Distinguished tensors and Poisson brackets in the multi-time Hamilton Geometry

Gheorghe Atanasiu and Mircea Neagu

Abstract. The aim of this paper is to present the main geometrical objects on the dual 1-jet vector bundle $J^{1*}(\mathcal{T}, M)$ (this is the polymomentum phase space of the De Donder-Weyl covariant Hamiltonian formulation of field theory) that characterize our approach of multi-time Hamilton geometry. In this direction, we firstly introduce the geometrical concept of a nonlinear connection N on the dual 1-jet space $J^{1*}(\mathcal{T}, M)$. Then, we construct its adapted bases of vector or covector fields and we compute the Poisson brackets of adapted d-vector fields. An almost product structure is naturally given.

M.S.C. 2000: 53B40, 53C60, 53C07.
Key words: dual 1-jet spaces, nonlinear connections, d-tensors, Poisson brackets, almost product structure.

1 Introduction

It is well known that the 1-jet spaces are the basic mathematical objects used in the study of classical and quantum field theories. For this reason, the differential geometry of 1-jet bundles was intensively studied by a lot of authors like, for example, (in chronological order) Saunders [24], Asanov [4], Neagu and Udrişte [23], [20], [22].

In the last decades, numerous physicists and geometers were preoccupied with the development of that so-called the *covariant Hamiltonian geometry of physical fields*, which is the *multi-parameter*, or *multi-time*, extension of the classical Hamiltonian formulation from Mechanics.

It is important to point out that the covariant Hamiltonian geometry of physical fields appears in the literature of specialty in three distinct variants:

1. the *multisymplectic* geometry, which is developed by Gotay, Isenberg, Marsden, Montgomery and their co-workers [11], [12] on a finite-dimensional multisymplectic phase space.

BSG Proceedings 16. The Int. Conf. of Diff. Geom. and Dynamical Systems (DGDS-2008) and The V-th Int. Colloq. of Mathematics in Engineering and Numerical Physics (MENP-5) - math. sections, August 29 - September 2, 2008, Mangalia, Romania, pp. 12-27.

2. the *polysymplectic* geometry, which is elaborated by Giachetta, Mangiarotti and Sardanashvily [9], [10], emphasizing the relations between the equations of first order Lagrangian field theory on fiber bundles and the covariant Hamilton equations on a finite-dimensional polysymplectic phase space.

3. the *De Donder-Weyl covariant Hamiltonian* geometry, intensively studied by Kanatchikov (please see the papers [13], [14], [15] and references therein) as opposed to the conventional field-theoretical Hamiltonian formalism, which requires the space + time decomposition and leads to the picture of a field as a mechanical system with infinitely many degrees of freedom.

From the perspective of geometers, we point out that, following the geometrical ideas *initially stated* by Asanov in the paper [4], a *multi-time Lagrange contravariant geometry on 1-jet spaces* (in the sense of distinguished linear connections, torsions and curvatures) was recently developed by Neagu and Udrişte in the works [20], [22] and [23]. This geometrical theory is a natural multi-time extension on 1-jet spaces of the already classical *Lagrange geometrical theory of the tangent bundle* elaborated by Miron and Anastasiei [17]. Note also that recent new geometrical developments, which relies on the multi-time Lagrange contravariant geometrical ideas from [20], are given by Udrişte and his co-workers in the papers [8] and [25].

On the other hand, suggested by the field theoretical extension of the basic structures of classical Analytical Mechanics within the framework of the De Donder-Weyl covariant Hamiltonian formulation, the studies of Miron [16], Atanasiu [5], [6] and others led to the development of the *Hamilton geometry of the cotangent bundle* exposed in the book [18]. This is the point start in our geometrical study, given in this paper, which is called by us the *multi-time covariant Hamilton geometry.* Note that the multi-time covariant Hamilton geometry is a natural multi-time generalization of the Hamilton geometry of the cotangent bundle from the book [18].

2 The dual 1-jet bundle $J^{1*}(\mathcal{T}, M)$

Let $\mathcal{T}$ and M be two real smooth manifolds of dimensions m, respectively n, whose local coordinates are $(t^a)_{a=\overline{1,m}}$, respectively $\left(x^i\right)_{i=\overline{1,n}}$.

Remark 2.1. i) Throughout this paper all geometrical objects and all mappings are considered of class C^∞, expressed by the words *differentiable* or *smooth.*

ii) Note that the indices a, b, c, d, e, f, g, h run over the set $\{1, 2, ..., m\}$, the indices i, j, k, l, p, q, r, s run over the set $\{1, 2, ..., n\}$, and the Einstein convention of summation is adopted all over this work.

Let us consider the 1-jet vector bundle

$$J^1(\mathcal{T}, M) \to \mathcal{T} \times M,$$

which has the coordinates (t^a, x^i, x^i_a) and the fibre type $\mathbb{R}^{mn}$.

Remark 2.2. From a physical point of view, we use the following terminology:

- The manifold $\mathcal{T}$ is regarded as a *temporal* manifold, or a *multi-time* manifold. Note that our temporal coordinates t^a are like the space-time variables in the De Donder-Weyl covariant Hamilton geometry [13], [15].
- The manifold M is regarded as a *spatial* one. Note that our spatial coordinates x^i denote the field variables in the De Donder-Weyl covariant Hamilton geometry [13], [14], [15].
- The coordinates x^i_a are regarded by us as *partial directions* or *partial velocities.* Note that, in the De Donder-Weyl covariant Hamilton geometry [13], [14], [15], the coordinates x^i_a are space-time derivatives (or first jets) of the field variables.
- The fibre bundle $J^1(\mathcal{T}, M) \to \mathcal{T}\times M$ is regarded as a *bundle of configurations* for "multi-time" physical events. This is because in the particular case when $\mathcal{T} = \mathbb{R}$ (i.e., the temporal manifold $\mathcal{T}$ coincides with the usual time axis represented by the set of real numbers $\mathbb{R}$), we recover the bundle of configurations which characterizes the classical non-autonomous mechanics. For more details, please see Abraham and Marsden [1] and Bucătaru and Miron [7].

In order to simplify the notations, we will use the notation $E = J^1(\mathcal{T}, M)$. We recall that the transformation of coordinates

$$\left(t^a, x^i, x^i_a\right) \longleftrightarrow \left(\tilde{t}^a, \tilde{x}^i, \tilde{x}^i_a\right),$$

induced from $\mathcal{T} \times M$ on the 1-jet space E, are given by

$$\left\{\begin{array}{ll} \tilde{t}^a = \tilde{t}^a\left(t^b\right), & \det\left(\dfrac{\partial \tilde{t}^a}{\partial t^b}\right) \neq 0, \\[2ex] \tilde{x}^i = \tilde{x}^i\left(x^j\right), & \det\left(\dfrac{\partial \tilde{x}^i}{\partial x^j}\right) \neq 0, \\[2ex] \tilde{x}^i_a = \dfrac{\partial \tilde{x}^i}{\partial x^j}\dfrac{\partial t^b}{\partial \tilde{t}^a}x^j_b. & \end{array}\right. \tag{2.1}$$

Remark 2.3. Let $\mathbb{R} \times TM$ be the trivial bundle over the base tangent space TM, whose coordinates induced by TM are $\left(t, x^i, y^i\right)$, $i = \overline{1,n}$, $n = \dim M$, $t \in \mathbb{R}$ being a temporal parameter. Then, the changes of coordinates on the trivial bundle $\mathbb{R}\times TM \to TM$ are given by

$$\left\{\begin{array}{l} \tilde{t} = t \\ \tilde{x}^i = \tilde{x}^i\left(x^j\right) \\ \tilde{y}^i = \dfrac{\partial \tilde{x}^i}{\partial x^j}y^j. \end{array}\right.$$

A time dependent Lagrangian function for M is a real valued function L on $\mathbf{E} = \mathbb{R} \times TM$. Such Lagrangians (called *rheonomic* or *non-autonomous*) are important for variational calculus and non-autonomous mechanics. A *rheonomic Lagrange geometry* on the trivial bundle $\mathbf{E} = \mathbb{R} \times TM$ was developed by Anastasiei, Kawaguchi and Miron [2], [3], [17].

More general, let us consider the 1-jet bundle $\mathbb{E} = J^1(\mathbb{R}, M) \equiv \mathbb{R} \times TM$ over the product manifold $\mathbb{R} \times M$. Then, the changes of coordinates on $\mathbb{E}$ have the form

$$\begin{cases} \tilde{t} = \tilde{t}\,(t) \\ \tilde{x}^i = \tilde{x}^i\left(x^j\right) \\ \tilde{y}^i = \dfrac{\partial \tilde{x}^i}{\partial x^j}\dfrac{dt}{d\tilde{t}}y^j. \end{cases}$$

These transformations of coordinates point out the *relativistic* character played by the time t in our non-autonomous mechanics. A *relativistic rheonomic Lagrange geometry* on the 1-jet bundle $\mathbb{E} = J^1(\mathbb{R}, M)$ was developed by Neagu in [21]. This geometry has many similarities with the geometry elaborated by Anastasiei, Kawaguchi and Miron, but they are however distinct ones.

Using the general theory of vector bundles, we can consider the dual 1-jet vector bundle $E^* = J^{1*}(\mathcal{T}, M)$, whose fibre type is also $\mathbb{R}^{mn}$. In fact, the construction of the dual 1-jet vector bundle relies on the substitution of coordinates $x^i_a \to p^a_i$, that is if $E = J^1(\mathcal{T}, M)$ is coordinated by $\left(t^a, x^i, x^i_a\right)$, then the corresponding dual 1-jet space $E^* = J^{1*}(\mathcal{T}, M)$ is coordinated by $\left(t^a, x^i, p^a_i\right)$. Moreover, the transformation of coordinates

$$\left(t^a, x^i, p^a_i\right) \leftrightarrow \left(\tilde{t}^a, \tilde{x}^i, \tilde{p}^a_i\right),$$

induced from $\mathcal{T} \times M$ on the dual 1-jet space E^*, are given by

$$\begin{cases} \tilde{t}^a = \tilde{t}^a\left(t^b\right), & \det\left(\dfrac{\partial \tilde{t}^a}{\partial t^b}\right) \neq 0, \\ \tilde{x}^i = \tilde{x}^i\left(x^j\right), & \det\left(\dfrac{\partial \tilde{x}^i}{\partial x^j}\right) \neq 0, \\ \tilde{p}^a_i = \dfrac{\partial x^j}{\partial \tilde{x}^i}\dfrac{\partial \tilde{t}^a}{\partial t^b}p^b_j. \end{cases} \tag{2.2}$$

According to Kanatchikov's physical terminology [14], we introduce

Definition 2.1. The coordinates p^a_i, $a = \overline{1, m}$, $i = \overline{1, n}$, are called polymomenta, and the dual 1-jet space E^* is called the polymomentum phase space.

Remark 2.4. Let $\mathbb{R} \times T^*M$ be the trivial bundle over the base cotangent space T^*M, whose coordinates induced by T^*M are $\left(t, x^i, p_i\right)$, $i = \overline{1, n}$, $n = \dim M$, $t \in \mathbb{R}$ being temporal parameter. Then, the changes of coordinates on the trivial bundle $\mathbb{R} \times T^*M \to T^*M$ are given by

$$\begin{cases} \tilde{t} = t \\ \tilde{x}^i = \tilde{x}^i\left(x^j\right) \\ \tilde{p}_i = \dfrac{\partial x^j}{\partial \tilde{x}^i}p_j. \end{cases}$$

A time dependent Hamiltonian function for M is a real valued function H on $\mathbf{E}^* = \mathbb{R} \times T^*M$. Such Hamiltonians (called *rheonomic* or *non-autonomous*) are important for covariant Hamiltonian approach of non-autonomous mechanics. A geometrization of these Hamiltonians on the trivial bundle $\mathbf{E}^* = \mathbb{R} \times T^*M$ is studied by Miron, Atanasiu and their co-workers [5], [6], [18].

More general, in our jet geometrical approach, we use a relativistic time t. This is because we have a change of coordinates (induced on $\mathbf{E}^* = \mathbb{R} \times T^*M$ by the product manifold $\mathbb{R} \times M$) of the form

$$\begin{cases} \tilde{t} = \tilde{t}\,(t) \\ \tilde{x}^i = \tilde{x}^i\left(x^j\right) \\ \tilde{p}_i = \dfrac{\partial x^j}{\partial \tilde{x}^i}\dfrac{d\tilde{t}}{dt} p_j. \end{cases}$$

The above two groups of transformations of coordinates are in use for the *relativistic rheonomic Hamilton geometry* that study the metrical structure

$$g^{ij}\,(t, x, p) = \frac{1}{2}\frac{\partial^2 H}{\partial p_i \partial p_j}.$$

Doing a transformation of coordinates (2.2) on E^*, we find the following results:

Proposition 2.1. *The local natural basis*

$$\left\{\frac{\partial}{\partial t^a}, \frac{\partial}{\partial x^i}, \frac{\partial}{\partial p_i^a}\right\}$$

of the Lie algebra of the vector fields $\chi\,(E^*)$ transforms by the laws:

$$\begin{aligned} \frac{\partial}{\partial t^a} &= \frac{\partial \tilde{t}^b}{\partial t^a}\frac{\partial}{\partial \tilde{t}^b} + \frac{\partial \tilde{p}_j^b}{\partial t^a}\frac{\partial}{\partial \tilde{p}_j^b}, \\ \frac{\partial}{\partial x^i} &= \frac{\partial \tilde{x}^j}{\partial x^i}\frac{\partial}{\partial \tilde{x}^j} + \frac{\partial \tilde{p}_j^b}{\partial x^i}\frac{\partial}{\partial \tilde{p}_j^b}, \\ \frac{\partial}{\partial p_i^a} &= \frac{\partial \tilde{t}^b}{\partial t^a}\frac{\partial x^i}{\partial \tilde{x}^j}\frac{\partial}{\partial \tilde{p}_j^b}; \end{aligned} \tag{2.3}$$

Proposition 2.2. *The local natural cobasis*

$$\left\{dt^a, dx^i, dp_i^a\right\}$$

of the Lie algebra of the covector fields $\chi^*\,(E^*)$ transforms by the laws:

$$\begin{aligned} dt^a &= \frac{\partial t^a}{\partial \tilde{t}^b} d\tilde{t}^b, \\ dx^i &= \frac{\partial x^i}{\partial \tilde{x}^j} d\tilde{x}^j, \\ dp_i^a &= \frac{\partial p_i^a}{\partial \tilde{t}^b} d\tilde{t}^b + \frac{\partial p_i^a}{\partial \tilde{x}^j} d\tilde{x}^j + \frac{\partial t^a}{\partial \tilde{t}^b}\frac{\partial \tilde{x}^j}{\partial x^i} d\tilde{p}_j^b. \end{aligned} \tag{2.4}$$

3 Nonlinear connections

Taking into account the complicated transformation rules (2.3) and (2.4), we need a *nonlinear connection* on the dual 1-jet space E^*, in order to construct some *adapted bases* whose transformation rules to be simpler (tensorial ones, for instance).

Let $u^* = \left(t^a, x^i, p_i^a\right) \in E^*$ be an arbitrary point and let us consider the differential map

$$\pi^*{}_{*,u^*} : T_{u^*}E^* \to T_{(t,x)}\left(\mathcal{T} \times M\right)$$

of the canonical projection

$$\pi^* : E^* \to \mathcal{T} \times M, \quad \pi^*\left(u^*\right) = (t, x),$$

together with its vector subspace

$$W_{u^*} = Ker\pi^*{}_{*,u^*} \subset T_{u^*}E^*.$$

Because the differential map $\pi^*{}_{*,u^*}$ is a surjection, we find that we have $\dim_{\mathbb{R}} W_{u^*} = mn$ and, moreover, a basis in W_{u^*} is determined by $\left\{\dfrac{\partial}{\partial p_i^a}\right\}$.

So, the map

$$\mathcal{W} : u^* \in E^* \to W_{u^*} \subset T_{u^*}E^*$$

is a differential distribution which is called the *vertical distribution* on the dual 1-jet vector bundle E^*.

Definition 3.1. A differential distribution

$$\mathcal{H} : u^* \in E^* \to H_{u^*} \subset T_{u^*}E^*,$$

which is supplementary to the vertical distribution $\mathcal{W}$, that is

$$T_{u^*}E^* = H_{u^*} \oplus W_{u^*}, \; \forall \; u^* \in E^*, \tag{3.1}$$

is called a nonlinear connection on E^* ($\mathcal{H}$ is also called the horizontal distribution on the dual 1-jet space E^*).

The above definition implies that

$$\dim_{\mathbb{R}} H_{u^*} = m + n, \; \forall \; u^* \in E^*,$$

and that the Lie algebra of the vector fields $\chi\left(E^*\right)$ can be decomposed in the direct sum

$$\chi\left(E^*\right) = \mathcal{S}\left(\mathcal{H}\right) \oplus \mathcal{S}\left(\mathcal{W}\right),$$

where $\mathcal{S}\left(\mathcal{H}\right)$ (resp. $\mathcal{S}\left(\mathcal{W}\right)$) is the set of differentiable sections on $\mathcal{H}$ (resp. $\mathcal{W}$).

Supposing that $\mathcal{H}$ is a fixed nonlinear connection on E^*, we have the isomorphism

$$\pi^*{}_{*,u^*}|_{H_{u^*}} : H_{u^*} \to T_{\pi^*(u^*)}\left(\mathcal{T} \times M\right),$$

which allows us to prove the following result:

Theorem 3.1. *(i) There exist the unique, linear independent, horizontal vector fields*

$$\frac{\delta}{\delta t^a}, \; \frac{\delta}{\delta x^i} \in \mathcal{S}\left(\mathcal{H}\right),$$

having the properties:

$$\begin{aligned} \pi^*{}_*\left(\frac{\delta}{\delta t^a}\right) &= \frac{\partial}{\partial t^a}, \\ \pi^*{}_*\left(\frac{\delta}{\delta x^i}\right) &= \frac{\partial}{\partial x^i}. \end{aligned} \tag{3.2}$$

(ii) The horizontal vector fields $\dfrac{\delta}{\delta t^a}$ *and* $\dfrac{\delta}{\delta x^i}$ *can be uniquely written in the form:*

$$\begin{aligned} \frac{\delta}{\delta t^a} &= \frac{\partial}{\partial t^a} - \underset{1}{N}{}^{(b)}_{(j)a}\frac{\partial}{\partial p^b_j}, \\ \frac{\delta}{\delta x^i} &= \frac{\partial}{\partial x^i} - \underset{2}{N}{}^{(b)}_{(j)i}\frac{\partial}{\partial p^b_j}. \end{aligned} \tag{3.3}$$

(iii) With respect to a transformation of coordinates (2.2) on E^**, the coefficients* $\underset{1}{N}{}^{(b)}_{(j)a}$ *and* $\underset{2}{N}{}^{(b)}_{(j)i}$ *obey the rules*

$$\begin{aligned} \underset{1}{\tilde{N}}{}^{(b)}_{(j)c}\frac{\partial \tilde{t}^c}{\partial t^a} &= \underset{1}{N}{}^{(c)}_{(k)a}\frac{\partial \tilde{t}^b}{\partial t^c}\frac{\partial x^k}{\partial \tilde{x}^j} - \frac{\partial \tilde{p}^b_j}{\partial t^a}, \\ \underset{2}{\tilde{N}}{}^{(b)}_{(j)k}\frac{\partial \tilde{x}^k}{\partial x^i} &= \underset{2}{N}{}^{(c)}_{(k)i}\frac{\partial \tilde{t}^b}{\partial t^c}\frac{\partial x^k}{\partial \tilde{x}^j} - \frac{\partial \tilde{p}^b_j}{\partial x^i}. \end{aligned} \tag{3.4}$$

(iv) To give a nonlinear connection $\mathcal{H}$ *on* E^* *is equivalent to give a set of local functions*

$$N = \left(\underset{1}{N}{}^{(b)}_{(j)a},\ \underset{2}{N}{}^{(b)}_{(j)i}\right)$$

which transform by the rules (3.4).

Proof. Let $\dfrac{\delta}{\delta t^a}, \dfrac{\delta}{\delta x^i} \in \chi(E^*)$ be vector fields on E^*, locally expressed by

$$\begin{aligned} \frac{\delta}{\delta t^a} &= A^b_a\frac{\partial}{\partial t^b} + A^j_a\frac{\partial}{\partial x^j} + A^{(b)}_{(j)a}\frac{\partial}{\partial p^b_j}, \\ \frac{\delta}{\delta x^i} &= X^b_i\frac{\partial}{\partial t^b} + X^j_i\frac{\partial}{\partial x^j} + X^{(b)}_{(j)i}\frac{\partial}{\partial p^b_j}, \end{aligned}$$

which verify the relations (3.2). Then, taking into account the local expression of the map $\pi^*{}_*$, we get

$$\begin{aligned} A^b_a &= \delta^b_a,\ A^j_a = 0,\ A^{(b)}_{(j)a} = -\underset{1}{N}{}^{(b)}_{(j)a}, \\ X^b_i &= 0,\ X^j_i = \delta^j_i,\ X^{(b)}_{(j)i} = -\underset{2}{N}{}^{(b)}_{(j)i}. \end{aligned}$$

These equalities prove the form (3.3) of the vector fields from Theorem 3.1, together with their linear independence. The uniqueness of the coefficients $\underset{1}{N}{}^{(b)}_{(j)a}$ and $\underset{2}{N}{}^{(b)}_{(j)i}$ is obvious.

Because the vector fields $\dfrac{\delta}{\delta t^a}$ and $\dfrac{\delta}{\delta x^i}$ are globally defined, we deduce that a change of coordinates (2.2) on E^* produces a transformation of the coefficients $\underset{1}{N}{}^{(b)}_{(j)a}$ and $\underset{2}{N}{}^{(b)}_{(j)i}$ by the rules (3.4).

Finally, starting with a set of functions

$$N = \left(\underset{1}{N}{}^{(b)}_{(j)a},\ \underset{2}{N}{}^{(b)}_{(j)i}\right)$$

which respect the rules (3.4), we can construct the horizontal distribution $\mathcal{H}$ taking

$$H_{u^*} = Span\left\{ \left.\frac{\delta}{\delta t^a}\right|_{u^*}, \left.\frac{\delta}{\delta x^i}\right|_{u^*} \right\}.$$

The decomposition $T_{u^*}E^* = H_{u^*} \oplus W_{u^*}$ is obvious now. □

Definition 3.2. (i) The geometrical entity $\underset{1}{N} = \left(\underset{1}{N}{}^{(b)}_{(j)a}\right)$ (resp. $\underset{2}{N} = \left(\underset{2}{N}{}^{(b)}_{(j)i}\right)$) is called a temporal (resp. spatial) nonlinear connection on E^*.

(ii) The set of the linear independent vector fields

$$\left\{\frac{\delta}{\delta t^a}, \frac{\delta}{\delta x^i}, \frac{\partial}{\partial p_i^a}\right\} \subset \chi\left(E^*\right) \tag{3.5}$$

is called the adapted basis of vector fields produced by the nonlinear connection

$$N = \left(\underset{1}{N}, \underset{2}{N}\right).$$

Example 3.1. Let us consider that $h_{ab}(t)$ (resp. $\varphi_{ij}(x)$) is a semi-Riemannian metric on $\mathcal{T}$ (resp. M) and let $\varkappa^a_{bc}(t)$ (resp. $\gamma^i_{jk}(x)$) be its Christoffel symbols. Setting

$$\begin{aligned} \underset{1}{\overset{0}{N}}{}^{(b)}_{(j)c} &= \varkappa^b_{ac} p^a_j, \\ \underset{2}{\overset{0}{N}}{}^{(b)}_{(j)k} &= -\gamma^i_{jk} p^b_i, \end{aligned} \tag{3.6}$$

we obtain that

$$\overset{0}{N} = \left(\underset{1}{\overset{0}{N}}{}^{(b)}_{(j)c},\ \underset{2}{\overset{0}{N}}{}^{(b)}_{(j)k}\right)$$

is a nonlinear connection on $E^* = J^{1*}\left(\mathcal{T}, M\right)$, which is called the *canonical nonlinear connection on* E^* *attached to the pair of semi-Riemannian metrics* $h_{ab}(t)$ *and* $\varphi_{ij}(x)$.

With respect to the coordinate transformations (2.2) the adapted basis (3.5) has its transformation laws extremely simple, namely (tensorial ones)

$$\begin{aligned}\frac{\delta}{\delta t^a} &= \frac{\partial \tilde{t}^b}{\partial t^a}\frac{\delta}{\delta \tilde{t}^b},\\ \frac{\delta}{\delta x^i} &= \frac{\partial \tilde{x}^j}{\partial x^i}\frac{\delta}{\delta \tilde{x}^j},\\ \frac{\partial}{\partial p_i^a} &= \frac{\partial \tilde{t}^b}{\partial t^a}\frac{\partial x^i}{\partial \tilde{x}^j}\frac{\partial}{\partial \tilde{p}_j^b},\end{aligned}\tag{3.7}$$

in contrast with the transformations (2.3).

The dual basis of the adapted basis (3.5) is given by

$$\left\{dt^a, dx^i, \delta p_i^a\right\} \subset \chi^*\left(E^*\right)\tag{3.8}$$

where

$$\delta p_i^a = dp_i^a + \underset{1}{N}{}_{(i)b}^{(a)}dt^b + \underset{2}{N}{}_{(i)j}^{(a)}dx^j.\tag{3.9}$$

Definition 3.3. The dual basis of covector fields given by (3.8) and (3.9) is called the adapted cobasis of covector fields attached to the nonlinear connection $N = \left(\underset{1}{N}, \underset{2}{N}\right)$.

It is obvious that we always have

$$\begin{array}{lll}\dfrac{\delta}{\delta t^a} \rfloor dt^b = \delta_a^b, & \dfrac{\delta}{\delta t^a} \rfloor dx^j = 0, & \dfrac{\delta}{\delta t^a} \rfloor \delta p_j^b = 0,\\ \dfrac{\delta}{\delta x^i} \rfloor dt^b = 0, & \dfrac{\delta}{\delta x^i} \rfloor dx^j = \delta_i^j, & \dfrac{\delta}{\delta x^i} \rfloor \delta p_j^b = 0,\\ \dfrac{\partial}{\partial p_i^a} \rfloor dt^b = 0, & \dfrac{\partial}{\partial p_i^a} \rfloor dx^j = 0, & \dfrac{\partial}{\partial p_i^a} \rfloor \delta p_j^b = \delta_a^b\delta_j^i.\end{array}$$

Moreover, with respect to (2.2), we obtain the following tensorial transformation rules:

$$\begin{aligned}dt^a &= \frac{\partial t^a}{\partial \tilde{t}^b}d\tilde{t}^b,\\ dx^i &= \frac{\partial x^i}{\partial \tilde{x}^j}d\tilde{x}^j,\\ \delta p_i^a &= \frac{\partial t^a}{\partial \tilde{t}^b}\frac{\partial \tilde{x}^j}{\partial x^i}\delta \tilde{p}_j^b.\end{aligned}\tag{3.10}$$

As a consequence of the preceding assertions, we find the following simple result:

Proposition 3.2. *i) The Lie algebra $\chi\left(E^*\right)$ of the vector fields on E^* decomposes in the direct sum*

$$\chi\left(E^*\right) = \chi\left(\mathcal{H}_{\mathcal{T}}\right) \oplus \chi\left(\mathcal{H}_M\right) \oplus \chi\left(\mathcal{W}\right),$$

where

$$\chi\left(\mathcal{H}_{\mathcal{T}}\right) = Span\left\{\frac{\delta}{\delta t^a}\right\},\ \chi\left(\mathcal{H}_M\right) = Span\left\{\frac{\delta}{\delta x^i}\right\},\ \chi\left(\mathcal{W}\right) = Span\left\{\frac{\partial}{\partial p_i^a}\right\}.$$

ii) The Lie algebra $\chi^*(E^*)$ *of the covector fields on* E^* *decomposes in the direct sum*

$$\chi^*(E^*) = \chi^*(\mathcal{H}_{\mathcal{T}}) \oplus \chi^*(\mathcal{H}_M) \oplus \chi^*(\mathcal{W}),$$

where

$$\chi^*(\mathcal{H}_{\mathcal{T}}) = Span\left\{dt^a\right\},\ \chi^*(\mathcal{H}_M) = Span\left\{dx^i\right\},\ \chi^*(\mathcal{W}) = Span\left\{\delta p_i^a\right\}.$$

Definition 3.4. The distribution $\mathcal{H}_{\mathcal{T}}$ (resp. $\mathcal{H}_M$) is called the $\mathcal{T}$-horizontal distribution (resp. M-horizontal distribution) on E^*.

Let $h_{\mathcal{T}}$, h_M and w be the $\mathcal{T}$- horizontal, M-horizontal and vertical, respectively, canonical projections associated to the decomposition of vector fields on E^*. Then, it is obvious that we have the relations:

$$\begin{aligned}
&h_{\mathcal{T}} + h_M + w = I,\\
&h_{\mathcal{T}}^2 = h_{\mathcal{T}},\ h_M^2 = h_M,\ w^2 = w,\\
&h_{\mathcal{T}} \circ h_M = h_M \circ h_{\mathcal{T}} = 0,\\
&h_{\mathcal{T}} \circ w = w \circ h_{\mathcal{T}} = 0,\\
&h_M \circ w = w \circ h_M = 0.
\end{aligned} \tag{3.11}$$

Starting with a vector field $X \in \chi(E^*)$, we denote

$$X^{\mathcal{H}_{\mathcal{T}}} = h_{\mathcal{T}} X, \quad X^{\mathcal{H}_M} = h_M X, \quad X^{\mathcal{W}} = wX.$$

Therefore, we have the unique decomposition

$$X = X^{\mathcal{H}_{\mathcal{T}}} + X^{\mathcal{H}_M} + X^{\mathcal{W}}, \tag{3.12}$$

where, using the adapted basis (3.5), we get

$$\begin{aligned}
X^{\mathcal{H}_{\mathcal{T}}} &= X^a \frac{\delta}{\delta t^a},\\
X^{\mathcal{H}_M} &= X^i \frac{\delta}{\delta x^i},\\
X^{\mathcal{W}} &= X_{(i)}^{(a)} \frac{\partial}{\partial p_i^a}.
\end{aligned}$$

By means of (3.7), we deduce that

$$\begin{aligned}
\tilde{X}^a &= \frac{\partial \tilde{t}^a}{\partial t^b} X^b,\\
\tilde{X}^i &= \frac{\partial \tilde{x}^i}{\partial x^j} X^j\\
\tilde{X}_{(i)}^{(a)} &= \frac{\partial \tilde{t}^a}{\partial t^b} \frac{\partial x^j}{\partial \tilde{x}^i} X_{(j)}^{(b)}.
\end{aligned}$$

Now, we can give the following important results:

Proposition 3.3. *The horizontal distribution $\mathcal{H}$ is integrable if and only if for any vector fields $X, Y \in \chi(E^*)$ we have*

$$\left[X^{\mathcal{H}_{\mathcal{T}}}, Y^{\mathcal{H}_{\mathcal{T}}}\right]^{\mathcal{W}} = 0, \ \ \left[X^{\mathcal{H}_{\mathcal{T}}}, Y^{\mathcal{H}_{M}}\right]^{\mathcal{W}} = 0, \ \ \left[X^{\mathcal{H}_{M}}, Y^{\mathcal{H}_{M}}\right]^{\mathcal{W}} = 0.$$

Proof. Indeed, the Poisson bracket between two $\mathcal{T}$-horizontal vector fields $X^{\mathcal{H}_{\mathcal{T}}}$, $Y^{\mathcal{H}_{\mathcal{T}}}$ (resp. M-horizontal vector fields $X^{\mathcal{H}_{M}}$, $Y^{\mathcal{H}_{M}}$) or between a $\mathcal{T}$-horizontal vector field $X^{\mathcal{H}_{\mathcal{T}}}$ and an M-horizontal vector field $Y^{\mathcal{H}_{M}}$ belongs to the horizontal distribution $\mathcal{H}$ if and only if the preceding three equations hold good. □

Proposition 3.4. *The vertical distribution $\mathcal{W}$ is always integrable.*

A similar theory can be done for 1-forms. With respect to the decomposition of covector fields on E^*, any 1-form $\omega \in \chi^*(E^*)$ can be uniquely written in the form

$$\omega = \omega^{\mathcal{H}_{\mathcal{T}}} + \omega^{\mathcal{H}_{M}} + \omega^{\mathcal{W}}, \tag{3.13}$$

where

$$\omega^{\mathcal{H}_{\mathcal{T}}} = \omega \circ h_{\mathcal{T}}, \ \omega^{\mathcal{H}_{M}} = \omega \circ h_{M}, \ \omega^{\mathcal{W}} = \omega \circ w.$$

In the adapted cobasis (3.8), we have in fact

$$\omega = \omega_a dt^a + \omega_i dx^i + \omega_{(a)}^{(i)} \delta p_i^a.$$

The components ω_a, ω_i, $\omega_{(a)}^{(i)}$ are transformed by (2.2) as follows:

$$\begin{aligned} \omega_a &= \frac{\partial \tilde{t}^b}{\partial t^a} \tilde{\omega}_b, \\ \omega_i &= \frac{\partial \tilde{x}^j}{\partial x^i} \tilde{\omega}_j, \\ \omega_{(a)}^{(i)} &= \frac{\partial \tilde{t}^b}{\partial t^a} \frac{\partial x^i}{\partial \tilde{x}^j} \omega_{(b)}^{(j)}. \end{aligned}$$

4 The algebra of distinguished tensor fields

In the study of differential geometry of the dual 1-jet space $E^* = J^{1*}(\mathcal{T}, M)$, a central role is played by the *distinguished tensors* or, briefly, *d-tensors.*

Definition 4.1. A geometrical object $T = \left(T_{bj(c)(l)\dots}^{ai(k)(d)\dots}\right)$ on E^* which, with respect to a transformation of coordinates (2.2) on E^*, verifies the transformation rules

$$T_{bj(c)(l)\dots}^{ai(k)(d)\dots} = \tilde{T}_{fq(g)(s)\dots}^{ep(r)(h)\dots} \frac{\partial t^a}{\partial \tilde{t}^e} \frac{\partial \tilde{t}^f}{\partial t^b} \frac{\partial x^i}{\partial \tilde{x}^p} \frac{\partial \tilde{x}^q}{\partial x^j} \left(\frac{\partial x^k}{\partial \tilde{x}^r} \frac{\partial \tilde{t}^g}{\partial t^c}\right) \left(\frac{\partial t^d}{\partial \tilde{t}^h} \frac{\partial \tilde{x}^s}{\partial x^l}\right) \dots$$

is called a distinguished tensor field or a d-tensor.

Remark 4.1. It is obvious that the vector fields $\delta/\delta t^a$, $\delta/\delta x^i$, $\partial/\partial p_i^a$ of the adapted basis are d-vector fields on E^*, as the adapted components X^a, X^i, $X_{(i)}^{(a)}$ of a vector field $X \in \chi(E^*)$. Also, the covector fields dt^a, dx^i, δp_i^a of the adapted cobasis are d-covector fields on E^*, as the adapted components $\omega_a, \omega_i, \omega_{(a)}^{(i)}$ of an 1-form $\omega \in \chi^*(E^*)$. It follows that the set

$$\left\{1, \frac{\delta}{\delta t^a}, \frac{\delta}{\delta x^i}, \frac{\partial}{\partial p_i^a}, dt^a, dx^i, \delta p_i^a\right\}$$

generates the algebra of the d-tensor fields over the ring of functions $\mathcal{F}(E^*)$.

Example 4.1. i) If $H : E^* \to \mathbb{R}$ is a Hamiltonian function depending on the polymomenta p_i^a, then the local components

$$G_{(a)(b)}^{(i)(j)} = \frac{1}{2}\frac{\partial^2 H}{\partial p_i^a \partial p_j^b},$$

represent a d-tensor field $\mathbb{G} = \left(G_{(a)(b)}^{(i)(j)}\right)$. If $\mathcal{T} = \mathbb{R}$ and H is a regular Hamiltonian function, then the d-tensor field $\mathbb{G}$ can be regarded as the fundamental metrical d-tensor $g^{ij}(t, x, p)$ from the rheonomic Hamilton geometry.

ii) Let us consider the d-tensor field

$$\mathbb{C}^* = \left(\mathbb{C}_{(i)}^{(a)}\right),$$

where

$$\mathbb{C}_{(i)}^{(a)} = p_i^a.$$

Particularly, for $\mathcal{T} = \mathbb{R}$, this d-tensor can be regarded as the classical Liouville-Hamilton vector field $\mathbb{C}^* = p_i(\partial/\partial p_i)$ on the cotangent bundle T^*M, which is used in the Hamilton geometry [18]. Consequently, our d-tensor field $\mathbb{C}^*$ on E^* is called *the Liouville-Hamilton d-tensor field of polymomenta.*

iii) Let $\varphi_{ij}(x)$ be a semi-Riemannian metric on the spatial manifold M. The geometrical object

$$\mathbb{H} = \left(H_{(i)jk}^{(a)}\right),$$

where

$$H_{(i)jk}^{(a)} = \varphi_{ij} p_k^a,$$

is a d-tensor field on E^*, which is called *the polymomentum Hamilton d-tensor attached to the semi-Riemannian metric* $\varphi_{ij}(x)$.

iv) Let $h_{ab}(t)$ be a semi-Riemannian metric on the temporal manifold $\mathcal{T}$. We can construct the d-tensor field

$$\mathbb{J} = \left(J_{(a)bj}^{(i)}\right),$$

where

$$J_{(a)bj}^{(i)} = h_{ab}\delta_j^i.$$

The d-tensor field $\mathbb{J}$ is called the *d-tensor of h-normalization* on the dual 1-jet vector bundle E^*.

5 The Poisson brackets. The almost product structure $\mathbb{P}$

In applications, the Poisson brackets of the d-vector fields

$$\left\{\frac{\delta}{\delta t^a}, \frac{\delta}{\delta x^i}, \frac{\partial}{\partial p_i^a}\right\}$$

from the adapted basis are very important. By a direct calculation, we obtain

Proposition 5.1. *The Poisson brackets of the d-vector fields of the adapted basis (3.5) are given by*

$$\begin{aligned}
&\left[\frac{\delta}{\delta t^b}, \frac{\delta}{\delta t^c}\right] = R^{(a)}_{(i)bc}\frac{\partial}{\partial p_i^a},\\
&\left[\frac{\delta}{\delta t^b}, \frac{\delta}{\delta x^k}\right] = R^{(a)}_{(i)bk}\frac{\partial}{\partial p_i^a},\\
&\left[\frac{\delta}{\delta t^b}, \frac{\partial}{\partial p_k^c}\right] = B^{(a)\ (k)}_{(i)b(c)}\frac{\partial}{\partial p_i^a},\\
&\left[\frac{\delta}{\delta x^j}, \frac{\delta}{\delta x^k}\right] = R^{(a)}_{(i)jk}\frac{\partial}{\partial p_i^a},\\
&\left[\frac{\delta}{\delta x^j}, \frac{\partial}{\partial p_k^c}\right] = B^{(a)\ (k)}_{(i)j(c)}\frac{\partial}{\partial p_i^a},\\
&\left[\frac{\partial}{\partial p_j^b}, \frac{\partial}{\partial p_k^c}\right] = 0,
\end{aligned} \tag{5.1}$$

where, if $\underset{1}{N}{}^{(a)}_{(i)b}$ and $\underset{2}{N}{}^{(a)}_{(i)j}$ are the local coefficients of the given nonlinear connection N, then

$$\begin{aligned}
&R^{(a)}_{(i)bc} = \frac{\delta \underset{1}{N}{}^{(a)}_{(i)b}}{\delta t^c} - \frac{\delta \underset{1}{N}{}^{(a)}_{(i)c}}{\delta t^b},\\
&R^{(a)}_{(i)bk} = \frac{\delta \underset{1}{N}{}^{(a)}_{(i)b}}{\delta x^k} - \frac{\delta \underset{2}{N}{}^{(a)}_{(i)k}}{\delta t^b},\\
&R^{(a)}_{(i)jk} = \frac{\delta \underset{2}{N}{}^{(a)}_{(i)j}}{\delta x^k} - \frac{\delta \underset{2}{N}{}^{(a)}_{(i)k}}{\delta x^j},\\
&B^{(a)\ (k)}_{(i)b(c)} = \frac{\partial \underset{1}{N}{}^{(a)}_{(i)b}}{\partial p_k^c}, \quad B^{(a)\ (k)}_{(i)j(c)} = \frac{\partial \underset{2}{N}{}^{(a)}_{(i)j}}{\partial p_k^c}.
\end{aligned} \tag{5.2}$$

Using the relations (5.1) and the Proposition 3.3, we get

Proposition 5.2. *The horizontal distribution $\mathcal{H}$ is integrable if and only if the following d-tensor fields vanish:*

$$R^{(a)}_{(i)bc} = 0,\ R^{(a)}_{(i)bk} = 0,\ R^{(a)}_{(i)jk} = 0. \tag{5.3}$$

Now, assuming that a nonlinear connection N is given, we can define an $\mathcal{F}(E^*)$-linear mapping $\mathbb{P} : \chi(E^*) \to \chi(E^*)$, putting

$$\mathbb{P}\left(X^{\mathcal{H}_\mathcal{T}}\right) = X^{\mathcal{H}_\mathcal{T}},\ \mathbb{P}\left(X^{\mathcal{H}_M}\right) = X^{\mathcal{H}_M},\ \mathbb{P}\left(X^{\mathcal{W}}\right) = -X^{\mathcal{W}},\ \forall\ X \in \chi(E^*). \tag{5.4}$$

Thus, $\mathbb{P}$ has the properties:

$$\mathbb{P}\circ\mathbb{P} = I, \quad \mathbb{P} = 2\left(h_\mathcal{T} + h_M\right) - I = I - 2w, \quad \operatorname{rank}\mathbb{P} = m + n + mn. \tag{5.5}$$

Obviously, we have

Theorem 5.3. *A nonlinear connection N on E^* is characterized by the existence of the* almost product structure $\mathbb{P}$ *on E^*, whose eigenspace corresponding to the eigenvalue -1 coincides with the linear space of the vertical distribution $\mathcal{W}$ on E^*.*

Moreover, taking into account that the Nijenhuis tensor of the almost product structure $\mathbb{P}$ is given by

$$\mathcal{N}_\mathbb{P}(X,Y) = \mathbb{P}^2[X,Y] + [\mathbb{P}X, \mathbb{P}Y] - \mathbb{P}[\mathbb{P}X, Y] - \mathbb{P}[X, \mathbb{P}Y],$$

we obtain that

$$\begin{aligned}
&\mathcal{N}_\mathbb{P}\left(X^{\mathcal{H}_\mathcal{T}}, Y^{\mathcal{H}_\mathcal{T}}\right) = 4w\left[X^{\mathcal{H}_\mathcal{T}}, Y^{\mathcal{H}_\mathcal{T}}\right], && \mathcal{N}_\mathbb{P}\left(X^{\mathcal{H}_\mathcal{T}}, Y^{\mathcal{H}_M}\right) = 4w\left[X^{\mathcal{H}_\mathcal{T}}, Y^{\mathcal{H}_M}\right],\\
&\mathcal{N}_\mathbb{P}\left(X^{\mathcal{H}_\mathcal{T}}, Y^{\mathcal{W}}\right) = 0, && \mathcal{N}_\mathbb{P}\left(X^{\mathcal{H}_M}, Y^{\mathcal{H}_M}\right) = 4w\left[X^{\mathcal{H}_M}, Y^{\mathcal{H}_M}\right],\\
&\mathcal{N}_\mathbb{P}\left(X^{\mathcal{H}_M}, Y^{\mathcal{W}}\right) = 0, && \mathcal{N}_\mathbb{P}\left(X^{\mathcal{W}}, Y^{\mathcal{W}}\right) = 0.
\end{aligned}$$

Therefore, we can formulate the following

Proposition 5.4. *The almost product structure $\mathbb{P}$ is integrable if and only if the horizontal distribution $\mathcal{H}$ is integrable.*

References

[1] R. Abraham, J. E. Marsden, *Foundations of Mechanics*, Benjamin, New York, 1978.

[2] M. Anastasiei, H. Kawaguchi, *A geometrical theory of time dependent Lagrangians. I. Non-linear connections, II. M-connections*, Tensor N. S. 48 (1989), 273-282, 283-293.

[3] M. Anastasiei, H. Kawaguchi, *A geometrical theory of time dependent Lagrangians. III. Applications*, Tensor N. S. 49 (1990), 296-304.

[4] G. S. Asanov, *Jet extension of Finslerian gauge approach*, Fortschritte der Physik 38, 8 (1990), 571-610.

[5] Gh. Atanasiu, *The invariant expression of Hamilton geometry*, Tensor N. S. 47, 3 (1988), 225-234.

[6] Gh. Atanasiu, F. C. Klepp, *Nonlinear connections in cotangent bundle*, Publ. Math., Debrecen, Hungary, 39, No. 1-2 (1991), 107-111.

[7] I. Bucătaru, R. Miron, *Finsler-Lagrange Geometry. Applications to Dynamical Systems*, Editura Academiei Române, Bucharest, 2007.

[8] I. Duca, C. Udrişte, *Some inequalities satisfied by periodical solutions of multi-time Hamilton equations*, Balkan Journal of Geometry and Its Applications 11, 1 (2006), 50-60.

[9] G. Giachetta, L. Mangiarotti, G. Sardanashvily, *Covariant Hamiltonian field theory*, http://arXiv.org/hep-th/9904062, (1999).

[10] G. Giachetta, L. Mangiarotti, G. Sardanashvily, *Polysymplectic Hamiltonian formalism and some quantum outcomes*, http://arXiv.org/hep-th/0411005, (2004).

[11] M. Gotay, J. Isenberg, J. E. Marsden, R. Montgomery, *Momentum maps and classical fields. Part I. Covariant field theory*, http://arXiv.org/physics/9801019v2, (2004).

[12] M. Gotay, J. Isenberg, J. E. Marsden, *Momentum maps and classical fields. Part II. Canonical analysis of field theories*, http://arXiv.org/math-ph/0411032, (2004).

[13] I. V. Kanatchikov, *Basic structures of the covariant canonical formalism for fields based on the De Donder-Weyl theory*, http://arXiv.org/hep-th/9410238, (1994).

[14] I. V. Kanatchikov, *On quantization of field theories in polymomentum variables*, AIP Conf. Proc. 453 (1998), 356-367; http://arXiv.org/hep-th/9811016, (1998).

[15] I. V. Kanatchikov, *On the canonical structure of the De Donder-Weyl covariant Hamiltonian formulation of field theory I. Graded Poisson brackets and equations of motion*, http://arXiv.org/hep-th/9312162, (1993).

[16] R. Miron, *Hamilton geometry*, An. Şt. "Al. I. Cuza" Univ., Iaşi, Romania, 35 (1989), 33-67.

[17] R. Miron, M. Anastasiei, *The Geometry of Lagrange Spaces: Theory and Applications*, Kluwer Academic Publishers, 1994.

[18] R. Miron, D. Hrimiuc, H. Shimada, S. V. Sabău, *The Geometry of Hamilton and Lagrange Spaces*, Kluwer Academic Publishers, 2001.

[19] R. Miron, M. S. Kirkovits, M. Anastasiei, *A geometrical model for variational problems of multiple integrals*, Proc. of Conf. on Diff. Geom. and Appl., Dubrovnik, Yugoslavia, June 26-July 3, (1988), 8-25.

[20] M. Neagu, *Riemann-Lagrange Geometry on 1-Jet Spaces*, Matrix Rom, Bucharest, 2005.

[21] M. Neagu, *The geometry of relativistic rheonomic Lagrange spaces*, Proceedings of Workshop on Global Analysis, Differential Geometry and Lie Algebras, No. 5 (2001); Editor: Prof. Dr. Grigorios Tsagas, University "Aristotle" of Thessaloniki, Greece, 142-169.

[22] M. Neagu, C. Udrişte, *A Riemann-Lagrange geometrization for metrical multi-time Lagrange spaces"*, Balkan Journal of Geometry and Its Applications 11, 1 (2006), 87-98.

[23] M. Neagu, C. Udrişte, *Torsion, curvature and deflection d-tensors on* $J^1(T, M)$, Balkan Journal of Geometry and Its Applications 6, 1 (2001), 29-45.

[24] D. Saunders, *The Geometry of Jet Bundles*, Cambridge University Press, New York, London, 1989.

[25] C. Udrişte, A. Pitea, J. Mihăilă, *Determination of metrics by boundary energy*, Balkan Journal of Geometry and Its Applications 11, 1 (2006), 131-143.

Authors' address:

Gheorghe Atanasiu and Mircea Neagu
University Transilvania of Brasov,
Faculty of Mathematics and Informatics,
Department of Algebra, Geometry and Differential Equations,
B-dul Eroilor, Nr. 29, BV 500036, Brasov, Romania.
Website: http://cs.unitbv.ro/˜geome/
Website: http://www.2collab.com/user:mirceaneagu
E-mail: gh_atanasiu@yahoo.com, mircea.neagu@unitbv.ro