L-moment problems

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Abstract. In this note we solve two L -moment problems. The first problem is a two dimensional complex L-moment problem on the closed unit disc in $\mathbb{C}$ having as data a complex sequence $\{y_{\alpha,\beta}\} \subset \mathbb{C}$. The second is a real L-moment problem on an arbitrary compact set in $\mathbb{R}$ having as data a real sequence $\{A_n\}_n \subset \mathbb{R}$.

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1 Introduction

This note is devoted to the L-moment problem. The L-moment problem consists in characterizing the sequence of moments $a_n = \int_{\mathbb{R}} t^n f(t)dt, n \in \mathbb{N}$ of a measurable function f (with prescribed support) which satisfies a boundedness condition such as $0 \leq f \leq L$ $a.e.dt$. The L-moment problem was formulated and completely solved by Akhiezer and Krein in the thirties [2]. The interest for the moments of a bounded measurable function on the real axis goes back to A. A.Markov in the ninth decade of the last century. The aim of the present paper is to study a two dimensional complex L-moment problem (in Section 2) and a one-dimensional real moment problem (in Section 3).

2 A complex L-moment problem

The L-complex moment problem consists in characterizing a double complex sequence $\{y_{\alpha,\beta}\}_{\alpha,\beta\in\mathbb{Z}^+}$ in order to represent the moments of a measurable function g on D_1 which satisfies a boundedness condition $0 \leq g \leq L$ $a.e.d\mu$ for a positive constant $L > 0$. We state and solve this L-complex moment problem in the following theorem.

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Theorem 1. *Let $D_1 = \{z \in \mathbb{C}, |z| \le 1\}$ be the closed unit disc endowed with a planar positive measure μ (e.g., the measure constructed in [11]). Let $\{y_{\alpha,\beta}\}_{\alpha\beta\in\mathbb{Z}^+}$ be a sequence of complex numbers with a finite number of non-zero terms. The following assertions are equivalent:*

(i) There exists a positive constant $M > 0$ such that for all finitely supported multi-sequence $\{\xi_\alpha\}_{\alpha\in\mathbb{N}} \subset \mathbb{C}$ and any $k \in \mathbb{N}$ we have:

$$0 \le \sum_{\alpha,\beta} \sum_{0\le\theta\le k} (-1)^\theta C_k^\theta y_{\alpha+\theta,\beta+\theta} \xi_\alpha \overline{\xi_\beta} \le$$

$$\le M \sum_{\alpha,\beta} \sum_{0\le\theta\le k} (-1)^\theta C_k^\theta c_{\alpha+\theta,\beta+\theta} \xi_\alpha \overline{\xi_\beta},$$

where $c_{\alpha,\beta}$ stands for

$$c_{\alpha,\beta} = \int_{D_1} z^\alpha \overline{z}^\beta d\mu(z).$$

(ii) There exists a real-valued bounded function $g \in L^\infty(D_1)$ with $0 \le g \le L$ a.e.μ for L a positive constant, such that:

$$y_{\alpha,\beta} = \int_{D_1} z^\alpha \overline{z}^\beta g(z) d\mu(z)\ \forall \alpha, \beta \in \mathbb{Z}_+.$$

Proof. (i)⇒(ii) Let $P_n(z,\overline{z}) = \{P(z,\overline{z}) = \sum a_{\alpha,\beta} z^\alpha \overline{z}^\beta, a_{\alpha,\beta} \in \mathbb{C}\}$ be the $\mathbb{C}$-vector space of polynomials with complex coefficients in $z, \overline{z}$. We define the linear functional $L(z^\alpha \overline{z^\beta}) = y_{\alpha,\beta}$ for all $\alpha, \beta \in \mathbb{Z}_+$ and extend L to $P_n(z,\overline{z})$ by linearity. From (i) we have $0 \le L(P(z,\overline{z})) \le \int_{D_1} P(z,\overline{z}) d\mu(z)$ for all polynomials $P(z,\overline{z}) = |Q(z)|^2(1-|z|^2)^k$ with $Q(z) = \sum \xi_\alpha z^\alpha$ an analytical polynomial and for any $k \in \mathbb{N}$. Let $f(z,\overline{z})$ be an arbitrary polynomial in $P_n(z,\overline{z})$; in this case $f(z,\overline{z}) = f_1(z,\overline{z}) + i f_2(z,\overline{z})$, with f_1, f_2 polynomials in $z, \overline{z}$ with real coefficients. If we take $z = x + iy$, we obtain $f_j(z,\overline{z}) = f_j^1(x,y) + i f_j^2(x,y), j = 1,2$ with f_j^k polynomials in real variable x, y with real coefficients for all $k = 1,2, j = 1,2$. From Cassier's paper [4] and from [10], we have the the decompositions: $f_1^1(x,y) = q_1(z,\overline{z}) - q_2(z,\overline{z})$ and $f_1^2(x,y) = q_3(z,\overline{z}) - q_4(z,\overline{z})$, similary $f_2^1(x,y) = q_5(z,\overline{z}) - q_6(z,\overline{z})$ and $f_2^2(x,y) = q_7(z,\overline{z}) - q_8(z,\overline{z})$ with $q_i(z,\overline{z}) = |P(z)|^2(1-|z|^2)^{k_i}$, for $P(z) \in P_\alpha = \{P, P(z) = \sum a_\alpha z^\alpha, a_\alpha \in \mathbb{C}\}$.

In this case

$$|L(f)| = |L(f_1) + iL(f_2)| \le |L(f_1)| + |L(f_2)|$$

and

$$(a) |L(f_1)| \le |L(f_1^1)| + |L(f_2^1)| = |L(q_1 - q_2)| + |L(q_3 - q_4)| \le$$

$$\le \int_{D_1} |q_1(z,\overline{z}) - q_2(z,\overline{z})| d\mu(z) + \int_{D_1} |q_3(z,\overline{z}) - q_4(z,\overline{z})| d\mu(z) \le$$

$$\le 2 \int_{D_1} |f_1| d\mu(z)$$

Similarly

$$(b) \quad |L(f_2)| \le 2 \int_{D_1} |f_1| d\mu(z)$$

From (a) and (b), we obtain:

$$(c)\ \ |L(f)| \leq 4 \int_{D_1} |f| d\mu(z)$$

for any $f \in P_n(z, \overline{z})$. We extend with the Hahn-Banach theorem the functional L on $L^1(D_1)$ preserving the inequality (c). In this case, L is a bounded functional on $L^1(D_1)$; it follows that there exists $g \in L^\infty(D_1)$ such that $L(f) = \int_{D_1} f(z)g(z)d\mu(z)$ for any $f \in L^1(D_1)$. Because $z^\alpha \overline{z}^\beta \in L^1(D_1)$ it follows that $L(z^\alpha, \overline{z^\beta}) = y_{\alpha,\beta} = \int_{D_1} z^\alpha \overline{z^\beta} g(z)d\mu(z)$. In this case we have

$$0 \leq L(|p(z)|^2(1-|z|^2)^k) = \int_{D_1} |P(z)|^2(1-|z|^2)^k g(z)d\mu(z)$$

for any analytical polynomial $P(z)$ and any $k \in \mathbb{N}$, we have also $0 \leq L(h) = \int_{D_1} h(z,\overline{z})g(z)d\mu(z)$ for any Hermitian polynomial $h(z, \overline{z})$ with $h(z, \overline{z}) \geq 0$ on D_1. In this case $0 \leq L(f) = \int_{D_1} f(z)g(z)d\mu(z)$ for any continuous function f on D_1 with $f(z) \geq 0$. It follows that $g(z) \geq 0$ $a.e.\mu$. Similarly, $g \leq 4$ $a.e.\mu$ on D_1. From both inequalities, we get $0 \leq g \leq 4$ $a.e.\mu$. Conversely: ii)⇒ *i*).

From (ii), we have

$$\begin{gathered}\sum_{\alpha,\beta} \sum_{0\leq\theta\leq k} (-1)^\theta C_k^\theta \int_{D_1} z^{\alpha+\theta} \overline{z}^{\beta+\theta} g(z) \xi_\alpha \overline{\xi_\beta} d\mu(z) = \\ = \int_{D_1} (1-|z|^2)^k |P(z)|^2 g(z)d\mu(z) \leq L \int_{D_1} (1-|z|^2)^k |P(z)|^2 d\mu(z) = \\ = L \sum_{\alpha,\beta} \sum_{0\leq\theta\leq k} (-1)^\theta C_k^\theta c_{\alpha+\theta,\beta+\theta} \xi_\alpha \overline{\xi_\beta}.\end{gathered}$$

3 A real L-moment problem

Another real L-moment problem appears when we connect the L-moment problem of a continuous, compact supported function on the real axis and the solution of the Dirichlet problem for bi-harmonic equation in the half plane. The Dirichlet problem is the following:

Theorem 2. *Let* $H = \{x + iy, x, y \in \mathbb{R}, y > 0\}$ *and* $c = \mathbb{R} = (-\infty, +\infty)$*. The Dirichlet problem (D.P.) requires to find a harmonic function* $U(x, y)$ *from* $C^2(H)$ *continuous on* $H \cup c$*, such that* $U|_{y=0} = U^*(x)$ *. The solution of the D.P. ([3],[7]) is:* $U(x, y) = \frac{y}{\pi} \int_{-\infty}^{+\infty} \frac{U^*(t)}{(x-t)^2+y^2} dt$.

It is also known from Cissotti's formula that:

Theorem 3. *If on the c axis, the real part* $U^*(x)$ *and the imaginary part* $V^*(x)$ *are known, under regularity conditions one can drive in* H *the holomorphic functions* $f(z) = \frac{1}{i\pi} \int_{-\infty}^{+\infty} \frac{U^*(t)}{t-z} dt + iK_1$ *or* $f(z) = \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{V^*(t)}{t-z} dt + K_2$ *whose real and imaginary parts respectively are solutions of the D.P. The real constants* K_1, K_2 *can be determined knowing the value of* $f(z)$ *at a single point in* H*.*

Indeed, we have

$$\mathbb{R}f(z) = \mathbb{R}\frac{1}{i\pi}\int_{-\infty}^{+\infty}\frac{U^*(t)}{t-z}dt + iK_2 = -\mathbb{R}\frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{U^*(t)}{t-(x+iy)}dt$$

$$= \mathbb{R}\frac{y}{\pi}\int_{-\infty}^{+\infty}\frac{U^*(t)}{(t-x)^2+y^2}dt.$$

In connection with the D.P. and the real L-moment problem we can prove the following theorem:

Theorem 4. *The function f represents the solution of the D.P. with data a compact supported continous function on $\mathbb{R}$ if and only if there exists a L -moment sequence $\{A_n\}_n$ for a continuous, compact supported function on $\mathbb{R}$ such that we have the representation:$f(z) = -\frac{1}{i\pi}\sum_{n=o}^{+\infty} A_n z^{-n-1}$ when $|z| > L > 0$ and $L \geq \{inf|suppf|, sup|suppf|\}$ with* inf *and* sup *being the lower, respectively the upper bound of the compact support of f.*

Proof. *Let $f(z)$ be the solution of the D.P. with data a function $U^* \in C_0^0(\mathbb{R})$. In this case, according to Cisotti's formula, the function*

$$f(z) = \frac{1}{i\pi}\int_{-\infty}^{+\infty}\frac{U^*(t)}{t-z}dt$$

is the solution of the D.P. We have for $|z| \geq max\{\inf|suppf|, \sup|suppf|\}$

$$f(z) = \frac{1}{i\pi}\int_{-\infty}^{+\infty}\frac{U^*(t)}{z(1-\frac{t}{z})}dt = -\frac{i}{\pi}\int_{-\infty}^{+\infty}\frac{U^*(t)}{z}\left(\sum_{n=0}^{+\infty}\frac{t^n}{z^n}dt\right)$$

$$= -\frac{1}{i\pi}\sum_{n=0}^{+\infty}\left[\int_{-\infty}^{+\infty}U^*(t)t^n dt\right]z^{-n-1}.$$

Taking $A_n = \int_{-\infty}^{+\infty} U^*(t)t^n dt$, we get the required representation.

Conversely, let $\{A_n\}_n$ be a L-moment problem sequence of a continuous, compact supported function on $\mathbb{R}$, $U^* \in C_0^0(\mathbb{R})$ that is $A_n = \int_{-\infty}^{+\infty} U^*(t)t^n dt$. Computing the expression

$$f(z) = -\frac{1}{i\pi}\sum_{n=0}^{+\infty}A_n z^{-n-1} = -\frac{1}{i\pi}\sum_{n=0}^{+\infty}[\int_{-\infty}^{+\infty}U^*(t)t^n dt]z^{-n-1} =$$

$$= -\frac{1}{i\pi}\int_{-\infty}^{+\infty}\frac{U^*(t)}{t-z}dt$$

for $|z| > L$, we get from Cissoti's formula, that $f(z)$ is a solution of the D.P. with data a function $U^* \in C_0^2(\mathbb{R})$.$\square$

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