Leafwise 2-jet cohomology on foliated Finsler manifolds

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Abstract. The tangent space of a Finsler manifold (M, F) is a Riemannian manifold with respect to the induced Sasaki-Finsler metric and admits a natural foliated structure F_V given by the vertical distribution (A. Bejancu and H.R. Farran, 2006). It is known that the Levi-Civita connection on the slit tangent space TM^0 induces a connection in the structural bundle. In this paper the Vaisman connection on (TM^0, F_V) is introduced; this induces connections on the structural and the transversal bundles. It is shown that the Levi-Civita and the Vaisman connections induce the same connections in the structural bundle iff (M, F) is a Landsberg manifold and that (TM^0, F_V) is a Reinhart space iff (M, F) is a Riemannian manifold. Further, on the foliated manifold (TM^0, F_V) is introduced and studied the space $J^{\{l,2\}}(TM^0)$ of leafwise 2-jets. A decomposition of this space is obtained, and the 1-dimensional Cech cohomology group of (TM^0, F_V) with coefficients in the sheaf of basic functions is expressed in terms of fields of leafwise 2-jets. We define the leafwise Mastrogiacomo cohomology group with respect to the connection ∇ induced in the structural bundle by a connection on (TM^0, F_V) ; as well it is shown that the cohomology group is isomorphic with the 1-dimensional Cech cohomology group of TM^0 with coefficients in the sheaf Ω_∇ of germs of functions f on TM^0, which satisfy $\nabla df = 0$. In particular, for the 4-dimensional 4-root space, it is proved that the sheaf Ω_∇ is isomorphic with the sheaf of basic functions on TM^0.

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1 Preliminaries

Generally speaking, jets are equivalence classes of maps between manifolds, maps which have the same derivative up to a specified order, [1], [8]. The second order

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jets are elements of $J^2(M,N)$, where M, N are two differentiable manifolds. In this paper we consider only the case $N = R$, hence the jets of the real differentiable functions on M. Supposing that M has a foliated structure, in [5] we introduced the leafwise 2-jets and the transversal 2-jets on M. These new types of 2-jets define the Mastrogiacomo cohomology groups on M, groups which are related with some Cech cohomology groups of the manifold M.

The tangent bundle of a Finsler manifold (M,F) is a Riemannian manifold with respect to the Sasaki-Finsler metric G, and it has a natural foliated structure F_V given by the vertical distribution, [3]. It is known, [3], that the Levi-Civita connection on TM^0 induces a connection in the structural bundle. In this paper we introduce the Vaisman connection, [10], on (TM^0, F_V), which induces connections on the structural and the transversal bundles. We show that the Levi-Civita and the Vaisman connections induce the same connections in the structural bundle iff (M,F) is a Landsberg manifold. Moreover, (TM^0, F_V) is a Reinhart space iff (M,F) is a Riemannian manifold.

On the foliated manifold (TM^0, F_V) we introduce and study the space of leafwise 2-jets, $J^{l,2}(TM^0)$. We obtain a decomposition of this space and we express the 1-dimensional Cech cohomology group of (TM^0, F_V) with coefficients in the sheaf of basic functions in terms of fields of leafwise 2-jets. If ∇ is the connection induced in the structural bundle by a connection on (TM^0, F_V), then we define the leafwise Mastrogiacomo cohomology group with respect to ∇. This cohomology group is isomorphic with the 1-dimensional Cech cohomology group of TM^0 with coefficients in the sheaf Ω_∇ of germs of functions f on TM^0 which satisfy $\nabla df = 0$.

At the end we present a particular case, where M is a 4-dimensional 4-root space, and we compute the sheaf Ω_∇, proving that it is isomorphic with the sheaf of basic functions on TM^0.

2 A natural foliation on the tangent bundle of a Finsler manifold

In this section we follow [4], [3] to present some questions about Finsler manifolds. Let M be an n-dimensional paracompact manifold and TM its tangent bundle. If E is a bundle over M, we denote by $\Gamma(E)$ the module of its smooth sections. If $(x^i)_{i=\overline{1,n}}$ are the local coordinates on M and $(y^i)_{i=\overline{1,n}}$ are the fiber coordinates, then $(x^i, y^i)_{i=\overline{1,n}}$ are the local coordinates on TM. It is well-known that the transformation of local coordinates on TM are

$$\widetilde{x}^{i_1} = \widetilde{x}^{i_1}(x^1, \ldots, x^n), \quad \widetilde{y}^{i_1} = \frac{\partial \widetilde{x}^{i_1}}{\partial x^i} y^i. \tag{2.1}$$

In this paper the indices take the values $i, j, k, i_1, j_1, ... = \overline{1,n}$ and we use the Einstein convention for summation.

We assume that M is a Finsler manifold, so there exists a function $F : TM \to [0, \infty)$ which vanishes only on the zero section on TM and is smooth on $TM^0 = TM \backslash \{0\}$, such that it is positive 1-homogeneous in y,

$$F(x, ky) = |k|\, F(x, y), \quad \forall k \in R,$$

and the matrix of coefficients

$$g_{ij}(x,y) = \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j}, \tag{2.2}$$

is positive definite at any point of the domain of the local chart.

Let F_V be the foliation on TM^0 determined by the fibers of $\pi : TM^0 \to M$, called the *vertical foliation*. The leaves are exactly $\{T_xM^0\}_{x\in M}$. The local coordinates $(x^i, y^i)_i$ are adapted to this foliation, which means that the leaves are locally defined by $x^i = constant$. The sections of the *structural* (also called vertical) bundle VTM^0 are locally spanned by $\{\frac{\partial}{\partial y^i}\}_i$. We consider the functions

$$G^i = \frac{1}{4}g^{ik}\left(\frac{\partial^2 F^2}{\partial y^k \partial x^h}y^h - \frac{\partial F^2}{\partial x^k}\right), \quad G_i^j = \frac{\partial G^j}{\partial y^i},$$

where the matrix (g^{ik}) is the inverse of the matrix described by(2.2).

There is a complementary bundle HTM^0 to VTM^0 in TTM^0, called the *transversal* (or horizontal) bundle of the foliation, whose sections are locally spanned by

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - G_i^j\frac{\partial}{\partial y^j}, \quad i = \overline{1,n}.$$

We have the decomposition

$$TTM^0 = HTM^0 \oplus VTM^0, \tag{2.3}$$

so every vector field X on TM^0 has a vertical part VX and an horizontal part HX. The relations

$$G\left(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\right) = G\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right) = g_{ij}, \quad G\left(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^j}\right) = 0,$$

define a Riemannian metric on TM^0, called the *Sasaki-Finsler metric*.

Let $\widetilde{\nabla}$ be the Levi-Civita connection of G. It induces a connection ∇ on VTM^0 given by

$$\nabla_X VY = V(\widetilde{\nabla}_X VY),$$

where $X, Y \in \Gamma(TTM^0)$ and V is the projection morphism of TTM^0 on VTM^0. Locally, ∇ has the following expression ([3]):

$$\nabla_{\frac{\partial}{\partial y^j}}\frac{\partial}{\partial y^i} = C_{ij}^k\frac{\partial}{\partial y^k}, \quad \nabla_{\frac{\delta}{\delta x^j}}\frac{\partial}{\partial y^i} = F_{ij}^k\frac{\partial}{\partial y^k}, \tag{2.4}$$

where

$$C_{ij}^k = \frac{1}{2}g^{kh}\frac{\partial g_{ij}}{\partial y^h}, \quad F_{ij}^k = \frac{1}{2}g^{hk}\left(\frac{\delta g_{hi}}{\delta x^j} + \frac{\delta g_{hj}}{\delta x^i} - \frac{\delta g_{ij}}{\delta x^h}\right). \tag{2.5}$$

Remark 2.1. M being a paracompact manifold, there is a Riemannian structure on M, let $a = (a_{ij})$ be the coefficients of its Riemannain metric. We have $F = \sqrt{a_{ij}y^iy^j}$ (hence $g_{ij} = a_{ij}$ don't depend on y), and, in general, (M, F) is Riemannian, iff

$$C_{ij}^k = 0, \quad \forall i, j, k = \overline{1,n}.$$

The decomposition (2.3) shows that for an arbitrary connection on TM^0, the torsion T has the decomposition

$$T(X,Y) = V(T(X,Y)) + H(T(X,Y)),$$

where H is the projection morphism of TTM^0 on HTM^0.

In the following we consider some notions from [10], related to foliated manifolds, in our particular case, (TM^0, F_V, G). First of these notions is the *Vaisman* connection ∇^v on a Riemannian foliated manifold. This is a connection on TM^0, uniquely defined by the conditions:

a) If $Y \in \Gamma(VTM^0))$ (respectively $\in \Gamma(HTM^0)$), then $\nabla^v_X Y \in \Gamma(VTM^0)$ (respectively $\in \Gamma(HTM^0)$) for every X;

b) if $X, Y, Z \in \Gamma(VTM^0)$ $(\Gamma(HTM^0))$, then $(\nabla^v_X G)(Y,Z) = 0$;

c) $V(T(X,Y)) = 0$ if at least one of the arguments is in $\Gamma(VTM^0)$ and $H(T(X,Y)) = 0$ if at least one of the arguments is in $\Gamma(HTM^0)$.

In [10, p. 188], the coefficients of ∇^v in an adapted chart are given and it is proved that:

Proposition 2.1. *Let (M,F) be a Finsler manifold. Then the Vaisman connection ∇^v on (TM^0, G, F_V), is locally expressed with respect to the adapted local basis $\{\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i}\}$ as follows:*

$$\nabla^v_{\frac{\delta}{\delta x^j}} \frac{\delta}{\delta x^i} = F^k_{ij} \frac{\delta}{\delta x^k}, \quad \nabla^v_{\frac{\partial}{\partial y^j}} \frac{\delta}{\delta x^i} = 0,$$

$$\nabla^v_{\frac{\delta}{\delta x^j}} \frac{\partial}{\partial y^i} = \frac{\partial G^k_j}{\partial y^i} \frac{\partial}{\partial y^k}, \quad \nabla^v_{\frac{\partial}{\partial y^j}} \frac{\partial}{\partial y^i} = C^k_{ij} \frac{\partial}{\partial y^k}. \tag{2.6}$$

An imediate consequence of the above proposition is:

Proposition 2.2. *If (M,F) is a Finsler manifold, then the Levi-Civita and the Vaisman connections on the foliated manifold (TM^0, G, F_V) induce the same connection on* the structural bundle *iff M is a Landsberg manifold.*

Proof: The relations (2.6) give the local expression of the induced connection on VTM^0. Taking into account relations (2.6) and (2.4), we have that the Levi-Civita and the Vaisman connections on the foliated manifold (TM^0, G, F_V) induce the same connection on the structural bundle iff

$$\frac{\partial G^k_j}{\partial y^i} = F^k_{ij}. \tag{2.7}$$

The first member of the above equality is usualy denoted by G^k_{ij} and a Finsler manifold where holds the relation (2.7), is a *Landsberg manifold* ([3]).

Remark 2.2. *In [3, p. 137, theorem 2.2], it is proved that a Finsler manifold is a Landsberg manifold if and only if the vertical foliation F_V on the Riemannian manifold (TM^0, G) is totally geodesic (that means all leaves are geodesic submanifolds). So, we can say that the Levi-Civita and Vaisman connection induce the same connection on the structural bundle VTM^0 if and only if the foliation F_V is totally geodesic.*

Definition 2.1. A Riemannian foliated manifold with Riemannian metric g is called a *Reinhart* space iff

$$(\nabla^v_X G)(Y, Z) = 0, \tag{2.8}$$

for all the sections X of the structural bundle and Y, Z sections of *the transversal bundle*, where the covariant derivative is taken with respect to the Vaisman connection of the manifold.

Proposition 2.3. *Let (M, F) be a Finsler manifold. The foliated manifold (TM^0, G, F_V) is a Reinhart space if and only if (M, F) is a Riemannian manifold.*

Proof. Let $X \in \Gamma(VTM^0)$ and $Y, Z \in \Gamma(HTM^0)$, and ∇^v be the Vaisman connection on (TM^0, G, F_V). In an adapted local chart we have

$$X = X^i \frac{\partial}{\partial y^i}, \quad Y = Y^j \frac{\delta}{\delta x^j}, \quad Z = Z^k \frac{\delta}{\delta x^k}.$$

By the linearity of ∇^v we can consider $X = \frac{\partial}{\partial y^i}$. Then we compute

$$\left(\nabla^v_{\frac{\partial}{\partial y^i}} G\right)(Y, Z) = \tfrac{\partial}{\partial y^i}\left(Y^j Z^k g_{jk}\right) - G\left(\nabla^v_{\frac{\partial}{\partial y^i}} Y^j \tfrac{\delta}{\delta x^j}, Z^k \tfrac{\delta}{\delta x^k}\right) -$$
$$-G\left(Y^j \tfrac{\delta}{\delta x^j}, \nabla^v_{\frac{\partial}{\partial y^i}} Z^k \tfrac{\delta}{\delta x^k}\right) = Y^j Z^k \tfrac{\partial g_{jk}}{\partial y^i}.$$

Hence the condition (2.8) for all $Y, Z \in \Gamma(HTM^0)$ is equivalent to $\frac{\partial g_{jk}}{\partial y^i} = 0$, so, by remark 2.1, it is equivalent to (M, F) Riemannian manifold.

Remark 2.3. *In [3, p. 137, theorem 2.1], it is proved that a Finsler manifold is a Riemannain manifold if and only if the Sasaki-Finsler metric G is bundle-like for the vertical foliation F_V of TM^0. So, we can say that the Sasaki-Finsler metric G is bundle-like for the vertical foliation F_V of TM^0 if and only if (TM^0, G, F_V) is a Reinhart space.*

Let $\Omega^0(TM^0)$ be the ring of real differentiable functions on TM^0 and $\Omega^{p,q}(TM^0)$ be the module of (p, q) forms on the foliated manifold (TM^0, G, F_V). Hence, $\theta \in \Omega^{p,q}(TM^0)$ is a $(p+q)$-form on (TM^0, G) which is potentially different from zero only if p arguments are sections of HTM^0 and q arguments are sections of VTM^0. By this definition, $\theta \in \Omega^{0,1}(TM^0)$ has the local expression

$$\theta = \theta_i \delta y^i, \tag{2.9}$$

where θ_i are differentiable functions on the domain of the local chart and $\{dx^i, \delta y^i\}$ is the dual basis of $\{\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i}\}$. Obviously, if $U, \widetilde{U}$ are two domains which overlap, then in their intersection

$$\widetilde{\theta}_{i_1} = \frac{\partial \tilde{y}^i}{\partial y^{i_1}}\theta_i = \frac{\partial \tilde{x}^i}{\partial x^{i_1}}\theta_i, \tag{2.10}$$

where we used also (2.1).

Let $d_{01} : \Omega^{p,q}(TM^0) \to \Omega^{p,q+1}(TM^0)$ be the foliated derivative on (TM^0, G, F_V). It is well-known, [10], that $d_{01}^2 = 0$. If $f \in \Omega^0(TM^0)$ and $\theta \in \Omega^{0,1}(TM^0)$, then we locally have

$$d_{01}f = \frac{\partial f}{\partial y^i}\delta y^i, \quad d_{01}\theta = \left(\frac{\partial \theta_i}{\partial y^j} - \frac{\partial \theta_j}{\partial y^i}\right)\delta y^j \wedge \delta y^i.$$

The foliated derivative define the leafwise cohomology of a foliated manifold. In this paper we present only the 1-dimensional leafwise cohomology group of (TM^0, G, F_V), which is interesting for our research.

We recall that a $(0,1)$-form θ is called d_{01} *− exact* if there exists $f \in \Omega^0(TM^0)$ such that $\theta = d_{01}f$ and it is called d_{01} *− closed* if $d_{01}\theta = 0$. The Poincare lemma assures that every d_{01}-closed form is d_{01}−locally exact.

Let $Z^{0,1}(TM^0)$ and $B^{0,1}(TM^0)$ be the groups of $(0,1)$-forms which are, respectively, d_{01}-closed and d_{01}-exact. Obviously, $B^{0,1}(TM^0)$ is a subgroup of $Z^{0,1}(TM^0)$. The quotient group

$$H^1(F_V) = \frac{Z^{0,1}(TM^0)}{B^{0,1}(TM^0)}, \tag{2.11}$$

is called the 1-dimensional *leafwise cohomology* group of the foliated manifold (TM^0, G, F_V).

We recall that $f \in \Omega^0(TM^0)$ is called *basic* if $d_{01}f = 0$, or, equivalent, if it is constant on the leaves, so it doesn't depend on x. Let Φ be the sheaf of germs of basic functions on (TM^0, G, F_V). An important result is the de Rham theorem:

Theorem 2.4. *The cohomology group $H^1(F_V)$ is isomorphic with the 1-dimensional Cech cohomology group of TM^0 with coefficients in the sheaf Φ:*

$$H^1(F_V) \approx \mathbf{H}^1(TM^0, \Phi).$$

3 Leafwise 2-jets on (TM^0, G, F_V)

If M is a differential real manifold, we know, [1], [8], that two functions $f, g \in \Omega^0(M)$ determine the same *2-jet* at a fixed point $x \in M$ if $f(x) = g(x) = 0$ and their first and second derivative are equal at x.

On a foliated manifold two differentiable real functions determine the same *leafwise 2-jet* at a point if they determine the same 2-jet in the leaf through that point ([5]). So, for (TM^0, G, F_V) we have the following definition:

Definition 3.1. We say that $f, g \in \Omega^0(TM^0)$ determine the same *leafwise 2-jet*, or *l,2-jet* at $(x,y) \in TM^0$ if they determine the same 2-jet at (x,y) in T_xM^0.

The relation "to determine the same l,2-jet at (x,y)" is an equivalence on $\Omega^0(TM^0)$. The equivalence class containing f is denoted by $j^{l,2}_{(x,y)}f$ and it is called the *l,2-jet of f at x*. By the above definition we have that the equality $j^{l,2}_{(x,y)}f = j^{l,2}_{(x,y)}\tilde{f}'$ is locally expressed by:

$$f(x,y) = \tilde{f}(x,y) = 0, \quad \frac{\partial f}{\partial y^i}(x,y) = \frac{\partial \tilde{f}}{\partial y^i}(x,y), \quad \frac{\partial^2 f}{\partial y^i \partial y^j}(x,y) = \frac{\partial^2 \tilde{f}}{\partial y^i \partial y^j}(x,y),$$

for every $i, j = \overline{1,n}$. These conditions have geometrical meaning and they can be considered as definition for "f, g determine the same l,2-jet at (x,y)".

For every $f, g \in \Omega^0(TM^0)$ and $\alpha \in \mathbb{R}$, we define the following operations:

$$\begin{aligned} &j^{l,2}_{(x,y)}f + j^{l,2}_{(x,y)}g = j^{l,2}_{(x,y)}(f+g), \\ &\alpha \cdot j^{l,2}_{(x,y)}f = j^{l,2}_{(x,y)}(\alpha \cdot f), \\ &j^{l,2}_{(x,y)}f \cdot j^{l,2}_{(x,y)}g = j^{l,2}_{(x,y)}(f \cdot g). \end{aligned}$$

It is easy to see that the above operations are well-defined. For example, we shall prove that the product doesn't depend on the representatives. Let $\tilde{f} \in j^{l,2}_{(x,y)}f$ and $\tilde{g} \in j^{l,2}_{(x,y)}g$. By the definition 3.1, it results $f(x,y) = g(x,y) = \tilde{f}(x,y) = \tilde{g}(x,y) = 0$, and

$$\begin{aligned} &\tfrac{\partial f}{\partial y^i}(x,y) = \tfrac{\partial \tilde{f}}{\partial y^i}(x,y), \quad \tfrac{\partial^2 f}{\partial y^i \partial y^j}(x,y) = \tfrac{\partial^2 \tilde{f}}{\partial y^i \partial y^j}(x,y), \\ &\tfrac{\partial g}{\partial y^i}(x,y) = \tfrac{\partial \tilde{g}}{\partial y^i}(x,y), \quad \tfrac{\partial^2 g}{\partial y^i \partial y^j}(x,y) = \tfrac{\partial^2 \tilde{g}}{\partial y^i \partial y^j}(x,y). \end{aligned}$$

Then we have

$$\begin{aligned} \frac{\partial (f\cdot g)}{\partial y^i}(x,y) &= \frac{\partial f}{\partial y^i}(x,y)\cdot g(x,y) + \frac{\partial g}{\partial y^i}(x,y)\cdot f(x,y) = \frac{\partial (\tilde{f}\cdot\tilde{g})}{\partial y^i}(x,y) = 0 \\ \frac{\partial^2 (f\cdot g)}{\partial y^i \partial y^j}(x,y) &= 2\frac{\partial f}{\partial y^i}(x,y)\cdot\frac{\partial g}{\partial y^i}(x,y) = 2\frac{\partial \tilde{f}}{\partial y^i}(x,y)\cdot\frac{\partial \tilde{g}}{\partial y^i}(x,y) = \frac{\partial^2 (\tilde{f}\cdot\tilde{g})}{\partial y^i \partial y^j}(x,y), \end{aligned}$$

which shows that $j^{l,2}_{(x,y)}(f\cdot g) = j^{l,2}_{(x,y)}(\tilde{f}\cdot\tilde{g})$, q.e.d.

Now let $f \in \Omega^0(TM^0)$ such that $f(x,y) = 0$ for a fixed point $(x,y) \in TM^0$. Let $(U, (x^i, y^i))$ be a local chart at (x,y), such that $y^i(x,y) = 0, i = \overline{1,n}$. In U, both the functions f and

$$y^i \frac{\partial f}{\partial y^i}(x,y) + \frac{1}{2} y^i y^j \frac{\partial^2 f}{\partial y^i \partial y^j}(x,y),$$

determine the same l,2-jet at (x,y), so we obtain the local expression of $j^{l,2}_{(x,y)}f$:

$$j^{l,2}_{(x,y)}f = \frac{\partial f}{\partial y^i}(x,y) j^{l,2}_{(x,y)}y^i + \frac{1}{2}\frac{\partial^2 f}{\partial y^i \partial y^j} j^{l,2}_{(x,y)}y^i \cdot j^{l,2}_{(x,y)}y^j.$$

Moreover, we say that it is a l,2-jet at (x,y) every expression locally given in $(U, (x^i, y^i))$ by

$$\omega^2_{(x,y)} = \omega_i(x,y)\cdot j^{l,2}_{(x,y)}y^i + \frac{1}{2}\omega_{ij}(x,y) j^{l,2}_{(x,y)}y^i \cdot j^{l,2}_{(x,y)}y^j, \tag{3.1}$$

where $\omega_i, \omega_{ij} \in \Omega^0(U)$ are satisfying $\omega_{ij} = \omega_{ji}$ and if $\tilde{U}$ is another domain of local chart at (x, y), then in $U \cap \tilde{U}$ these functions satisfy

$$\tilde{\omega}_{i_1} = \omega_i \frac{\partial y^i}{\partial \tilde{y}^{i_1}}, \quad \tilde{\omega}_{i_1 j_1} = \omega_i \frac{\partial^2 y^i}{\partial \tilde{y}^{i_1} \partial \tilde{y}^{j_1}} + \omega_{ij} \frac{\partial y^i}{\partial \tilde{y}^{i_1}} \frac{\partial y^j}{\partial \tilde{y}^{j_1}}.$$

Taking into account (2.1), the transformation of the component functions of a l,2-jet at (x, y) are:

$$\tilde{\omega}_{i_1} = \omega_i \frac{\partial x^i}{\partial \tilde{x}^{i_1}}, \quad \tilde{\omega}_{i_1 j_1} = \omega_{ij} \frac{\partial x^i}{\partial \tilde{x}^{i_1}} \frac{\partial x^j}{\partial \tilde{x}^{j_1}}. \tag{3.2}$$

We denote by $J^{l,2}_{(x,y)}(TM^0)$ the space of l,2-jets at (x, y) and we can see that this is a real algebra. The space $J^{l,2}(TM^0) = \cup_{(x,y) \in TM^0} J^{l,2}_{(x,y)}(TM^0)$ is a fiber bundle over TM^0, where the fiber coordinates are $(\omega_i, \omega_{ij})_{1 \le i < j \le n}$.

Definition 3.2. An l,2-jet at (x, y) locally given by (3.1) is called *homogeneous* if $\omega_i = 0$ for all $i = \overline{1, n}$.

The relations (3.2) show that the notion of homogeneous l,2-jet has geometrical meaning. We denote by $\Theta^{l,2}_{(x,y)}(TM^0)$ the space of homogenous l,2-jets at (x, y). The space $\Theta^{l,2}(TM^0) = \cup_{(x,y) \in TM^0} \Theta^{l,2}_{(x,y)}(TM^0)$ is a fiber bundle over TM^0, where the fiber coordinates are $(\omega_{ij})_{1 \le i < j \le n}$.

The smooth sections of $J^{l,2}(TM^0)$, $\Theta^{l,2}(TM^0)$ are called the *fields of l,2-jets*, and the *fields of homogeneous l,2-jets* (or h,l,2-jets) on TM^0, respectively. The spaces of these sections are denoted by $J^2(F_V)$, $\Theta^2(F_V)$, respectively. We have the inclusion $\Theta^2(F_V) \subset J^2(F_V)$.

Let $j^{l,2}$ be the map which assigns to every $f \in \Omega^0(TM^0)$ its field of l,2-jets

$$j^{l,2} : \Omega^0(TM^0) \to J^2(F_V). \tag{3.3}$$

The local expression of this map is

$$j^{l,2} f = \frac{\partial f}{\partial y^i} j^{l,2} y^i + \frac{1}{2} \frac{\partial^2 f}{\partial y^i \partial y^j} j^{l,2} y^i \cdot j^{l,2} y^j. \tag{3.4}$$

The changing rules for the fields of l,2-jets of the coordinates are:

$$j^{l,2} \tilde{y}^{i_1} = \frac{\partial \tilde{y}^{i_1}}{\partial y^i} j^{l,2} y^i + \frac{1}{2} \frac{\partial^2 \tilde{y}^{i_1}}{\partial y^i \partial y^j} j^{l,2} y^i \cdot j^{l,2} y^j,$$

$$j^{l,2} \tilde{y}^{i_1} \cdot j^{l,2} \tilde{y}^{j_1} = \frac{\partial \tilde{y}^{i_1}}{\partial y^i} \frac{\partial \tilde{y}^{j_1}}{\partial y^j} j^{l,2} y^i \cdot j^{l,2} y^j.$$

Taking into account relations (2.1), the above transformations become:

$$j^{l,2} \tilde{y}^{i_1} = \frac{\partial \tilde{x}^{i_1}}{\partial x^i} j^{l,2} y^i, \quad j^{l,2} \tilde{y}^{i_1} \cdot j^{l,2} \tilde{y}^{j_1} = \frac{\partial \tilde{x}^{i_1}}{\partial x^i} \frac{\partial \tilde{x}^{j_1}}{\partial x^j} j^{l,2} y^i \cdot j^{l,2} y^j. \tag{3.5}$$

Let ω^2 be an arbitrary field of l,2-jets on TM^0, locally given by

$$\omega^2 = \omega_i \cdot j^{l,2} y^i + \frac{1}{2} \omega_{ij} j^{l,2} y^i \cdot j^{l,2} y^j,$$

where the relations (3.2) hold. The relations (3.2) and (3.5) show that the expressions

$$\omega^1 = \omega_i \cdot j^{l,2} y^i, \quad \omega_o^2 = \frac{1}{2} \omega_{ij} j^{l,2} y^i \cdot j^{l,2} y^j,$$

have geometrical meaning. Moreover, ω_o^2 is a field of h,l,2-jets. We call ω^1 the *1-type part* of ω^2 and ω_o^2 the *homogeneous part* of ω^2. it results that every field of l,2-jets on TM^0 is a sum of its 1-type and homogeneous parts:

$$\omega^2 = \omega^1 + \omega_o^2. \tag{3.6}$$

Let $J^{l,2,1}_{(x,y)}(TM^0)$ the set of all the expressions $\omega_i(x,y) \cdot j^{l,2}_{(x,y)} y^i$ satisfying (3.2) and (3.5). The space

$$J^{l,2,1}(TM^0) = \cup_{(x,y) \in TM^0} J^{l,2,1}_{(x,y)}(TM^0)$$

is a fiber bundle over TM^0, and a subbundle of $J^{l,2}(TM^0)$. The space of its sections is denoted by $J^{2,1}(F_V)$.

Using the relation (3.6) it is easy to verify that

Proposition 3.1. *The bundle of l,2-jets on the foliated manifold TM^0, F_V is the direct sum of the bundles $J^{l,2,1}(TM^0)$ and $\Theta^{l,2}(TM^0)$.*

An immediate consequence is the decomposition

$$J^2(F_V) = J^{2,1}(F_V) \oplus \Theta^2(F_V).$$

Proposition 3.2. *The space $J^{2,1}(F_V)$ is a $\Omega^0(TM^0)$-module, isomorphic with the module $\Omega^{0,1}(TM^0)$ of $(0,1)$-forms on TM^0.*

Proof. Let $\xi : \Omega^{0,1}(TM^0) \to J^{2,1}(F_V)$ be the map which assigns to every $(0,1)$-form θ locally given by (2.9) the field of l,2-jets ω^2 locally given by $\theta_i \cdot j^{l,2} y^i$. From (2.10), (3.2) and (3.5), it results that ξ is well-defined. Also, it is an isomorphism of modules.

Remark 3.1. Ones could say that every field of l,2-jets on TM^0 is congruent modulo $\Omega^{0,1}(TM^0)$ to every field of h,l,2-jets on the same manifold.

We can see that the map ξ from the proof of Proposition 3.2 satisfies

$$(\xi \circ d_{01}) f = (j^{l,2} f)^1, \quad \forall f \in \Omega^0(TM^0),$$

where the second member is the 1-type part of the field of l,2-jets of the function f. Let $\delta^{l,2}$ be the map who assigns to every $f \in \Omega^0(TM^0)$ the homogeneous part of $j^{l,2} f$:

$$\delta^{l,2} : \Omega^{0,1}(TM^0) \to \Theta^2(F_V), \quad \delta^{l,2} f = (j^{l,2} f)_o. \tag{3.7}$$

By the relations (3.3), (3.4), (3.6)-(3.7), we have

Proposition 3.3. *The map $j^{l,2}$ satisfies*

$$j^{l,2} = \xi \circ d_{01} + \delta^{l,2}.$$

Now, we define the following map

$$D : \Omega^{0,1}(TM^0) \to \Theta^2(F_V),$$

given locally by

$$D(\theta_i \delta y^i) = \frac{1}{4}\left(\frac{\partial \theta_i}{\partial y^j} + \frac{\partial \theta_j}{\partial y^i}\right) \cdot j^{l,2}y^i \cdot j^{l,2}y^j.$$

By a direct calculation, D has geometrical meaning. Moreover, for an arbitrary $f \in \Omega^0(TM^0)$, we can compute in every local chart

$$(D \circ d_{01})f = D\left(\frac{\partial f}{\partial y^i}\delta y^i\right) = \frac{1}{2}\frac{\partial^2 f}{\partial y^i \partial y^j} j^{l,2}y^i \cdot j^{l,2}y^j = (j^{l,2}f)_o = \delta^{l,2}f.$$

If we denote by ζ the map $\xi + D$, then $\zeta : \Omega^{0,1}(TM^0) \to J^2(F_V)$ is satisfying by proposition 3.3 the following relation

$$j^{l,2} = \zeta \circ d_{01}. \tag{3.8}$$

4 Mastrogiacomo leafwise cohomology on (TM^0, G, F_V)

In this section we use the fields of leafwise 2-jets from the previous section for defining two cohomology groups of the foliated manifold (TM^0, G, F_V), one of them being essentially dependent by a given connection on TM^0.

Definition 4.1. We say that a field of l,2-jets $\omega^2 \in J^2(F_V)$ is $j^{l,2}$-*exact* if $\omega^2 = j^{l,2}f$ for some $f \in \Omega^0(TM^0)$, and it is $j^{l,2}$-*closed* if it is locally $j^{l,2}$-exact.

We denote by $E^2(F_V)$, $C^2(F_V)$ the spaces of fields of l,2-jets on TM^0 which are $j^{l,2}$-exact, respectively $j^{l,2}$-closed. Obviously, $E^2(F_V) \subset C^2(F_V)$. We call the *Mastrogiacomo leafwise cohomology* group of TM^0 the quotient group

$$H^{l,2}(F_V) = \frac{C^2(F_V)}{E^2(F_V)}.$$

Proposition 4.1. *If ζ is the map from the relation (3.8), then we have*

$$\zeta\left(Z^{0,1}(TM^0)\right) = C^2(F_V), \quad \zeta\left(B^{0,1}(TM^0)\right) = E^2(F_V),$$

where the spaces $Z^{0,1}(TM^0)$, $B^{0,1}(TM^0)$ are from the relation (2.11).

Proof: Let $\theta \in Z^{0,1}(TM^0)$, which means that it is a $(0,1)$-form satisfying $d_{01}\theta = 0$. The Poincare lemma assures that in every domain U from TM^0 there exists $f_U \in \Omega^0(U)$ such that in U we have $\theta = d_{01}f_U$. We compute in U

$$\zeta(\theta) = (\zeta \circ d_{01})(f_U) = j^{l,2}f_U$$

using relation (3.8). It results that $\zeta(\theta)$ is locally $j^{l,2}$-exact, so it belongs to $C^2(F_V)$.

Considering now an arbitrary $\omega^2 \in C^2(F_V)$, we have that in every domain U from TM^0 there is a differentiable function f_U such that $\omega^2 = j^{l,2} f_U$. Let θ be the $(0,1)$-form locally given in U by $\theta = d_{01} f_U$. This form is globally defined because the field of l,2-jets ω^2 is globally defined. Moreover, $d_{01}\theta = 0$ and $\zeta(\theta) = \omega^2$, so $C^2(F_V) \subset \zeta\left(Z^{0,1}(TM^0)\right)$, which entails the first equality. The second equality follows by an analogous argument, q.e.d..

An immediate consequence is:

Theorem 4.2. *The map ζ induces an isomorphism between the 1-leafwise cohomology $H^1(F_V)$ and the Mastrogiacomo leafwise cohomology group $H^{l,2}(F_V)$.*

Proof. The map

$$\zeta^* : H^1(F_V) \to H^{l,2}(F_V), \quad \zeta^*([\theta]) = [\zeta(\theta)]^{l,2},$$

is well-defined by Proposition 4.1 and it is an isomorphism of groups.

From Theorems 2.1 and 4.1, it results:

Theorem 4.3. *The Mastrogiacomo leafwise cohomology group of the foliated manifold (TM^0, G, F_V) and the 1-dimensional Cech cohomology group of TM^0 with coefficients in the sheaf Φ of basic functions are isomorphic.*

In the following, let $\tilde{\nabla}$ be the Levi-Civita connection of the metric G on TM^0 and ∇ the connection induced by $\tilde{\nabla}$ in the structural bundle, from the section 2. The following map is well-defined:

$$\delta_{\nabla}^{l,2} : \Omega^{0,1}(TM^0) \to \Theta^{2,1}(F_V),$$

locally given by

$$\delta_{\nabla}^{l,2} f = \frac{1}{2}\left(\frac{\partial^2 f}{\partial y^i \partial y^j} - C_{ij}^k \frac{\partial f}{\partial y^k}\right) j^{l,2} y^i \cdot j^{l,2} y^j, \tag{4.1}$$

where C_{ij}^k are the coefficients of the restriction of ∇ to the leaves, given by (2.5). Indeed, if U and $\tilde{U}$ are two domanis of local charts which overlap, then we have in $U \cap \tilde{U}$, [4]:

$$\tilde{C}_{i_1 j_1}^{k_1} \cdot \frac{\partial x^k}{\partial \tilde{x}^{k_1}} = C_{ij}^k \cdot \frac{\partial x^i}{\partial \tilde{x}^{i_1}} \cdot \frac{\partial x^j}{\partial \tilde{x}^{j_1}}.$$

Using the above transformation, by a direct calculation results that the second member of (4.1) satisfies the conditions (3.2) (here $\omega_i = 0$), so $\delta_{\nabla}^{l,2} f$ is a field of h,l,2-jets on TM^0.

Definition 4.2. We say that a field of h,l,2-jets $\omega^2 \in \Theta^2(F_V)$ is $\delta_{\nabla}^{l,2}$-*exact* if $\omega^2 = \delta_{\nabla}^{l,2} f$ for some $f \in \Omega^0(TM^0)$, and it is $\delta_{\nabla}^{l,2}$-*closed* if it is locally $\delta_{\nabla}^{l,2}$-exact.

We denote by $E_{\nabla}^2(F_V)$, $C_{\nabla}^2(F_V)$ the spaces of fields of h,l,2-jets on TM^0 which are $\delta_{\nabla}^{l,2}$-exact, $\delta_{\nabla}^{l,2}$-closed, respectively. Obviously, $E_{\nabla}^2(F_V) \subset C_{\nabla}^2(F_V)$. We call the

Mastrogiacomo leafwise cohomology group with respect to ∇ of TM^0 the quotient group

$$H^{l,2}_{\nabla}(F_V) = \frac{C^{2}_{\nabla}(F_V)}{E^{2}_{\nabla}(F_V)}.$$

Let Ω_{∇} be the sheaf of germs of differentiable functions on TM^0 which satisfy $\nabla df = 0$. Locally, this condition is equivalent with the following system:

$$\frac{\partial^2 f}{\partial y^i \partial y^j} - C^k_{ij}\frac{\partial f}{\partial y^k} = 0, \quad (\forall) i, j = \overline{1, n}. \tag{4.2}$$

The integrability conditions $\frac{\partial^3 f}{\partial y^i \partial y^j \partial y^h} = \frac{\partial^3 f}{\partial y^h \partial y^i \partial y^j}$ yield that the solutions of (4.2) have to satisfy

$$S^l_{ijh}\frac{\partial f}{\partial y^l} = 0, \quad (\forall) i, j, h = \overline{1, n},$$

where

$$S^l_{ijh} = \frac{\partial C^l_{ij}}{\partial y^h} - \frac{\partial C^l_{jh}}{\partial y^i} + C^k_{ij}C^l_{kh} - C^k_{jh}C^l_{ki},$$

are the curvature coefficients of the restriction of ∇ along the leaves. On the other hand, taking into account the well-known relations, [4],

$$y^i C^k_{ij} = 0,$$

from the system (4.2) it follows

$$y^i \frac{\partial^2 f}{\partial y^i \partial y^j} = 0, \quad (\forall) j = \overline{1, n}.$$

which is equivalent to

$$Lf - f \in \Phi(TM^0),$$

where L is the vertical Liouville vector field on TM^0,

$$L = y^i \frac{\partial}{\partial y^i}.$$

For an arbitrary foliated manifold we proved in [7] that the Mastrogiacomo leafwise cohomology with respect to a connection ∇ and the 1-dimensional Cech cohomology group with coefficients in the sheaf Ω_{∇} of that manifold, are isomorphic ([7], theorem 4.1, p.83). So, we have

Theorem 4.4. *If* (M, F) *is a Finsler manifold, for the foliated manifold* (TM^0, G, F_V) *the following isomorphism holds:*

$$H^{l,2}_{\nabla}(F_V) \approx \mathbf{H}^1(TM^0, \Omega_{\nabla}).$$

In the following we determine the Mastrogiacomo leafwise cohomology group of TM^0 with respect to ∇ in two particular cases.

(I) If the Finsler manifold (M, F) is Riemannian, which is equivalent from remark (2.1) to relation $C^k_{ij} = 0$ for all indices i, j, k, and ∇ is the induced connection on the structural bundle by the Levi-Civita connection on TM^0, then the sheaf Ω_∇ contains the germs of functions which are satisfying

$$\frac{\partial^2 f}{\partial y^i \partial y^j} = 0, \quad (\forall) i, j = \overline{1, n}.$$

The solutions of the above system are $f = \alpha_i y^i + \alpha$, with α_i, α basic functions. Hence, we have the isomorphism of shaves $\Omega_\nabla \approx \Phi^{n+1}$, where Φ is the sheaf of germs of basic functions on TM^0. So, by Theorem 4.3, it results

$$H^{l,2}_\nabla(F_V) \approx \mathbf{H}^1(TM^0, \Phi^{n+1}).$$

Taking into account theorem 2.1, we obtain that the 1-leafwise cohomology group of (TM^0, G, F_V) is a proper subgroup of $H^{l,2}_\nabla(F_V)$.

(II) Let (M, F) be a 4-dimensional 4-roots space, where the fundamental functions F is locally given by $F = \sqrt[4]{y^1 y^2 y^3 y^4}$. We suppose that F is well-defined on TM. By a straightforward calculation we obtain $g_{ii} = -\frac{F^2}{8}\frac{1}{(y^i)^2}$, $g_{ij} = \frac{F^2}{8}\frac{1}{y^i y^j}$, and further

$$C^i_{ii} = -\frac{3}{8(y^i)^2}, \quad C^k_{ii} = \frac{y^k}{8(y^i)^2}, \quad C^i_{ik} = \frac{1}{8y^k}, \quad C^k_{ij} = -\frac{y^k}{8y^i y^j},$$

for all indices $i, j, k \in \{1, 2, 3, 4\}$ such that $i \neq j \neq k \neq i$.

Then the system (4.2) becomes

$$\begin{cases} 8(y^i)^2 \dfrac{\partial^2 f}{\partial (y^i)^2} + 4y^i \dfrac{\partial f}{\partial y^i} - Lf = 0 \\[2ex] 8y^i y^j \dfrac{\partial^2 f}{\partial y^i y^j} + 2y^i \dfrac{\partial f}{\partial y^i} + 2y^j \dfrac{\partial f}{\partial y^j} - Lf = 0 \end{cases}$$

for all $i, j \in \{1, 2, 3, 4\}$, where the repetition of the same index does not means summation. So, this system has 10 equations. The integrability conditions give that a solution f has to satisfy

$$y^1 \frac{\partial f}{\partial y^1} = y^2 \frac{\partial f}{\partial y^2} = y^3 \frac{\partial f}{\partial y^3} = y^4 \frac{\partial f}{\partial y^4}, \tag{4.3}$$

hence the system becomes

$$\frac{\partial^2 f}{\partial (y^i)^2} = 0, \quad \frac{\partial^2 f}{\partial y^i \partial y^j} = 0, \quad (\forall) i, j \in \{1, 2, 3, 4\}, i \neq j.$$

The solutions are $f = \alpha_1 y^1 + \alpha_2 y^2 + \alpha_3 y^3 + \alpha_4 y^4 + \alpha$, where α_i, α are basics functions, so they do not depend on y. These solutions have to satisfy as well the relations (4.3), hence

$$\alpha_1(x) y^1 = \alpha_2(x) y^2 = \alpha_3(x) y^3 = \alpha_4(x) y^4, \quad (\forall) x \in M.$$

It results that $\alpha_i = 0$ for all $i \in \{1, 2, 3, 4\}$. We have then $\Omega_\nabla \approx \Phi$.

For this particular Finsler manifold we obtain

$$H^{l,2}_{\nabla}(F_V) \approx \mathbf{H}^{1}(TM^{0}, \Phi).$$

Taking into account theorem 2.1, it follows that the Mastrogiacomo leafwise cohomology group with respect to ∇ is isomorphic with the 1-leafwise cohomology group of the foliated manifold (TM^0, G, F_V).

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