# Normal semi-invariant submanifolds of paraquaternionic space forms and mixed 3-structures

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**Abstract.** In this paper we study the normal semi-invariant submanifolds of paraquaternionic space forms and obtain some properties; in particular, we prove the non-existence of a mixed geodesic foliated normal semi-invariant submanifold in a non-flat paraquaternionic space form. Also, we give some conditions for the existence of a mixed 3-structure on a normal semi-invariant submanifold of a paraquaternionic Kähler manifold.

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## 1 Introduction

The paraquaternionic structures, firstly named quaternionic structures of second kind, have been introduced by P. Libermann [13] in 1954. The differential geometry of manifolds endowed with this kind of structures is a very interesting subject and these manifolds have been intensively studied by many authors [1], [5], [6], [7], [10], [19], [20].

On the other hand, the concept of normal semi-invariant submanifold has been introduced by Bejancu [3]. In the present note we study normal semi-invariant submanifolds of paraquaternionic space forms. The paper is organized as follows: in Section 2 one reminds basic definitions and fundamental properties of normal semi-invariant submanifolds of paraquaternionic Kähler manifolds. In Section 3 we obtain some properties for normal semi-invariant submanifolds in paraquaternionic space forms and prove the non-existence of mixed geodesic foliated normal semi-invariant submanifolds in non-flat paraquaternionic space forms. In the last Section we show the existence of a mixed 3-structure on a normal semi-invariant submanifold of a paraquaternionic Kähler manifold under some conditions.

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## 2 Preliminaries

Let $\overline{M}$ be a smooth manifold. We say that a rank-3 subbundle $\sigma$ of $End(T\overline{M})$ is an almost paraquaternionic structure on $\overline{M}$ if a local basis $\{J_1, J_2, J_3\}$ exists on sections of $\sigma$, such that for all $\alpha \in \{1,2,3\}$ we have:

$$J_\alpha^2 = -\epsilon_\alpha Id,\ J_\alpha J_{\alpha+1} = -J_{\alpha+1}J_\alpha = \epsilon_{\alpha+2}J_{\alpha+2} \tag{2.1}$$

where $\epsilon_1 = 1,\ \epsilon_2 = \epsilon_3 = -1$ and the indices are taken from $\{1,2,3\}$ modulo 3.

Let $(\overline{M}, \overline{g})$ be a semi-Riemannian manifold and $\sigma$ an almost paraquaternionic structure on $\overline{M}$. The metric $\overline{g}$ is said to be adapted to the paraquaternionic structure $\sigma$ if it satisfies:

$$\overline{g}(J_\alpha X, J_\alpha Y) = \epsilon_\alpha \overline{g}(X,Y),\ \alpha \in \{1,2,3\}, \tag{2.2}$$

for all vector fields $X, Y$ on $\overline{M}$ and any local basis $\{J_1, J_2, J_3\}$ of $\sigma$, where $\epsilon_1 = 1,\ \epsilon_2 = \epsilon_3 = -1$. In this case, $(\overline{M}, \sigma, \overline{g})$ is said to be an almost paraquaternionic hermitian manifold.

It is clear that any almost paraquaternionic hermitian manifold is of dimension $4m,\ m \geq 1$, and any adapted metric is necessarily of neutral signature $(2m, 2m)$.

Moreover, if the Levi-Civita connection of $\overline{g}$ satisfies the following conditions for all $\alpha \in \{1,2,3\}$:

$$\overline{\nabla}_X J_\alpha = -\epsilon_\alpha[\omega_{\alpha+2}(X)J_{\alpha+1} - \omega_{\alpha+1}(X)J_{\alpha+2}] \tag{2.3}$$

where $\epsilon_1 = 1,\ \epsilon_2 = \epsilon_3 = -1$ and the indices are taken from $\{1,2,3\}$ modulo 3, for any vector field $X$ on $\overline{M}$, $\omega_1, \omega_2, \omega_3$ being local 1-forms over the open for which $\{J_1, J_2, J_3\}$ is a local basis of $\sigma$, then $(\overline{M}, \sigma, \overline{g})$ is said to be a paraquaternionic Kähler manifold (see [7]). If $\omega_1 = \omega_2 = \omega_3 = 0$, then $(\overline{M}, \sigma, \overline{g})$ is said to be a locally para-hyper-Kähler manifold.

We remark that any paraquaternionic Kähler manifold is an Einstein manifold, provided that $dim M > 4$ (see [5], [7], [10], [19]).

Next, let $(M, g)$ be a non-degenerate submanifold of a paraquaternionic Kähler manifold $(\overline{M}, \overline{g}, \sigma)$, with $g = \overline{g}_{|M}$. Then $(M, g)$ is called a normal semi-invariant submanifold of $(\overline{M}, \overline{g}, \sigma)$ if there exists a non-degenerate vector subbubdle $Q$ of the normal bundle $TM^\perp$ such that:

(i) $J_\alpha(Q_p) = Q_p, \forall p \in M,\ \forall \alpha \in \{1,2,3\}$;

(ii) $J_\alpha(Q_p^\perp) \subset T_pM, \forall p \in M,\ \forall \alpha \in \{1,2,3\}$, where $Q^\perp$ is the complementary orthogonal bundle to $Q$ in $TM^\perp$ (see [3]).

If $(M, g)$ is a normal semi-invariant submanifold of a paraquaternionic Kähler manifold $(\overline{M}, \overline{g}, \sigma)$, then we set $D_{\alpha p} = J_\alpha(Q_p^\perp)$. We consider $D_{1p} \oplus D_{2p} \oplus D_{3p} = D_p^\perp$ and $3s$-dimensional distribution $D^\perp : p \mapsto D_p^\perp$ globally defined on $M$, where $s = dim D_p^\perp$. We denote by $D$ the complementary orthogonal distribution to $D^\perp$ in $TM$. We remark that $D$ is called the paraquaternionic distribution because it is invariant with respect to the action of $J_\alpha$, $\alpha \in \{1,2,3\}$ (see [3]).

Let $(M, g)$ be a normal semi-invariant submanifold of a paraquaternionic Kähler manifold $(\overline{M}, \sigma, \overline{g})$. Then we say that:

(i) $M$ is $D$-geodesic if:

$$B(X,Y)=0, \forall X,Y\in\Gamma(D).$$

(ii) $M$ is $D^{\perp}$-geodesic if:

$$B(X,Y)=0, \forall X,Y\in\Gamma(D^{\perp}).$$

(iii) $M$ is mixed geodesic if:

$$B(X,Y)=0, \forall X\in\Gamma(D), Y\in\Gamma(D^{\perp}).$$

We recall now the following results concerning the integrability of the distributions $D$ and $D^{\perp}$.

**Theorem 2.1.** [3] *Let $(M,g)$ be a normal semi-invariant submanifold of a paraquaternionic Kähler manifold $(\overline{M},\overline{g},\sigma)$. Then the following assertions are equivalent:*
*(i) The paraquaternionic distribution $D$ is integrable.*
*(ii) $M$ is $D$-geodesic.*
*(iii) The second fundamental form $h$ of $M$ satisfies:*

$$h(X,J_\alpha Y)=h(Y,J_\alpha X),\ \forall X,Y\in\Gamma(D),\ \alpha\in\{1,2,3\}.$$

**Theorem 2.2.** [3] *Let $(M,g)$ be a normal semi-invariant submanifold of a paraquaternionic Kähler manifold $(\overline{M},\overline{g},\sigma)$. Then the following assertions are equivalent:*
*(i) The distribution $D^{\perp}$ is integrable.*
*(ii) $h(X,Y)\in\Gamma(Q)$, $\forall\ X\in\Gamma(D),\ Y\in\Gamma(D^{\perp})$.*

**Remark 2.3.** *If $(M,g)$ is a normal semi-invariant submanifold of a paraquaternionic Kähler manifold $(\overline{M},\overline{g},\sigma)$, then for any $X\in\Gamma(D),\ Y\in\Gamma(TM)$ and $W\in\Gamma(Q)$ we can easily see that we have:*

$$\overline{g}(h(J_\alpha X,Y),W)=\overline{g}(J_\alpha h(X,Y),W), \tag{2.4}$$

*for all $\alpha\in\{1,2,3\}$.*

A normal semi-invariant submanifold $(M,g)$ of a paraquaternionic Kähler manifold $(\overline{M},\overline{g},\sigma)$ is said to be foliated if $D$ and $D^{\perp}$ are both non-null involutive distributions on $M$. Concerning the foliations $\mathcal{F}$ and $\mathcal{F}^{\perp}$ determined by the distributions $D$ and $D^{\perp}$ on $M$, provided they are integrable, Bejancu obtained that they are always totally geodesic (see [3]).

**Theorem 2.4.** *Let $(M,g)$ be a normal semi-invariant submanifold of a paraquaternionic Kähler manifold $(\overline{M},\overline{g},\sigma)$. Then the following assertions are equivalent:*
*(i) $M$ is foliated.*
*(ii) $h(X,Y)\in\Gamma(Q),\ \forall X\in\Gamma(D),\ Y\in\Gamma(TM)$.*

*Proof:* From (2.4) we derive (see also [2]):

$$h(X,Y)\in\Gamma(Q^{\perp}),\ \forall X,Y\in\Gamma(D)$$

and the proof is now clear from Theorems 2.1 and 2.2. □

# 3 Normal semi-invariant submanifolds of paraquaternionic space forms

Let $(\overline{M}, \sigma, \overline{g})$ be an almost hermitian paraquaternionic manifold. If $X \in T_p\overline{M}$, $p \in \overline{M}$, then the 4-plane $PQ(X)$ spanned by $\{X, J_1X, J_2X, J_3X\}$ is called a paraquaternionic 4-plane. A 2-plane in $T_p\overline{M}$ spanned by $\{X, Y\}$ is called half-paraquaternionic if $PQ(X) = PQ(Y)$.

Let $\pi = Sp\{X, Y\}$ be a plane tangent to $\overline{M}$ at a point $p \in \overline{M}$. The sectional curvature of $\pi$, denoted $K(\pi)$, is defined by:

$$K(\pi) = \frac{\overline{R}(X,Y,X,Y)}{\overline{g}(X,X)\overline{g}(Y,Y) - \overline{g}(X,Y)^2}. \tag{3.1}$$

It is clear that this definition makes sense only for non-degenerate plane, i.e. those satisfying: $\overline{g}(X,X)\overline{g}(Y,Y) - \overline{g}(X,Y)^2 \neq 0$

The sectional curvature for a half-paraquaternionic 2-plane is called sectional curvature. A paraquaternionic Kähler manifold of constant paraquaternionic sectional curvature is said to be a paraquaternionic space form.

It is well-known that a paraquaternionic Kähler manifold $(\overline{M}, \sigma, \overline{g})$ is a paraquaternionic space form, denoted $\overline{M}(c)$, if and only if its curvature tensor is:

$$\begin{aligned}\overline{R}(X,Y)Z &= \frac{c}{4}\{\overline{g}(Z,Y)X - \overline{g}(X,Z)Y + \sum_{\alpha=1}^{3}\epsilon_\alpha[\overline{g}(Z,J_\alpha Y)J_\alpha X \\ &\quad -\overline{g}(Z,J_\alpha X)J_\alpha Y + 2\overline{g}(X,J_\alpha Y)J_\alpha Z]\}\end{aligned} \tag{3.2}$$

for all vector fields $X, Y, Z$ on $M$ and any local basis $\{J_1, J_2, J_3\}$ of $\sigma$.

**Remark 3.1.** *Let $\overline{M}(c)$ be a paraquaternionic space form. Then, from (3.2) we deduce that the sectional curvature of 2-plane spanned by the pseudo-orthonormal vectors $X, Y \in T_p\overline{M}$, $p \in \overline{M}$, is:*

$$\overline{K}(X \wedge Y) = \frac{c}{4\epsilon_X\epsilon_Y}[\epsilon_X\epsilon_Y + 3\sum_{\alpha=1}^{3}\epsilon_\alpha\overline{g}(J_\alpha X, Y)^2], \tag{3.3}$$

*where $\epsilon_X = g(X,X)$, $\epsilon_Y = g(Y,Y)$, $\epsilon_1 = 1$, and $\epsilon_2 = \epsilon_3 = -1$.*

**Lemma 3.2.** *Let $(M, g)$ be a normal semi-invariant submanifold of a paraquaternionic space form $(\overline{M}(c), \sigma, \overline{g})$. Then, the sectional curvature of 2-plane spanned by the orthonormal vectors $X, Y \in T_pM$, $p \in M$, is:*

$$\begin{aligned}K(X \wedge Y) &= \frac{c}{4\epsilon_X\epsilon_Y}[\epsilon_X\epsilon_Y + 3\sum_{\alpha=1}^{3}\epsilon_\alpha\overline{g}(J_\alpha X, Y)^2] \\ &\quad +\overline{g}(h(X,X), h(Y,Y)) - \overline{g}(h(X,Y), h(X,Y)).\end{aligned} \tag{3.4}$$

*Proof.* The assertion follows from (3.3) and Gauss equation. □

**Lemma 3.3.** *Let $(M,g)$ be a normal semi-invariant submanifold of a paraquaternionic space form $(\overline{M}(c),\sigma,\overline{g})$. Then the sectional curvature of 2-plane spanned by the space-like or time-like unit vectors $X, J_\alpha X \in D_p,\ p\in M$, is:*

$$\begin{aligned} H_{J_\alpha}(X) &= -\frac{c}{2}+\overline{g}(h(X,X),h(J_\alpha X,J_\alpha X)) \\ &\quad -\overline{g}(h(X,J_\alpha X),h(X,J_\alpha X)). \end{aligned} \tag{3.5}$$

*Proof.* The formula follows immediately from (3.4). □

**Proposition 3.4.** *Let $(M,g)$ be a normal semi-invariant submanifold of a paraquaternionic space form $(\overline{M}(c),\sigma,\overline{g})$. If the paraquaternionic distribution $D$ is integrable, then $H_{J_\alpha}(X) = -\frac{c}{2}$, for any space-like or time-like unit vector field $X\in\Gamma(D)$ and $\alpha\in\{1,2,3\}$.*

*Proof.* The assertion is clear from Theorem 2.1 and (3.5). □

**Theorem 3.5.** *There are no mixed geodesic foliated normal semi-invariant submanifold of a paraquaternionic space form $\overline{M}(c)$, with $c\neq 0$.*

*Proof.* Let $M$ be a foliated normal semi-invariant submanifold of $\overline{M}(c)$. If we take $X\in\Gamma(D)$ and $W\in\Gamma(Q^\perp)$, from (3.2) we obtain:

$$[\overline{R}(X,J_\alpha X)J_\alpha W]^\perp = \frac{c}{2}\epsilon_\alpha g(X,X)W \tag{3.6}$$

where $\perp$ denotes the normal component, and so we have:

$$\begin{aligned} \overline{R}(X,J_\alpha X,W,J_\alpha W) &= \overline{g}([\overline{R}(X,J_\alpha X)J_\alpha W]^\perp,W) \\ &= \frac{c}{2}\epsilon_\alpha g(X,X)\overline{g}(W,W) \end{aligned} \tag{3.7}$$

From Codazzi equation we have:

$$\begin{aligned} \overline{R}(X,J_\alpha X,W,J_\alpha W) &= \overline{g}((\nabla^\perp_X h)(J_\alpha X,J_\alpha W)-(\overline{\nabla}_{J_\alpha X}h)(X,J_\alpha W),W) \\ &= \overline{g}(\nabla^\perp_X h(J_\alpha X,J_\alpha W),W)-\overline{g}(\nabla^\perp_{J_\alpha X}h(X,J_\alpha W),W) \end{aligned} \tag{3.8}$$

because the normal connection $\nabla^\perp$ satisfies:

$$(\nabla^\perp_U h)(V,Z)=\nabla^\perp_U h(V,Z)-h(\nabla_U V,Z)-h(V,\nabla_U Z),\ \forall U,V,Z\in\Gamma(TM).$$

On the other hand we have:

$$\begin{aligned} \overline{g}(\nabla^\perp_X h(J_\alpha X,J_\alpha W),W) &= -\overline{g}(h(J_\alpha X,J_\alpha W),\overline{\nabla}_X W) \\ &= \epsilon_\alpha\overline{g}(h(J_\alpha X,J_\alpha W),-J_\alpha(\overline{\nabla}_X J_\alpha)W+J_\alpha\overline{\nabla}_X J_\alpha W) \\ &= \epsilon_\alpha\overline{g}(h(J_\alpha X,J_\alpha W),J_\alpha(\nabla_X J_\alpha W+h(X,J_\alpha W))) \end{aligned}$$

and so we can easily derive:

$$\overline{g}(\nabla^\perp_X h(J_\alpha X,J_\alpha W),W)=\epsilon_\alpha\overline{g}(h(J_\alpha X,J_\alpha W),h(J_\alpha X,J_\alpha W))). \tag{3.9}$$

Similarly we find:

$$\overline{g}(\nabla^{\perp}_{J_\alpha X} h(X, J_\alpha W), W) = -\epsilon_\alpha \overline{g}(h(J_\alpha X, J_\alpha W), h(J_\alpha X, J_\alpha W))). \tag{3.10}$$

From (3.8), (3.9) and (3.10) we obtain:

$$\overline{R}(X, J_\alpha X, W, J_\alpha W) = 2\epsilon_\alpha \overline{g}(h(J_\alpha X, J_\alpha W), h(J_\alpha X, J_\alpha W))). \tag{3.11}$$

Now, from (11) and (15) we derive:

$$cg(X, X)\overline{g}(W, W) = 4\overline{g}(h(J_\alpha X, J_\alpha W), h(J_\alpha X, J_\alpha W))). \tag{3.12}$$

If $M$ is a mixed geodesic, from (3.12) we derive $c = 0$ and the proof is now complete. □

**Corollary 3.6.** *There are no totally geodesic foliated normal semi-invariant submanifold of a paraquaternionic space form $\overline{M}(c)$, with $c \neq 0$.*

*Proof.* The assertion follows from Theorem 3.5. □

# 4 Normal semi-invariant submanifolds and mixed 3-structures

**Definition 4.1.** Let $M$ be a differentiable manifold equipped with a triple $(\phi, \xi, \eta)$, where $\phi$ is a a field of endomorphisms of the tangent spaces, $\xi$ is a vector field and $\eta$ is a 1-form on $M$. If we have:

$$\phi^2 = -\epsilon I + \eta \otimes \xi, \ \eta(\xi) = \epsilon \tag{4.1}$$

then we say that:
(i) $(\phi, \xi, \eta)$ is an almost contact structure on $M$, if $\epsilon = 1$ (cf. [4]).
(ii) $(\phi, \xi, \eta)$ is a Lorentzian almost paracontact structure on $M$, if $\epsilon = -1$ (cf. [15]).

We remark that many authors also include in the above definition the conditions that $\phi\xi = 0$ and $\eta \circ \phi = 0$, although these are deducible from the conditions (4.1) (see [4]).

**Definition 4.2.** ([9]) Let $M$ be a differentiable manifold which admits an almost contact structure $(\phi_1, \xi_1, \eta_1)$ and two Lorentzian almost paracontact structures $(\phi_2, \xi_2, \eta_2)$ and $(\phi_3, \xi_3, \eta_3)$, satisfying the following conditions:

$$\eta_\alpha(\xi_\beta) = 0, \forall \alpha \neq \beta, \tag{4.2}$$

$$\phi_\alpha(\xi_\beta) = -\phi_\beta(\xi_\alpha) = \epsilon_\gamma \xi_\gamma, \tag{4.3}$$

$$\eta_\alpha \circ \phi_\beta = -\eta_\beta \circ \phi_\alpha = \epsilon_\gamma \eta_\gamma, \tag{4.4}$$

$$\phi_\alpha \circ \phi_\beta - \eta_\beta \otimes \xi_\alpha = -\phi_\beta \circ \phi_\alpha + \eta_\alpha \otimes \xi_\beta = \epsilon_\gamma \phi_\gamma, \tag{4.5}$$

where $(\alpha, \beta, \gamma)$ is an even permutation of (1,2,3) and $\epsilon_1 = 1, \epsilon_2 = \epsilon_3 = -1$.

Then the manifold $M$ is said to have a mixed 3-structure $(\phi_\alpha, \xi_\alpha, \eta_\alpha)_{\alpha\in\{1,2,3\}}$. Moreover, if the manifold $M$ admits a semi-Riemannian metric $g$ such that:

$$g(\phi_\alpha X, \phi_\alpha Y) = \epsilon_\alpha g(X,Y) - \eta_\alpha(X)\eta_\alpha(Y), \tag{4.6}$$

and

$$g(X, \xi_\alpha) = \eta_\alpha(X) \tag{4.7}$$

for all $X, Y \in \Gamma(TM)$ and $\alpha \in \{1,2,3\}$, then we say that $M$ has a metric mixed 3-structure and $g$ is called a compatible metric.

**Definition 4.3.** A metric mixed 3-structure $(\phi_\alpha, \xi_\alpha, \eta_\alpha, g)_{\alpha\in\{1,2,3\}}$ on a manifold $M$ is said to be:
(i) a mixed 3-Sasakian structure if $(\phi_1, \xi_1, \eta_1, g)$ is a Sasakian structure, i.e. (cf. [4]):

$$(\nabla_X \phi_1)Y = g(X,Y)\xi_1 - \eta_1(Y)X$$

and $(\phi_2, \xi_2, \eta_2, g)$, $(\phi_3, \xi_3, \eta_3, g)$ are Lorentzian para-Sasakian structures, i.e. (cf. [15]):

$$(\nabla_X \phi_2)Y = g(\phi_2 X, \phi_2 Y)\xi_2 + \eta_2(Y)\phi_2^2 X,$$
$$(\nabla_X \phi_3)Y = g(\phi_3 X, \phi_3 Y)\xi_3 + \eta_3(Y)\phi_3^2 X,$$

for all vector fields $X, Y$ on $M$, where $\nabla$ is the Levi-Civita connection of $g$.
(ii) a mixed 3-cosymplectic structure if:

$$(\nabla_X \phi_\alpha)(Y) = 0,\ (\nabla_X \eta_\alpha)(Y) = 0,\ \alpha \in \{1,2,3\},$$

for all vector fields $X, Y$ on $M$.

**Theorem 4.4.** *Let $(M,g)$ be a normal semi-invariant submanifold of a paraquaternionic Kähler manifold $(\overline{M}, \sigma, \overline{g})$ such that $dimQ^\perp = 1$ and $\overline{g}_{|Q^\perp}$ is positive definite. Then $M$ admits a metric mixed 3-structure.*

*Proof.* For any $X \in \Gamma(TM)$ and $\alpha \in \{1,2,3\}$ we have the decomposition:

$$J_\alpha X = \phi_\alpha X + F_\alpha X,$$

where $\phi_\alpha X$ and $F_\alpha X$ are the tangent part and the normal part of $J_\alpha X$, respectively. We can remark that, in fact, $F_\alpha X \in \Gamma(Q^\perp),\ \forall X \in \Gamma(TM)$.

Since $dimQ^\perp = 1$ and $\overline{g}_{|Q^\perp}$ is positive definite, we have $Q^\perp = < N >$, where $N$ is a unit space-like vector field and we can see that we have the decomposition:

$$J_\alpha X = \phi_\alpha X + \eta_\alpha(X)N, \tag{4.8}$$

where:

$$\eta_\alpha(X) = \overline{g}(J_\alpha X, N). \tag{4.9}$$

We define now the vector field $\xi_\alpha$ by:

$$\xi_\alpha = -J_\alpha N,$$

for any $\alpha \in \{1,2,3\}$.

By straightforward computation it follows now that $(\phi_\alpha, \xi_\alpha, \eta_\alpha, g)_{\alpha\in\{1,2,3\}}$ is a metric mixed 3-structure on $M$. □

**Theorem 4.5.** *Let $(M, g)$ be a normal semi-invariant submanifold of a paraquaternionic Kähler manifold $(\overline{M}, \sigma, \overline{g})$ such that $dimQ^{\perp} = 1$ and $\overline{g}_{|Q^{\perp}}$ is positive definite. If $(\nabla_X \phi_\alpha)Y = 0,\ \forall X, Y \in \Gamma(D)$ and $\alpha \in \{1, 2, 3\}$, then the distribution $D$ is integrable.*

*Proof.* For all $X, Y \in \Gamma(D)$ and $\alpha \in \{1, 2, 3\}$, since the paraquaternionic distribution $D$ is invariant under $\phi_\alpha$, we can always find $Z \in \Gamma(D)$ with $Y = \phi_\alpha Z$ and we obtain (see also [8]):

$$g([X, Y], \xi_\alpha) = 2g(\nabla_X \phi_\alpha Z, \xi_\alpha) = 2g(\phi_\alpha(\nabla_X Z), \xi_\alpha) = 0,$$

which implies:

$$[X, Y] \in \Gamma(D),\ \forall X, Y \in \Gamma(D).$$

Thus $D$ is integrable. □

**Corollary 4.6.** *Let $(M, g)$ be a normal semi-invariant submanifold of a paraquaternionic Kähler manifold $(\overline{M}, \sigma, \overline{g})$ with $dimQ^{\perp} = 1$ and $\overline{g}_{|Q^{\perp}}$ positive definite, such that the canonical metric mixed 3-structure is mixed 3-cosymplectic. Then the paraquaternionic distribution $D$ is integrable.*

*Proof.* The assertion follows from the above Theorem. □

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