Higher-dimensional black hole Geometric Thermodynamics

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Abstract. The pseudo-Riemannian Geometry is utilized in Mathematical Optimization, Thermodynamics or in Statistics as an important tool for recent research. Given a pseudo-Riemannian manifold (M, g) and a smooth function $f : M \to R$, whose Hessian with respect to g is non-degenerate, one can define on M the associated pseudo-Riemannian Hessian metric $h = \text{Hess}_g f$. In the following, we apply this methodology for describing the geometrical properties of some interesting mathematical objects like the higher dimensional Reissner-Nördstrom black holes.
The paper is organized as follows. Section 1 reviews the history of pseudo-Riemannian Geometry and points out which Hessian is more convenient for physical problems. Section 2 gives the Christoffel symbols and the system of geodesics of pseudo-Riemannian manifold $(M, h = \text{Hess}_g f)$, establishes the relation between the components of the curvature tensors field of (M, h) and (M, g), and determines the PDEs representing the coincidence between the Christoffel symbols of (M, h) and the Christoffel symbols of (M, g). The last section presents the comparison of null-length curves trajectories obtained with the two metrics for a 5-dimensional RN black hole.

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1 Weinhold and Ruppeiner metrics

Suppose the entropy S and other extensive variables N^a of the system (electric charge, angular momentum, etc) are coordinates of a point in the Euclidean space $(R^n, \delta = (\delta_{ij}))$. The Euclidean space $(R^n = \{(S, N^a)\}, \delta = (\delta_{ij}))$ is the most simple Riemannian manifold used to analyze a thermodynamic system. But, as it is well-known this space is flat. Accepting the mass (energy) M like a function $M : R^n \to R_+$,

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in 1975, F. Weinhold [16] introduced in thermodynamics a piecewise pseudo- Riemannian fundamental tensor as the Hessian of the mass (energy) $h^W = \text{Hess}_\delta M$. Taking as starting structure a pseudo-Riemannian manifold $\left(U \subset R^n, h^W\right)$, Weinhold developed a new point of view on thermodynamics. The papers of N. Pidokrajt [8] and T. Sarkar, G. Sengupta, B. N. Tiwari [10] underline that Weinhold geometry is useful for geometric approach to black hole thermodynamics. Now, suppose entropy S is a function of mass M and other extensive variables N^a of the system. Geometrically, it appears like the flat manifold $(R^n = \{(M, N^a)\}, \delta = (\delta_{ij}))$. In 1979, G. Ruppeiner [9] suggested to study thermodynamics using the minus Hessian of the entropy $h^R = -\text{Hess}_\delta S$ as a piecewise pseudo Riemannian fundamental tensor, i.e., he changed the initial geometrical structure (R^n, δ) with $\left(U \subset R^n, h^R\right)$. Of course, Ruppeiner accepted that when the entropy is expressed in terms of other extensive parameters, it basically tells us the thermal physics of the system. The Ruppeiner-Riemannian geometrical model allows the inclusion of the the- ory of fluctuations into the axioms of equilibrium thermodynamics in the sense that there exist equilibrium states and the distance between them is related to the fluctuation phenomena. In other words, the Ruppeiner-Riemannian geometry is connected to the underlying statistical mechanical model; for example, the flatness is equivalent to non-interacting statistics. The Weinhold geometry is a conformal counterpart of Ruppeiner geometry on a constant level set relating the variables S, M, N^a. More precisely, the arc elements are related by $ds_2^R = \frac{1}{T} ds_2^W$, where T is the temperature.

J. Donato [2] describes the connection between entropy and curvature. Since black holes are self-gravitating systems, they do exhibit negative specific heats. Also, the black holes should be treated in microcanonical ensemble. Globally, the Weinhold and Ruppeiner fundamental tensors have variable signature. Consequently the Ruppeiner theory of black holes cannot always be associated to the probability distribution in thermodynamic fluctuation theory.

The previous theory shows that for some applications is more convenient to replace the Euclidean space (R^n, δ) with pseudo-Riemannian manifold $(U \subset R^n, g)$, where the fundamental tensor g has a piecewise constant signature on R^n. Starting with a function $f : R^n \to R$, whose Hessian with respect to the initial piecewise pseudo-Riemannian fundamental tensor g is non-degenerate, we can introduce a pseudo-Riemannian Hessian manifold $(U \subset R^n, h = \text{Hess}_g f)$ which gives information about the properties of the system described by the function f and the fundamental tensor g. Also, exploiting the relation between the Christoffel symbols of Levi-Civita connection produced by the fundamental tensor h and the Christoffel symbols of Levi-Civita connection produced by the fundamental tensor g or the relation between the com- ponents of the corresponding curvature tensors, we can produce pseudo-Riemannian fundamental tensor with special properties [15]: metrics with negative curvature, metrics with zero curvature, complete or non-complete metrics, metrics capable to produce convexity, exact sequences of pseudo-Riemannian metrics, metrics representing fluctuation phenomena in thermodynamics, etc.

In our opinion, the pseudo-Riemannian Hessian manifold $(V \subset R^n, k = \text{Hess}_h f)$ is the most suitable to describe the fundamental properties of thermodynamic systems based on function f and the metric k since it re-inforces the information obtained from $(U \subset R^n, h = \text{Hess}_\delta f)$. Indeed, Weinhold and Ruppeiner started with constant state potentials δ_{ij} (i.e, forcing for a flat space) and built the first order nonconstant state

potentials h_{ij} as components of a nonconstant metric. This new metric determines a space $(U \subset R^n, h = \text{Hess}_\delta f)$ which is more appropriate for thermodynamics theories. Starting with Weinhold-Ruppeiner geometrical structure $(U \subset R^n, h = \text{Hess}_\delta f)$ and preserving the thermodynamic function f, we build a new fundamental tensor k. Its components are the second order nonconstant state potentials k_{ij}. Unlike the fundamental tensor h obtained using an apriori flat space, the fundamental tensor k emerges from the natural Weinhold-Ruppeiner space. So, if this conjecture should prove correct, we must keep one eye on the local skewness of the function f and one to the associated curvature produced by the fundamental tensor $k = \text{Hess}_h f$. The state potentials $\delta_{ij}, h_{ij}, k_{ij}$ are different from the usual thermodynamic potentials.
In differential geometry, Hessian metrics are samples of Riemannian or pseudo-Riemannian metrics. For example, Kähler metrics $\frac{\partial^2 f}{\partial z^i \partial \bar{z}^j}$ on complex manifolds are Riemannian metrics. The geometry of Hessian manifolds was studied by S. Amari [1], J. Duistermaat[3], N. Hitchin [4], H. Kito [6], Y. Nesterov -M. J. Tood [7], H. Shima [11], H. Shima - K. Yagi [12], B. Totaro [13], C. Udrişte - G. Bercu [15], etc.

2 Geometry via the fundamental tensor field

In this section we recall known formulas in the geometry derived from a fundamental tensor field [15]. A *fundamental tensor field* of type (0,2) on the smooth manifold R^n is a smooth symmetric differential 2-form g on R^n such that $g_x, x \in R^n$ is nondegenerate on $T_x R^n$. We call (R^n, g) an *Einsenhart manifold*.

Let g_{ij} be the components of the fundamental tensor g. The curve $\gamma(t) = \left(x^i(t)\right)_{i=1,\dots,n}$ in R^n is called null length curve if $g_{ij}(\gamma(t))\,\dot{x}^i(t)\,\dot{x}^j(t) = 0$. This definition is proper only in the regions where the fundamental tensor g is not (positive or negative) definite.

The Christoffel symbols G^k_{ij} of the Levi-Civita connection determined by the components g_{ij}, g^{ij} are

$$G^k_{ij} = \frac{1}{2} g^{kl} \left(\frac{\partial g_{li}}{\partial x^j} + \frac{\partial g_{jl}}{\partial x^i} - \frac{\partial g_{li}}{\partial x^j} \right).$$

They determine the second order differential equations

$$\ddot{x}^i + G^k_{ij} \dot{x}^j \dot{x}^k = 0, i = 1, \dots, n,$$

which describe the autoparallel lines (geodesics) $\gamma(t) = \left(x^i(t)\right)_{i=1,\dots,n}$. Also, they determine the components

$$G^l_{ijk} = \frac{\partial G^l_{ki}}{\partial x^j} - \frac{\partial G^l_{ji}}{\partial x^k} + G^r_{ki} G^l_{jr} - G^r_{ji} G^l_{kr}$$

of the curvature tensor field G.

If $f : R^n \to R$ is a smooth function, then the second covariant derivative

$$\text{Hess}_g f = \left(\frac{\partial^2 f}{\partial x^i \partial x^j} - G^k_{ij} \frac{\partial f}{\partial x^k} \right) dx^i \otimes dx^j$$

is called the Hessian of f. Notations

$$f_{,i} = \frac{\partial f}{\partial x^i};\; f_{,ij} = \frac{\partial^2 f}{\partial x^i \partial x^j} - G^m_{ij} f_{,m}; \quad f_{,ijk} = \frac{\partial f_{,ij}}{\partial x^k} - G^l_{ki} f_{,lj} - G^l_{kj} f_{,li}$$

Suppose that Hessian $h = \mathrm{Hess}_g f$ is non-degenerate. Then, h is a piecewise pseudo-Riemannian fundamental tendor that produces the Christoffel symbols H^k_{ij}.

2.1. Proposition. *Let $f^{pk}_{,}$ be the contravariant components of the fundamental tensor $h_{pk} = f_{,pk}$ and G^m_{ijk} be the components of the curvature tensor field produced by the fundamental tensor g_{ij}. Then, the Christoffel symbols H^k_{ij} are given by the following formulas*

$$H^p_{ij} = G^p_{ij} + \frac{1}{2} f^{kp}_{,} \left[f_{,ijk} + \left(G^m_{ikj} + G^m_{jki} \right) f_{,m} \right].$$

2.2. Corollary. *The equality $H^p_{ij} = G^p_{ij}$ is equivalent to the PDEs*

$$f_{,ijk} + \left(G^l_{ikj} + G^l_{jki} \right) f_{,l} = 0,$$

for all $i, j, k = 1, \ldots, n$.

2.3. Proposition. *Let h_{ij} be the components of the fundamental tensor $h = Hess_g f$ and k_{ij} be the components of the fundamental tensor $k = Hess_h f$. Then,*

$$k_{ij} = h_{ij} - \frac{1}{2} f^{pk}_{,} \left[f_{,ijk} + \left(G^l_{ikj} + G^l_{jki} \right) f_{,l} \right] f_{,p}.$$

An extensive presentation of the curvature tensor field H components based on the fundamental tensor $h = \mathrm{Hess}_g f$ is to be found in [14].

3 Higher dimensional Reissner-Nördstrom black hole

A Reissner-Nördstrom black hole is formed from non-rotating but electrically-charged matter.

Let M be the mass (energy), Q be the electric charge and S be the entropy of the black hole. The d-dimensional Reissner-Nördstrom black hole ($d > 4$) is characterized by [8]

$$S = \left[M + M\sqrt{1 - \frac{d-2}{2\,(d-3)} \frac{Q^2}{M^2}} \right]^{\frac{d-2}{d-3}}.$$

Solving with respect to the mass M, we find

$$M = \frac{S^{\frac{d-3}{d-2}}}{2} + \frac{d-2}{4\,(d-3)} \frac{Q^2}{S^{\frac{d-3}{d-2}}}.$$

Now we take the Euclidean manifold $\left(\{ (S, Q) \,| S > 0 \} \subset R^2, \delta \right)$.

Weinhold point of view [8],[10],[16]. The piecewise pseudo-Riemannian-Weinhold fundamental tensor is given by the Hessian matrix $h = \mathrm{Hess}_\delta M$, i.e.,

$$h = \begin{bmatrix} -\frac{d-3}{2(d-2)^2 S^{\frac{d-1}{d-2}}} - \frac{Q^2(-2d+5) S^{\frac{-3d+7}{d-2}}}{4(d-2)} & -\frac{Q S^{\frac{-2d+5}{d-2}}}{2} \\ -\frac{Q S^{\frac{-2d+5}{d-2}}}{2} & \frac{d-2}{2(d-3) S^{\frac{d-3}{d-2}}} \end{bmatrix}$$

The degeneration curves of the matrix h are given by the d-spanned family

$$S-\left[\frac{d-2}{2d-6}Q^2\right]^{\frac{d-2}{2d-6}}=0.$$

The signature of the Hessian h is

$$signature\,(h)=\begin{cases}(+,+) & for\ S>0, S-\left[\frac{d-2}{2d-6}Q^2\right]^{\frac{d-2}{2d-6}}<0\\ (+,-) & for\ S>0, S-\left[\frac{d-2}{2d-6}Q^2\right]^{\frac{d-2}{2d-6}}>0,\\ & S-\left[\frac{(d-2)(2d-5)}{2(d-3)}Q^2\right]^{\frac{d-2}{2d-6}}<0\\ (-,+) & for\ S>0, S-\left[\frac{(d-2)(2d-5)}{2(d-3)}Q^2\right]^{\frac{d-2}{2d-6}}.\end{cases}$$

The Euclidean model $\left(\{(S,Q)\,|S>0\}\subset R^2,\delta\right)$ is changed into a pseudo-Riemannian manifold $\left(U\subset\{(S,Q)\,|S>0\}\subset R^2, h=\mathrm{Hess}_\delta M\right)$. The null length curves are described by the differential equations

$$-\left[\frac{d-3}{2(d-2)^2S^{\frac{d-1}{d-2}}}+\frac{Q^2(-2d+5)S^{\frac{-3d+7}{d-2}}}{4(d-2)}\right]dS^2-QS^{\frac{-2d+5}{d-2}}dSdQ+\frac{d-2}{2(d-3)S^{\frac{d-3}{d-2}}}dQ^2=0$$

Now let us compute the Christoffel symbols of the fundamental tensor h. We find the following expressions

$$\begin{aligned}H^1_{11}=&\ -\left[\left(2d^3-11d^2+19d-10\right)Q^2S^{\frac{-2d+6}{d-2}}-2d^2+8d-6\right]/\\ &\left\{2S\,(d-2)\left[-2d+6+(d-2)\,Q^2S^{\frac{-2d+6}{d-2}}\right]\right\}\\ H^1_{12}=&\ (d-2)^2\,QS^{\frac{-2d+6}{d-2}}/\left[-2d+6+(d-2)\,Q^2S^{\frac{-2d+6}{d-2}}\right]\\ H^1_{22}=&-(d-2)^2\,S^{\frac{-d+4}{d-2}}/\left[-2d+6+(d-2)\,Q^2S^{\frac{-2d+6}{d-2}}\right]\end{aligned}$$

$$\begin{aligned}H^2_{11}=&\ -(d-3)\,Q\left[\left(2d^3-13d^2+28d-20\right)Q^2S^{\frac{-2d+6}{d-2.0}}+2d^2-14d+24\right]/\\ &\left\{2S^2\,(d-2)^2\left[-2d+6+(d-2)\,Q^2S^{\frac{-2d+6}{d-2}}\right]\right\}\\ H^2_{12}=&\ (d-3)\left[\left(2d^2-9d+10\right)Q^2S^{\frac{-2d+6}{d-2}}+2d-6\right]/\\ &\left\{2S\,(d-2)\left[-2d+6+(d-2)\,Q^2S^{\frac{-2d+6}{d-2}}\right]\right\}\\ H^2_{22}=&-QS^{\frac{-2d+6}{d-2}}\,(d-3)\,(d-2)\,/\left[-2d+6+(d-2)\,Q^2S^{\frac{-2d+6}{d-2}}\right]\end{aligned}$$

These symbols determine the geodesic equations

$$\begin{aligned}\tfrac{d^2S}{dt^2}+H^1_{11}\left(\tfrac{dS}{dt}\right)^2+H^1_{12}\tfrac{dS}{dt}\tfrac{dQ}{dt}+H^1_{22}\left(\tfrac{dQ}{dt}\right)^2=0\\ \tfrac{d^2Q}{dt^2}+H^2_{11}\left(\tfrac{dS}{dt}\right)^2+H^2_{12}\tfrac{dS}{dt}\tfrac{dQ}{dt}+H^2_{22}\left(\tfrac{dQ}{dt}\right)^2=0.\end{aligned}$$

The non-zero components of the Riemann curvature tensor are

$$H_{1212} = \frac{(d-3)S^{\frac{-3d+7}{d-2}}}{2(d-2)\left[-2d+6+(d-2)Q^2S^{\frac{-2d+6}{d-2}}\right]}$$
$$H_{1221} = H_{2112} = -H_{1212},\ H_{2121} = H_{1212}$$

Physical characteristics of Weinhold-Pidokrajt geometrical model: *the fundamental tensor $h = Hess_\delta M$ is piecewise pseudo-Riemannian; the components h_{ij} of the fundamental tensor h may be assimilated to state potentials; the Christoffel symbols and the curvature tensor diverge in the extremal limit and along the curves where the fundamental tensor changes signature; the manifold $(U \subset \{(S,Q)\,|S>0\} \subset R^2$, $h = Hess_\delta M)$ is non-flat; the sign of the sectional curvature depends on the sign of* $\det(h)$ *and the sign of H_{1212}.*

Our point of view. We start with the pseudo-Riemannian manifold $\left(U \subset \{(S,Q)\,|S>0\} \subset R^2, h = \mathrm{Hess}_\delta M\right)$. First, let us compute the covariant derivative of the differential $a = dM$ of the function M (a covariant vector) with respect to h, i.e., we compute the Hessian $k = \mathrm{Hess}_h M$. In this way, we obtain a new pseudo-Riemannian manifold $\left(V \subset \{(S,Q)\,|S>0\} \subset R^2, k = \mathrm{Hess}_h M\right)$.

Since $a = dM = (a_1, a_2)$, where $a_1 = \frac{d-3}{2(d-2)S^{\frac{1}{d-2}}} - \frac{Q^2}{4S^{\frac{2d-5}{d-2}}}$, $a_2 = \frac{(d-2)Q^2}{2(d-3)S^{\frac{d-3}{d-2}}}$, we find

$$k = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix}$$

where

$$\begin{aligned} k_{11} = \ & \Big[\left(2d^4-15d^3+41d^2-48d+20\right)Q^4S^{\frac{-3d+9}{d-2}} + (-4d^3+36d^2-108d+ \\ & 108)S^{\frac{d-3}{d-2}} + \left(4d^4-36d^3+112d^2-136d+48\right)Q^2S^{\frac{-d+3}{d-2}}\Big] / \\ & \left\{8S^2(d-2)^2\left[-2d+6+(d-2)Q^2S^{\frac{-2d+6}{d-2}}\right]\right\} \end{aligned}$$

$$\begin{aligned} k_{22} = \ & \Big[\left(2d^3-20d^2+62d-60\right)S^{\frac{d-1}{d-2}} + \left(d^3-5d^2+8d-4\right)Q^2S^{\frac{-d+5}{d-2}}\Big] / \\ & \left\{4S^2(d-3)\left[-2d+6+(d-2)Q^2S^{\frac{-2d+6}{d-2}}\right]\right\} \end{aligned}$$

$$\begin{aligned} k_{21} = k_{12} = \ & -Q\left[\left(d^2-3d+2\right)Q^2S^{\frac{-2d+7}{d-2}} + \left(2d^2-12d+18\right)S^{\frac{1}{(d-2)}}\right] / \\ & \left\{4S^2\left[-2d+6+(d-2)Q^2S^{\frac{-2d+6}{d-2}}\right]\right\} \end{aligned}$$

The degeneration curves of the matrix k are given by the d-spanned family

$$\begin{gathered} \left(4d^5-8d^4-232d^3+1424d^2-3036d+2232\right)Q^2S^{\frac{2}{d-2}} + \\ \left(-2d^5+30d^4-158d^3+378d^2-416d+168\right)Q^4S^{\frac{-2d+8}{d-2}} + \\ \left(-d^5+8d^4-25d^3+38d^2-28d+8\right)Q^6S^{\frac{-4d+14}{d-2}} + \\ \left(8d^5-136d^4+912d^3-3024d^2+4968d-3240\right)S^2 = 0 \end{gathered}$$

The signature of the Hessian k is not constant and this is why k represents a piecewise pseudo-Riemannian fundamental tensor. The null length curves are described by the differential equations

$$k_{11}dS^2 + (k_{12}+k_{21})\,dSdQ + k_{22}dQ^2 = 0$$

Now, let us compute the Christoffel symbols of the metric k.

$$\begin{aligned}K^1_{11} = \;& \Big[\big(-16d^8 + +336d^7 - 2784d^6 + 11200d^5 - 18896d^4 - 12784d^3 + \\ & 105984d^2 - 161568d + 84672\big)\,Q^2 S^{\frac{2}{d-2}} + \\ & \big(-8d^8 + 120d^7 - 648d^6 + 1072d^5 + 16416d^4 - 21096d^3 + \\ & 43080d^2 - 42048d + 16416\big)\,Q^4 S^{\frac{-2d+8}{d-2}} + \\ & \big(2d^8 - 27d^7 + 157d^6 - 513d^5 + 1029d^4 - 1296d^3 + \\ & 1000d^2 - 432d + 80\big)\,Q^8 S^{\frac{-6d+20}{d-2}} + \\ & \big(4d^8 - 108d^7 + 1084d^6 - 5692d^5 + 17616d^4 - 33396d^3 + \\ & 38032d^2 - 23904d + 6336\big)\,Q^6 S^{\frac{-4d+14}{d-2}} + \\ & \big(16d^7 - 336d^6 + 2960d^5 - 14160d^4 + 39600d^3 - 64368d^2 + \\ & 55728d - 19440)\,S^2\Big] \Big/ \Big\{2S\,(d-2)\left[-2d + 6 + (d-2)\,Q^2 S^{\frac{-2d+6}{d-2}}\right] \\ & \Big[\big(4d^5 - 8d^4 - 232d^3 + 1424d^2 - 3036d + 2232\big)\,Q^2 S^{\frac{2}{d-2}} + \\ & \big(-2d^5 + 30d^4 - 158d^3 + 378d^2 - 416d + 168\big)\,Q^4 S^{\frac{-2d+8}{d-2}} + \\ & \big(-d^5 + 8d^4 - 25d^3 + 38d^2 - 28d + 8\big)\,Q^6 S^{\frac{-4d+14}{d-2}} + \\ & \big(8d^5 - 136d^4 + 912d^3 - 3024d^2 + 4968d - 3240\big)\,S^2\Big]\Big\}\end{aligned}$$

$$\begin{aligned}K^1_{12} = \;& -Q\Big[\big(d^7 - 12d^6 + 61d^5 - 170d^4 + 280d^3 - 272d^2 + 144d - 32\big)\,Q^6 S^{\frac{-5d+18}{d-2}} + \\ & \big(-4d^7 + 56d^6 - 264d^5 + 272d^4 + 1724d^3 - \\ & 6504d^2 + 8784d - 4320\big)\,Q^2 S^{\frac{-d+6}{d-2}} + \\ & \big(-8d^7 + 152d^6 - 1168d^5 + 4592d^4 - 9416d^3 + \\ & 8520d^2 + 864d - 4320\big)\,S^{\frac{d}{d-2}} + \\ & \big(2d^7 - 50d^6 + 438d^5 - 1918d^4 + 4696d^3 - \\ & 6528d^2 + 4800d - 1440\big)\,Q^4 S^{\frac{-3d+12}{d-2}}\Big] \Big/ \Big\{S\left[-2d + 6 + (d-2)\,Q^2 S^{\frac{-2d+6}{d-2}}\right] \\ & \Big[\big(8d^5 - 136d^4 + 912d^3 - 3024d^2 + 4968d - 3240\big)\,S^2 + \\ & \big(-2d^5 + 30d^4 - 158d^3 + 378d^2 - 416d + 168\big)\,Q^4 S^{\frac{-2d+18}{d-2}} + \\ & \big(-d^5 + 8d^4 - 25d^3 + 38d^2 - 28d + 8\big)\,Q^6 S^{\frac{-4d+14}{d-2}} + \\ & \big(4d^5 - 8d^4 - 232d^3 + 1424d^2 - 3036d + 2232\big)\,Q^2 S^{\frac{2}{d-2}}\Big]\Big\}\end{aligned}$$

$$\begin{aligned}K^1_{22} = \;& (d-2)\Big[\big(-4d^6 + 48d^5 - 168d^4 - 64d^3 + 1596d^2 - 3312d + 2160\big)\,Q^2 S^{\frac{4}{d-2}} + \\ & \big(-8d^6 + 136d^5 - 928d^4 + 3248d^3 - 6120d^2 + 5832d - 2160\big)\,Q^4 S^{\frac{-2d+10}{d-2}} + \\ & \big(d^6 - 10d^5 + 41d^4 - 88d^3 + 104d^2 - 64d + 16\big)\,Q^6 S^{\frac{-4d+16}{d-2}}\Big] \Big/ \\ & \Big\{S\left[-2d + 6 + (d-2)\,Q^2 S^{\frac{-2d+6}{d-2}}\right] \\ & \Big[\big(8d^5 - 136d^4 + 912d^3 - 3024d^2 + 4968d - 3240\big)\,S^2 + \\ & \big(-2d^5 + 30d^4 - 158d^3 + 378d^2 - 416d + 168\big)\,Q^4 S^{\frac{-2d+18}{d-2}} + \\ & \big(-d^5 + 8d^4 - 25d^3 + 38d^2 - 28d + 8\big)\,Q^6 S^{\frac{-4d+14}{d-2}} + \\ & \big(4d^5 - 8d^4 - 232d^3 + 1424d^2 - 3036d + 2232\big)\,Q^2 S^{\frac{2}{d-2}}\Big]\Big\}\end{aligned}$$

$$K^2_{11} = Q(d-3)\Big[\big(-16d^7+384d^6-3290d^5+22080d^4-74160d^3+148608d^2-164592d+77760\big)S+\big(-d^8+136d^7-1024d^6+4536d^5-13120d^4+25504d^3-32280d^2+23904d-7776\big)Q^4S^{\frac{-3d+10}{d-2}}\big(-16d^8+272d^7-1856d^6+6256d^5-9488d^4-368d^3+20160d^2-22032d+6048\big)Q^2S^{\frac{-d+4}{d-2}}+\big(4d^8-92d^7+868d^6-4472d^5+13888d^4-26732d^3+31176d^2-20112d+5472\big)Q^6S^{\frac{-5d+16}{d-2}}+\big(2d^8-29d^7+182d^6-645d^5+1400d^4-1944d^3+1648d^2-748d+160\big)Q^8S^{\frac{-7d+22}{d-2}}\Big] \Big/ \Big\{2S(d-2)^2\Big[-2d+6+(d-2)Q^2S^{\frac{-2d+6}{d-2}}\Big]\Big[\big(8d^5-136d^4+912d^3-3024d^2+4968d-3240\big)S^2+\big(-2d^5+30d^4-158d^3+378d^2-416d+168\big)Q^4S^{\frac{-2d+18}{d-2}}+\big(-d^5+8d^4-25d^3+38d^2-28d+8\big)Q^6S^{\frac{-4d+14}{d-2}}+\big(4d^5-8d^4-232d^3+1424d^2-3036d+2232\big)Q^2S^{\frac{2}{d-2}}\Big]\Big\}$$

$$K^2_{12} = -(d-3)\Big[\big(4d^7-84d^6+688d^5-2932d^4+7116d^3-9896d^2+7312d-2208\big)Q^6S^{\frac{-4d+14}{d-2}}+\big(-8d^7+120d^6-776d^5+2840d^4-6448d^3+9200d^2-7680d+2880\big)Q^4S^{\frac{-2d+8}{d-2}}+\big(-16d^6+320d^5-2640d^4+11520d^3-28080d^2+36288d-19440\big)S^2+\big(2d^7-25d^6+132d^5-381d^4+648d^3-648d^2+352d-80\big)Q^8S^{\frac{-6d+20}{d-2}}+\big(-16d^7+240d^6-1456d^5+4512d^4-7312d^3+5328d^2-432d-864\big)Q^2S^{\frac{2}{d-2}} \Big/ \Big\{\Big[-2d+6+(d-2)Q^2S^{\frac{-2d+6}{d-2}}\Big]\Big[\big(8d^5-136d^4+912d^3-3024d^2+4968d-3240\big)S^2+\big(-2d^5+30d^4-158d^3+378d^2-416d+168\big)Q^4S^{\frac{-2d+18}{d-2}}+\big(-d^5+8d^4-25d^3+38d^2-28d+8\big)Q^6S^{\frac{-4d+14}{d-2}}+\big(4d^5-8d^4-232d^3+1424d^2-3036d+2232\big)Q^2S^{\frac{2}{d-2}}\Big]2S(d-2)\Big\}$$

$$K^2_{22} = Q(d-3)\Big[\big(-4d^6+48d^5-248d^4+736d^3-1364d^2+1488d-720\big)Q^2S^{\frac{-d+6}{d-2}}+\big(2d^6-38d^5+258d^4-850d^3+1468d^2-1272d+432\big)Q^4S^{\frac{-3d+12}{d-2}}+\big(-8d^6+104d^5-512d^4+1104d^3-648d^2-1080d+1296\big)S^{\frac{d}{d-2}}+\big(d^6-10d^5+41d^4-88d^3+104d^2-64d+16\big)Q^4S^{\frac{-5d+18}{d-2}}\Big] \Big/ \Big\{S\Big[-2d+6+(d-2)Q^2S^{\frac{-2d+6}{d-2}}\Big]\Big[\big(-d^5+8d^4-25d^3+38d^2-28d+8\big)Q^6S^{\frac{-4d+14}{d-2}}+\big(-2d^5+30d^4-158d^3+378d^2-416d+168\big)Q^4S^{\frac{-2d+18}{d-2}}+\big(8d^5-136d^4+912d^3-3024d^2+4968d-3240\big)S^2+\big(4d^5-8d^4-232d^3+1424d^2-3036d+2232\big)Q^2S^{\frac{2}{d-2}}\Big]\Big\}$$

The geodesic equations are of the form

$$\frac{d^2S}{dt^2}+K^1_{11}\left(\frac{dS}{dt}\right)^2+K^1_{12}\frac{dS}{dt}\frac{dQ}{dt}+K^1_{22}\left(\frac{dQ}{dt}\right)^2=0$$
$$\frac{d^2Q}{dt^2}+K^2_{11}\left(\frac{dS}{dt}\right)^2+K^2_{12}\frac{dS}{dt}\frac{dQ}{dt}+K^2_{22}\left(\frac{dQ}{dt}\right)^2=0.$$

The non-zero components of the Riemannian curvature tensor are

$$K_{1212} = [(-4d^{10} + 148d^9 - 2140d^8 + 16868d^7 - 82148d^6 + 260876d^5 - 548724d^4 + 752940d^3 - 639144d^2 + 296784d - 54432)Q^4 S^{\frac{-d+7}{d-2}} + (8d^{10} - 264d^9 + 3888d^8 - 33664d^7 + 189840d^6 - 728784d^5 + 1929312d^4 - 3478464d^3 + 4088232d^2 - 2828520d + 874800)S^{\frac{3d-5}{d-2}} + (-2d^{10} + 8d^9 + 326d^8 - 4528d^7 + 28154d^6 - 103432d^5 + 242386d^4 - 367248d^3 + 348624d^2 - 188352d + 44064)Q^6 S^{\frac{-3d+13}{d-2}} + (0.5d^{10} - 4.5d^9 - 22d^8 + 521d^7 - 3473.5d^6 + 12807.6d^5 - 29481d^4 + 43484d^3 - . 40032d^2 + 20952d - 4752)Q^8 S^{\frac{-5d+19}{d-2}} + (-2d^{10} - 5d^9 + 44.5d^8 - 232d^7 + 784.25d^6 - 1795d^5 + 2815d^4 - 2984d^3 + 2044d^2 - 816d + 144)Q^{10} S^{\frac{-7d+25}{d-2}} + (4d^{10} - 80d^9 + 528d^8 + 160d^7 - 23096d^6 + 158688d^5 - 570240d^4 + 1243896d^3 - 1657260d^2 + 12480048d - 408240)Q^2 S^{\frac{d+1}{d-2}} + 6.4 \cdot 10^{-7}(6d^7 - 5d^6)Q^{12} S^{\frac{-9d+31}{d-2}}] / \Big\{ S^4 (d-2)^2 [-2d + 6 + (d-2)Q^2 S^{\frac{-2d+6}{d-2}}]^3 [(-d^5 + 8d^4 - 25d^3 + 38d^2 - 28d + 8)Q^6 S^{\frac{-4d+14}{d-2}} + (-2d^5 + 30d^4 - 158d^3 + 378d^2 - 416d + 168)Q^4 S^{\frac{-2d+18}{d-2}} + (8d^5 - 136d^4 + 912d^3 - 3024d^2 + 4968d - 3240)S^2 + (4d^5 - 8d^4 - 232d^3 + 1424d^2 - 3036d + 2232)Q^2 S^{\frac{2}{d-2}}] \Big\}$$

$$K_{1221} = K_{2112} = -K_{1212}, \; K_{2121} = K_{1212}$$

Physical characteristics of our geometrical model: the fundamental tensor $k = \text{Hess}_h M$ is piecewise pseudo-Riemannian; the components of the fundamental tensor k_{ij} are like state potentials; the Christoffel symbols and the curvature tensor diverge in the extremal limit and along the curves where the fundamental tensor changes signature; the manifold $\left(V \subset \{(S,Q)\,|S>0\} \subset R^2, k = \text{Hess}_h M\right)$ is non-flat; the sectional curvature has piecewise constant sign depending on the sign of $det\,(k)$ and the sign of K_{1212}.

4 Comparison between null length curves generated by the h_{ij} and k_{ij} metrics for a 5-dimensional black hole

In the theory of General Relativity the null-length curves are associated with light propagation, hence they are called light paths. Based on the previous results [5] for the 4-dimensional Reissner-Nördstrom black-hole, we are able to present a comparison between the trajectories in (S, Q) space of the null length curves equations generated by the two fundamental metrics. In the particular case of five dimensions, the degeneration curves and the solutions of the null length curves equations presented in Section 3 are represented in Figure Fig.1a. The behaviour of light rays for two arbitrary initial conditions in the case of h_{ij} metric is totally different from their behaviour associated with k_{ij} metric. In the first case, represented in Fig.1a we observe not only an intersection with the degeneration curves that leads to complex solutions but a path convergence towards the S axis which, in our opinion, is a non-physical solution. A more appropriate behaviour, similar to the classical light cone [8], is illustrated in figure Fig.1b. In this case, we notice the self-parallel, non-intersecting, character of the null length curves.

Hence, we consider that the pseudo-Riemannian manifold ($V \subset \{(S,Q)|S > 0\} \subset R^2$, $k = \mathrm{Hess}_h M$) has the appropriate characteristics required to sustain a thermodynamic geometric approach of such black holes.

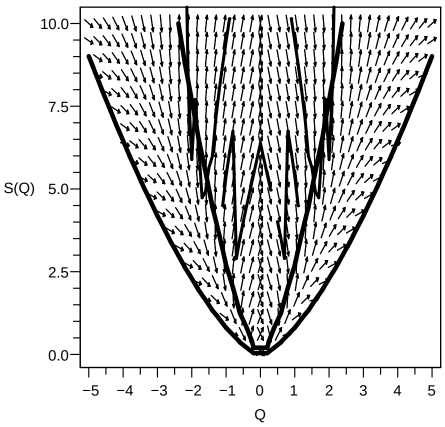

Fig.1a. Null-length curves.

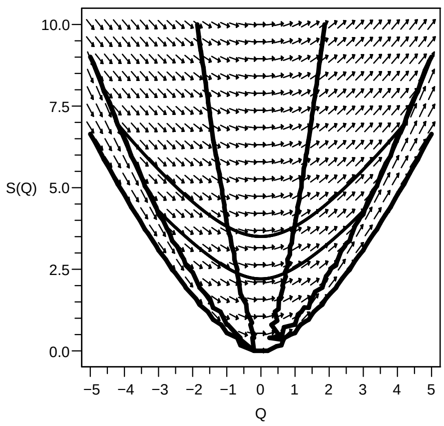

Fig.1b. Degeneration curves.

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