A remark on the Gauss-Bonnet theorem in Finsler geometry

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Abstract. Here it is shown, every n-dimensional compact constant isotropic L-curvature space and J-curvature space (as well as Landsberg surface) with non-zero flag curvature is Riemannian. As a consequence of this work, not only known results on the Gauss-Bonnet theorem on Finsler surfaces become trivial, but it will be so for a larger class of n-dimensional Finsler manifolds as well, namely constant isotropic L-curvature and J-curvature spaces.

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1 Introduction

In 1944, S. S. Chern presented a simple and intrinsic proof of the Gauss-Bonnet theorem for closed Riemannian manifolds [9]. Here, we quote several attempts to extend Gauss-Bonnet's theorem to the Finsler setting. A. Lichnerowicz's 1948 [12], H. Busemann [8] and P. Dazord [10] 1971 are authors of more significant results. There are also many other notable works, for instance, Rund [16], M. Matsumoto [14] in 1990 and Z. Shen [17] published in 1996 etc. After a period of fifty years, Chern came back to this theorem and in a joint paper with D. Bao [3] gave an extension of Gauss-Bonnet's theorem to Finsler surfaces as follows:

Theorem A. ([4] or [5]) *Let (M, F) be a compact connected Landsberg surface without boundary. Denote the common value of the Riemannian arc lengths of all its indicatrices by L. Then*

$$\frac{1}{L}\int_M K\sqrt{g}dx^1 \wedge dx^2 = \chi(M),$$

where the scalar K is Gaussian curvature, and $\chi(M)$ is the Euler characteristic of M.

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As a consequence of the present work, these attempts are actually without success, due to the fact that the Finsler space under consideration is trivially a Riemannian space. As a matter of fact the assumption of compactness is not only too strong for Landsberg surfaces and reduces these spaces to Riemannian ones, but also it trivializes a more general class of n-dimensional Finsler metrics, namely, constant isotropic L-curvature and J-curvature metrics. In the other hand the compactness assumption in Gauss-Bonnet's theorem is essential and can not be dropped.
The class of relatively Isotropic L-curvature and J-curvature metrics which generalizes the class of Landsberg metrics is an interesting class of Finsler metrics studied by many authors, not to be mentioned here. There are many examples of these metrics in the literature, see for instance Z. Shen [18]. Our goal is to deal with these two families of Finsler metrics and prove the following theorem.

Theorem 1.1. *Let (M, F) be an n-dimensional compact connected non-zero constant isotropic J-curvature or (resp. L-curvature) manifold, then it is Riemannian.*

This theorem provides in some senses an extension of the following result due to H. Akbar-Zadeh, see [1] page 129.

Theorem B. *Let (M, F) be a mean Landsberg surface. If F is of non-zero flag curvature, then it is Riemannian.*
The compactness assumption in Theorem 1 is essential and can not be dropped.

One remarkable feature of this result is that the Theorem 1 makes trivial some well-known results on the Gauss-Bonnet theorem on Finsler surfaces recently appeared in the text books on Finsler geometry, see for instance [4]. As another consequence of this result, the Gauss-Bonnet theorem is not only trivial for Landsberg surfaces but it will be so for a more general family of n-dimensional Finsler manifolds namely constant isotropic L-curvature and J-curvature spaces as well.

2 Preliminaries

Let M be an n-dimensional differentiable manifold and let $\pi : TM_o := TM\backslash\{0\} \to M$ denote its slit tangent bundle. A *Finsler structure* F on M is a nonnegative function on TM which is positively y-homogeneous of degree one with positive definite fundamental tensor $g := g_{ij}dx^i \otimes dx^j$ on π^*TM, where $g_{ij} := \frac{1}{2}[F^2]_{y^iy^j}(x, y)$. The *Cartan* tensor C is a symmetric tensor defined by $C(U, V, W) = C_{ijk}(y)U^iV^jW^k$, where $U = U^i\frac{\partial}{\partial x^i}$, $V = V^i\frac{\partial}{\partial x^i}$, $W = W^i\frac{\partial}{\partial x^i}$ and $C_{ijk} = \frac{1}{4}[F^2(y)]_{y^iy^jy^k}(y)$. The Cartan tensor characterizes Riemannian manifolds by $C = 0$. Consider the contracted Cartan tensor $I(U) = I_i(y)U^i$ with the components $I_i = g^{jk}C_{ijk}$. The tensor I is called *mean Cartan* tensor. By means of Deicke's theorem, $I = 0$ if and only if F is Riemannian. By definition, the *Landsberg curvature* L (resp. the *mean Landsberg curvature* J) is the covariant derivative of the Cartan tensor C (resp. *the mean Cartan tensor* I). A Finsler metric is said to be a *Landsberg metric* (resp. *mean Landsberg metric*) if $L = 0$ (resp. if $J = 0$).
The Finsler structure F is said to be of *relatively isotropic L-curvature*(resp. *relatively isotropic J-curvature*) if there is a non-zero scalar function $c(x)$ on M such that

$$L + c(x)FC = 0,$$
$$J + c(x)FI = 0,$$

respectively. To differentiate the Cartan tensor or mean Cartan tensor along geodesics, we need linearly parallel vector fields along a geodesic. The geodesics of F are characterized locally by

$$\frac{d^2x^i}{dt^2} + 2G^i\Big(x, \frac{dx}{dt}\Big) = 0,$$

where, $G^i = \frac{1}{4}g^{ik}\Big\{2\frac{\partial g_{pk}}{\partial x^q} - \frac{\partial g_{pq}}{\partial x^k}\Big\}y^p y^q$. Let $\alpha(t)$ be a geodesic parameterized by arc length on M, recall that a vector field $U(t) := U^i(t)(\partial/\partial x^i)|_{\alpha(t)}$ along the geodesic $\alpha(t)$ is said to be linearly parallel if the covariant derivative $D_{\dot\alpha(t)}$ along $\alpha(t)$ satisfies $D_{\dot\alpha(t)}U(t) = 0$ or equivalently $\frac{dU^i(t)}{dt} + \Gamma^i_{jk}U^j(t)U^k(t) = 0$, where, $G^i = \frac{1}{2}\Gamma^i_{jk}y^j y^k$. More precisely, consider the linearly parallel vector fields $U(t), V(t)$ and $W(t)$ along $\alpha(t)$, then we have

$$\begin{aligned} L_y(u, v, w) &:= \frac{d}{dt}C_{\dot\alpha(t)}((U(t), V(t), W(t)))|_{t=0}, \\ J_y(u) &:= \frac{d}{dt}I_{\dot\alpha(t)}(U(t))|_{t=0}, \end{aligned}$$

where, $u = U(0), v = V(0), w = W(0)$ and $y = \dot\alpha(0) \in T_xM$. If the non-zero scalar function $c(x)$ is constant then the Finsler manifold (M, F) is said to be a *constant isotropic mean Landsberg space*. In some literatures the Cartan and Landsberg tensors are denoted by A and $\dot A$ respectively and L-curvature (resp. J-curvature) metric is called mean Landsberg (resp. mean Cartan) metric. Throughout this paper, we make use of Einstein convention and set the Berwald connection on Finsler manifolds. The h- and v- covariant derivatives of a Finsler tensor fields are denoted here by "|" and "," respectively. We address the reader to [7] or [15], for the most general definition of Berwald connections and to [6] and [11] for some special vector fields on Finsler manifolds, and [13] for some Finsler metrics of order 2.

3 Proof of the main Theorem

Theorem 3.1. *Let (M, F) be an n-dimensional compact connected non-zero constant isotropic J-curvature or (resp. L-curvature) manifold, then it is Riemannian.*

Proof. Let (M, F) be a Finsler manifold, $p \in M$ and $y \in T_pM$. Let $\alpha(t) : t \in (-\infty, \infty) \to \alpha(t) \in M$ be a geodesic parameterized by arc length on M passing through $p = \alpha(0)$ and having the tangent vector $\frac{d\alpha}{dt}|_{t=0} = y$. To differentiate the Cartan tensor along $\alpha(t)$, we consider the linearly parallel vector fields $U(t), V(t)$ and $W(t)$ along $\alpha(t)$. On the other hand, by a direct computation (see for instance [1] or [18]) one can verify that for two linearly parallel vector fields $U(t)$ and $V(t)$ along $\alpha(t)$, we have

$$\frac{d}{dt}g_{\dot\alpha(t)}(U(t), V(t)) = 0.$$

In this sense, a g-orthonormal basis for π^*TM remains g-orthonormal at every point $(x(t), y(t))$ along the geodesics $\alpha(t)$. Therefore, if $U(t)$ is a parallel vector field along the geodesic $\alpha(t)$ with $U(0) = u$, by assuming $I(t) = I(U(t))$ and $J(t) = J(U(t))$ the equation $J_i = I_{i|k}y^k$ along geodesics can be written in the following form;

$$J(t) = \frac{dI}{dt}.$$

By definition, for a constant isotropic J-curvature or (resp. L-curvature) manifold there is a non-zero constant scalar c on M such that

$$J + c\ FI = 0,$$
$$L + c\ FC = 0,$$

respectively. Therefore these equations can be written along geodesics as the following ordinary differential equations;

$$\frac{dI}{dt} + c\ I(t) = 0, \tag{3.1}$$

$$\frac{dC}{dt} + c\ C(t) = 0, \tag{3.2}$$

where c is a non-zero constant scalar. The general solution of the equation (3.1), is given by

$$I(t) = I(0)\ e^{-ct}. \tag{3.3}$$

For $v \in TM_0$, assume that the norm of the mean Cartan torsion is $\|I\|_v := sup\ I(U)$, where the supremum is taken over all unit vectors of π_v^*TM. Suppose that $S_xM = \{w \in T_xM,\ F(w) = 1\}$ is the indicatrix and $\|I\| = sup_{v\in SM}\|I\|_v$, where $SM = \bigcup_{x\in M} S_xM$. Since M is compact the norm $\|I\|$ is bounded. On the other hand M is compact and therefore geodesically complete and the parameter t takes all the values in $(-\infty, +\infty)$. Letting $t \to +\infty$ or $t \to -\infty$, then Eq. (3.3) implies that $I(0) = 0$. In fact as t approaches to $t \to -\infty$ the left hand side of the equation is bounded and the right hand side is infinity, so Eq. (3.3) can be hold only if the coefficient $I(0)$ vanishes. Replacing it on the equation (3.3), we obtain $I(t) = 0$. Thus the mean Cartan torsion $I(u)$ vanishes along any arbitrary geodesic and therefore the mean Cartan tensor I vanishes identically. Hence, by straight forward computation due to Deicke, F is Riemannian. Similarly, the general solution of the equation (3.2) implies that the Cartan tensor C identically vanishes. This completes the proof of the theorem. □

We should remark that, if (M, F) is of zero flag curvature $K = \lambda = 0$ then it is Minkowskian [1]. The method used in this paper for solving a differential equation in Finsler geometry, is presented first in the H. Akbar-Zadeh's works see for example [1]. For other applications of this technics one can refer to [2], [7], [18], etc.

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