subsection 6.2.

6.2 Special ϵcomplex and special ϵKähler manifolds

Let us recall the (mathematical) definition of special ϵcomplex and special ϵKähler manifolds. For the physical motivation in super-symmetric field-theories we refer to the references in [1, 10]. Euclidean super-symmetric field-theories were subject of [4].

Definition 6. [1, 10, 4] A special ϵcomplex manifold (M, J, ∇) is an ϵcomplex manifold endowed with a flat torsion-free connection ∇, such that

$$(\nabla_X J)Y = (\nabla_Y J)X, \ \forall X, Y \in TM. \tag{6.8}$$

A special ϵKähler manifold (M, J, ∇, ω) is a special ϵcomplex manifold (M, J, ∇) endowed with a J-invariant ∇-parallel symplectic structure ω. The ϵKähler metric $g = \epsilon\omega(J\cdot, \cdot)$ is called the special ϵKähler metric of (M, J, ∇, ω).

A tt*-bundle (E, D, S) is called special if for each θ the connection D^θ is torsion-free and special, i.e. $(D^\theta_X J)Y = (D^\theta_Y J)X$.

Theorem 5. *[5, 21] There exists a one-to-one correspondence*

$$\Phi : \left\{ \begin{array}{c} \textit{special } \epsilon\textit{Kähler manifolds} \\ (M, J, \nabla, g) \end{array} \right\}$$
$$\to \left\{ \begin{array}{c} \textit{special metric } \epsilon tt^*\textit{-bundles } (TM, D, S, g) \\ \textit{over } \epsilon\textit{Hermitian manifolds } (M, J, g), \\ \textit{s.t. } \{S_X, J\} = 0, \ \forall X \in TM, \ \textit{and } \ DJ = 0 \end{array} \right\},$$

where the map Φ is defined by

$$\Phi(M, J, \nabla, g) = (TM, D := \nabla - S, S = -\frac{1}{2}\epsilon J \nabla J, g)$$

and its inverse is given by $\Phi^{-1}(TM, D, S, g) = (M, J, \nabla := D + S, g)$.

Remark 4. Any special ϵcomplex manifold (M, J, ∇) carries a canonical torsion-free ϵcomplex connection which coincides with the connection D as defined in Theorem 5 (cf. Theorem 4).
If (M, J, ∇, g) is a special ϵKähler manifold, then the connection D coincides with the Levi-Civita connection ∇^g of g.

Consider a simply connected special ϵKähler manifold (M^{2n}, J, ∇, g).
Any ∇-parallel frame s of volume 1 defines an ϵpluriharmonic map

$$G^{(s)} : M \to SL(2n, \mathbb{R})/SO(p, q),$$

where $(p, q) = (2k, 2l)$ is the symmetric signature of the metric g. Note, that it holds $p = q$ in the para-complex case.

Any simply connected special ϵKähler manifold M was realized by [1] and [4] as a non-degenerate ϵholomorphic Lagrangian immersion $(V = T^*\mathbb{C}_\epsilon^n = \mathbb{C}_\epsilon^{2n}, \Omega = \Omega_{can})$:

$$\varphi : M \to V = T^*\mathbb{C}_\epsilon^n = \mathbb{C}_\epsilon^{2n},$$

which is unique up to the action of the group $\mathrm{Aff}_{Sp(\mathbb{R}^{2n})}(V)$.
The metric is given by $g = \mathrm{Re}(\varphi^*\gamma)$ with $\gamma := \hat{\imath}\Omega(\cdot, \bar{\cdot})$. Lagrangian means $\varphi^*\Omega = 0$ and non-degenerate means that with the above definition g defines a metric, i.e. is non-degenerate. The immersion φ induces a map

$$L : M \to Gr_0^{k,l}(\mathbb{C}_\epsilon^{2n}), \quad p \quad \mapsto \quad d\varphi_p \, T_pM \subset V,$$

into the Grassmannian $Gr_0^{k,l}(\mathbb{C}_\epsilon^{2n})$ of ϵcomplex Lagrangian subspaces $W \subset V$, such that in the complex case $\gamma_{|W}$ has hermitian signature (k,l). Recall, that there is no notion of signature in the para-complex case. In fact, the map L is dual to the Gauss map:

$$L^\perp : M \quad \to \quad Gr_0^{l,k}(\mathbb{C}_\epsilon^{2n}), \quad p \mapsto L(p)^\perp = \overline{L(p)} \cong_\Omega L(p)^*.$$

Theorem 6. *[5, 21] Let (TM, D, S, g) be the metric ϵtt*-bundle associated to a simply connected special ϵKähler manifold (M^{2n}, J, ∇, g).*
Then there exists a ∇-parallel frame s such that the ϵpluriharmonic map $G^{(s)}$ takes values in the totally geodesic submanifold

$$Sp(\mathbb{R}^{2n})/U^\epsilon(k,l) \subset SL(2n, \mathbb{R})/SO(2k, 2l)$$

and coincides with the dual Gauss map L.

6.3 Nearly Kähler manifolds

Definition 7. A nearly ϵKähler manifold (M, J, g) is an almost ϵcomplex manifold (M, J) endowed with a pseudo-Riemannian metric g, such that
i) J is skew-symmetric with respect to g and
ii) $(\nabla^g_X J)Y = -(\nabla^g_Y J)X, \ \forall X, Y \in TM$.

Using Theorem 4 then yields the following proposition.

Proposition 3. *Let (M, g) be a Levi-Civita flat nearly ϵKähler manifold, then the data (TM, D, S, ω) defines a symplectic ϵtt*-bundle.*

Remark, that the connection $D := \nabla^g - S$ with $S := -\frac{1}{2}\epsilon J\nabla J$ is the unique ϵcomplex metric connection with totally skew-symmetric torsion [11, 14]. This connection is called the canonical connection.
The question if on a Levi-Civita flat nearly ϵKähler manifold (M, J, g) the data (TM, D, S, g) can define a metric ϵtt*-bundle was answered negatively by [20, 22] in the sense that it forces S to vanish.

For the related ϵpluriharmonic maps it follows:

Proposition 4. *[20, 22] A Levi-Civita flat nearly ϵKähler manifold (M, g) defines an ϵpluriharmonic map $\phi : M \to SO(2k, 2l)/U^{\epsilon}(k, l)$ (Recall, that $k = l$ in the para-complex case).*

The map ϕ is essentially the ϵcomplex structure in a Levi-Civita flat frame.
In the complex case there exist the following classification results for Levi-Civita flat nearly Kähler manifolds and the para-complex version is work in progress.
We denote by $V = \mathbb{C}^{k,l}$ the complex vector space $(\mathbb{C}^n, J_{can})$, $n = k + l$, endowed with the standard J_{can}-invariant pseudo-Euclidian scalar product g_{can} of symmetric signature $(2k, 2l)$.

Theorem 7. *[6] Let η be a constant three-form on a connected open set $U \subset V = \mathbb{C}^{k,l}$ containing 0 which satisfies $\eta_X\, \eta_Y = 0$, and $\{\eta_X, J_{can}\} = 0, \forall X, Y$. Denote by $(x^i)_{i=1}^{2n}$ flat (real) linear coordinates of U and by $\partial_i = \frac{\partial}{\partial x^i}$.*
Then there exists a unique almost complex structure

$$J = \exp\left(2\sum_{i=1}^{2n} x^i\, \eta_{\partial_i}\right) J_{can}$$

on U such that $J_0 = J_{can}$, and $M(U, \eta) := (U, g = g_{can}, J)$ is a flat nearly pseudo-Kähler manifold.
Any flat nearly pseudo-Kähler manifold is locally isomorphic to a flat nearly pseudo-Kähler manifold of the form $M(U, \eta)$.

We will say that a complex three-form $\zeta \in \Lambda^3(\mathbb{C}^m)^*$ is regular if

$$\mathrm{span}\{\zeta(Z, W, \cdot) | Z, W \in \mathbb{C}^m\} = (\mathbb{C}^m)^*.$$

The constraints on the three-form can be solved as follows:

Theorem 8. *[6] There exists a one-to-one correspondence between $GL(m, \mathbb{C})$-orbits on $\Lambda^3_{reg}(\mathbb{C}^m)^* \subset \Lambda^3(\mathbb{C}^m)^*$ and isomorphism classes of complete flat simply connected nearly pseudo-Kähler manifolds of real dimension $4m \geq 12$ and without pseudo-Kähler de Rham factor.*

References

[1] D. V. Alekseevsky, V. Cortés and C. Devchand, *Special complex manifolds*, J. Geom. Phys. 42 (2002), 85-105.

[2] W. Bertram, *The Geometry of Jordan and Lie-Structures,* Springer Lecture Notes in Mathematics vol. 1754 (2000).

[3] S. Cecotti and C. Vafa, *Topological-antitopological fusion,* Nuclear Physics B 367 (1991), 351-461.

[4] V. Cortés, C. Mayer, T. Mohaupt and F. Saueressig, *Special Geometry of Euclidean Supersymmetry I: Vector Multiplets,* J. High Energy Phys. JHEP03(2004)028, hep-th/0312001.

[5] V. Cortés and L. Schäfer, *Topological-antitopological fusion equations*, pluriharmonic maps and special Kähler manifolds, in 'Complex, Contact and Symmetric Manifolds', eds. O. Kowalski, E. Musso and D. Perrone, Progress in Mathematics 234, Birkhäuser 2005, 59–74.

[6] V. Cortés and L. Schäfer, *Flat nearly Kähler manifolds*, Ann. Global Anal. Geom. 32, no 4 (2007), 379-389.

[7] V. Cruceanu and P. Fortuny, P. M. Gadea, *A survey on paracomplex geometry,* Rocky Mt. J. Math. 26 (1996), no.1, 83-115.

[8] B. Dubrovin, *Geometry and integrability of topological-antitopological fusion,* Commun. Math. Phys. 152 (1992), 539-564.

[9] F. Etayo, R. Santamaria and U. R. Trias, *The geometry of a Bi-Lagrangian manifold*, Differ. Geom. Appl. 24 (2006), No. 1, 33-59.

[10] D.S. Freed, *Special Kähler manifolds*, Commun. Math. Phys. 203 (1999), no. 1, 31-52.

[11] Th. Friedrich and S. Ivanov, *Parallel spinors and connections with skew-symmetric torsion in string theory*, Asian J. Math. 6 (2002), no. 2, 303–335.

[12] A. Gray and L. M. Hervella, *The sixteen classes of almost Hermitian manifolds and their linear invariants,* Ann. Mat. Pura Appl. (4) 123 (1980), 35–58.

[13] C. Hertling, *tt*-geometry, Frobenius manifolds, their connections, and the construction for singularities*, J. Reine Angew. Math. 555 (2003), 77-161.

[14] S. Ivanov and S. Zamkovoy, *Para-Hermitian and Para-Quaternionic manifolds*, Diff. Geom. Appl. 23 (2005), 205–234.

[15] S. Kobayashi and K. Nomizu, *Foundations of differential geometry Volume I/II,* Interscience Publishers 1963/1969.

[16] P. Libermann, *Sur les structures presque paracomplexes,* C. R. Acad. Sci., Paris 234 (1952), 2517-2519.

[17] P. A. Nagy, *Connexions with totally skew-symmetric torsion and nearly-Kaehler geometry*, arXiv:0709.1231.

[18] L. Schäfer, *tt*-bundles and pluriharmonic maps*, Annals of Gl. An. and Geom 28, no. 3 (2005), 285-300.

[19] L. Schäfer, *Harmonic bundles topological-antitoplological fusion and the related puriharmonic maps*, J. of Geom. and Phys. 56, no. 5 (2006), 830-842.

[20] L. Schäfer, tt*-geometry on the tangent bundle of an almost complex manifold, J. of Geom. and Phys. 57 (2007), 999-1014.

[21] L. Schäfer, *tt*-bundles in para-complex geometry, special para-Kähler manifolds and para-pluriharmonic maps*, Diff. Geom. Appl. 24 (2006), 60–89.

[22] L. Schäfer, *Para-tt*-bundles on the tangent-bundle of an almost para-complex manifold*, Annals of Gl. An. and Geom. 32, no 2 (2007), 125-145.

[23] L. Schäfer, *A geometric construction of (para-)pluriharmonic maps into GL(2r)/Sp(2r)*, to appear in Balkan J. of Geom. and Its Appl. vol. 13 (2008).

[24] C.T. Simpson: *Higgs-bundles and local systems,* Institut des hautes Etudes scientifiques - Publication mathematiques, No. 75 (1992), 5-95.

Author's address:

Lars Schäfer
Department Mathematik, Universität Hamburg,
Bundesstraße 55, D-20146 Hamburg, Germany.
E-mail: schaefer@math.uni-hamburg.de