Synchronization and cryptography using chaotic dynamical systems

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Abstract. In this paper we considered a two-dimensional chaotic dynamical system and Rabinovich-Fabrikant chaotic dynamical system. Chaotic behavior has been proved by computing Lyapunov exponents. Synchronization phenomenon has been studied from analytical and numerical point of view, in case of classical dynamical systems. With a chaotic system being synchronized, we pointed out a type of encryption and decryption information using as secret keys the synchronized components of chaotic dynamical system and a key that we considered to be randomly generated, for each symbol at different moment of time.

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1 Introduction

Geometric mechanical systems have been studied lately using the theory of Lagrange, Hamiltonian and Poisson systems ([8], [12]). The study of dynamical systems was reduced to analyzing the solutions in a bounded region of the space and these solutions were in a stationary state or in an oscillatory state, but there is another type of behavior, called chaotic behavior [7]. Some requirements for a dynamical system to exhibit chaotic behavior are that the system must involve nonlinearity, the dependency on initial conditions. Chaotic behavior of a dynamical system can be proved using numerical methods, but there is a class for which analytical methods were developed to show that they are chaotic (using Shil′nikov scenario).

In this paper we presented chaotic dynamical systems in $\mathbb{R}^2$ and $\mathbb{R}^3$. Chaotic behavior is given by Lyapunov exponents. Here we presented e method of computing Lyapunov exponents, QR decomposition method, using orthogonal matrices. The method is illustrated by two examples, one in $2-$dimensional space and the other in $\mathbb{R}^3$ (Rabinovich-Fabrikant dynamical system). A way to synchronize two coupled

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dynamical systems, with the right side polynomial functions, is presented in Section 3. We defined a new type of synchronization, since for a 3-degree polynomial function we could not apply the existing techniques for synchronization. In the case presented above, it is quite hard to find a Lyapunov function for a dynamical system with its right side polynomial function of degree 3. We began with a drive system and a response system that results from the drive system, but in the right side it has a control vector. The error vector is determined, and also conditions that have to be fulfilled such that the error system to be asymptotically stable. In the last section we gathered the facts presented along this paper and we applied them in secure communication. We began with a chaotic system, we synchronized it with a coupled dynamical system (response system) and then we used the error system for encryption and decryption certain text, using secret randomly generated keys.

2 Lyapunov exponents for dynamical systems in $\mathbb{R}^2$ and $\mathbb{R}^3$

Lyapunov exponents are qualitative elements in the study of chaotic dynamical systems. We will determine Lyapunov exponents using QR continuous method using orthogonal matrices. We will use this method because it has some advantages that the other methods do not have [6]: a minimum number of parameters is used, rescalation is not necessary and numerical errors will never lead to lose of orthogonality [5].

Let us consider the dynamical system in $\mathbb{R}^n$ in the following form:

$$\dot{x}(t) = X(x(t)), \tag{2.1}$$

where $x = (x_1, x_2, ..., x_n)^T$, X is a vector field on $\mathbb{R}^n$. Let $x_0(t)$ be the orbit of the dynamical system (2.1) with the initial condition $x(0) = x_0$ and let $Z(t) = x(t) - x_0(t)$ be the deviation from the trajectory $x_0(t)$. The differential equation obtained from the linearization of the dynamical system (2.1) along the orbit is:

$$\dot{Z}(t) = DX(x_0(t))Z(t), \tag{2.2}$$

where DX is the Jacobi matrix of the vector field X. If Z_{in} is an initial condition for (2.2), then the solution of (2.2) is $Z(t) = M(t)Z_{in}$, where $M(t)$ is the tangent application. From (2.2) we have that $M(t)$ satisfies the following relation:

$$\dot{M}(t) = DX(x_0(t))M(t). \tag{2.3}$$

Let $\bigwedge$ be the matrix given by $\bigwedge = \lim\limits_{t\to\infty}(M(t)M^T(t))^{1/2t}$. For the dynamical system (2.1) the Lyapunov exponents are given as logarithm of eigenvalues of the the matrix $\bigwedge$ [5].

We consider the QR method for determining the Lyapunov exponents, using orthogonal matrices. Let Q be an orthogonal matrix and R an upper triangular matrix with positive elements on the main diagonal. Matrix $M(t)$ can be written on the form $M(t) = Q(t)R(t)$. From (2.3) follows that

$$Q^T(t)\dot{Q}(t) + \dot{R}(t)R^{-1}(t) = Q^T(t)DX(x_0(t))Q(t). \tag{2.4}$$

We notice that $Q^T(t)\dot{Q}(t)$ is an antisymmetric matrix and $\dot{R}(t)R^{-1}(t)$ is an upper triangular matrix.

The matrix $Q(t)$ can be written as a product of $\frac{n(n-1)}{2}$ elementary matrices from $SO(n)$, as $Q(t) = Q^{1\,2}(\theta_1(t)) \cdot ... \cdot Q^{n-1\,n}(\theta_{\frac{n(n-1)}{2}}(t))$, and the matrix $R(t)$ as

$$R(t) = \begin{bmatrix} e^{\lambda_1(t)} & r_{11}(t) & ... & r_{1n}(t) \\ 0 & e^{\lambda_2(t)} & ... & r_{2n}(t) \\ 0 & 0 & ... & e^{\lambda_n(t)} \end{bmatrix}. \tag{2.5}$$

Let S be the matrix given by the relation $S(t) = Q^T(t)DX(x_0(t))Q(t)$. The dynamical system (2.4) can be written now in the form

$$\begin{array}{l} \dot{\lambda}_i(t) = S_{ii}(t),\ i = 1, 2, ..., n. \\ S_{ij} = (Q^T(t)\dot{Q}(t))_{ij},\ i > j \\ (Q^T(t)\dot{Q}(t))_{ij} + \dot{r}_{ij}(t) = S_{ij}(t),\ i < j. \end{array} \tag{2.6}$$

Let $\lambda_i(t)$, $i = 1, 2, ..., n$, be the solution for (2.6). Lyapunov exponents are given by the following relations:

$$L_i = \lim_{t\to\infty} \frac{\lambda_i(t)}{t},\ i = 1, 2, ..., n. \tag{2.7}$$

A sufficient condition for a dynamical system to be chaotic is the existence of a positive Lyapunov exponent.

Let us consider a dynamical system in $\mathbb{R}^2$ given by:

$$\begin{cases} \dot{x}_1(t) = X_1(x_1, x_2), \\ \dot{x}_2(t) = X_2(x_1, x_2). \end{cases} \tag{2.8}$$

The dynamical system (2.6) can be written as:

(2.9)

$$\begin{cases} \dot{x}_1(t) = X_1(x_1, x_2), \\ \dot{x}_2(t) = X_2(x_1, x_2), \\ \dot{\theta}(t) = -\frac{1}{2}(X_{11} - X_{22})sin(2\theta(t)) + X_{12}sin^2\theta(t) - X_{21}cos^2\theta(t), \\ \dot{\lambda}_1(t) = X_{11}cos^2\theta(t) + X_{22}sin^2\theta(t) - \frac{1}{2}(X_{12} + X_{21})sin(2\theta(t)), \\ \dot{\lambda}_2(t) = X_{11}sin^2\theta(t) + X_{22}cos^2\theta(t) + \frac{1}{2}(X_{12} + X_{21})sin(2\theta(t)), \\ \dot{r}_{12}(t) = \frac{1}{r_{12(t)}}\Big(1 - \frac{1}{2}(X_{11} + X_{22})sin(2\theta(t)) + X_{21}sin^2\theta(t) - X_{12}cos^2\theta(t)\Big)e^{\lambda_1(t)+\lambda_2(t)}. \end{cases}$$

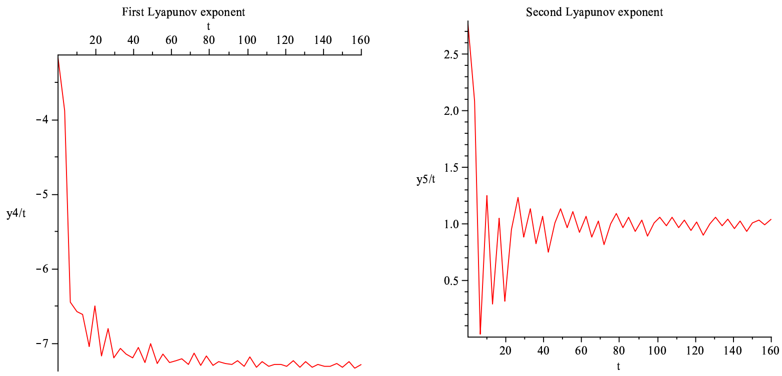
Example 2.1. For the dynamical system

$$\begin{cases} \dot{x}_1(t) = x_2(t), \\ \dot{x}_2(t) = -x_1(t) - dx_2(t) + dx_1(t)^2x_2(t), \end{cases} \tag{2.10}$$

by applying *(2.9)* we get:

$$\begin{cases} \dot{x}_1(t) = x_2(t), \\ \dot{x}_2(t) = -x_1(t) - dx_2(t) + dx_1(t)^2x_2(t), \\ \dot{\lambda}_1(t) = d(x_1(t)^2 - 1)sin^2\theta(t) - dx_1(t)x_2(t)sin(2\theta(t)), \\ \dot{\lambda}_2(t) = d(x_1(t)^2 - 1)cos^2\theta(t) + dx_1(t)x_2(t)sin(2\theta(t)), \\ \dot{\theta}_1(t) = 1 - 2dx_1(t)x_2(t)cos^2\theta(t) + \frac{1}{2}d(x_1(t)^2 - 1)sin(2\theta(t)), \\ \dot{r}_{12}(t) = \frac{d}{r_{12}(t)}(\frac{1}{2}(x_1(t)^2 - 1)sin(2\theta(t)) + 2x_1(t)x_2(t)sin^2\theta(t)). \end{cases} \tag{2.11}$$

With Maple 11 we get the following figures that represent Lyapunov exponents for *(2.10)*. They are presented for the initial conditions $x_1(0) = 0.8$ and $x_2(0) = 1$.

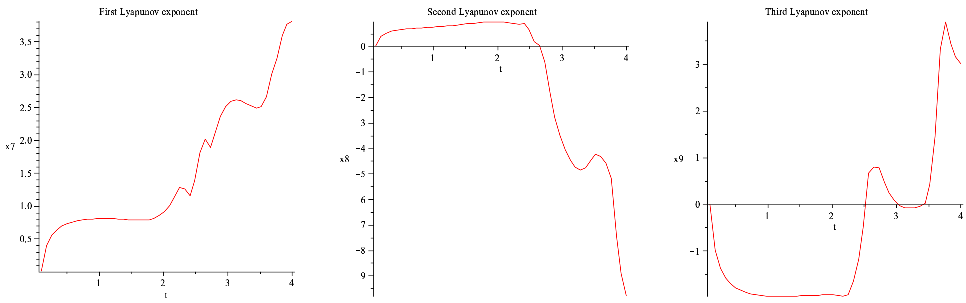


The dynamical system *(2.10)* has a positive Lyapunov exponent, so we may conclude that it is chaotic. The value of this Lyapunov exponent is 1.04.

Example 2.2. For the Rabinovich-Fabrikant dynamical system,

$$\left\{\begin{array}{l} \dot{x_1} = \gamma x_1 - x_2 + x_2(x_3 + x_1^2), \\ \dot{x_2} = x_1 + \gamma x_2 + x_1(3x_3 - x_1^2), \\ \dot{x_3} = -2\alpha x_3 - 2x_1x_2x_3, \end{array}\right. \tag{2.12}$$

relations *(2.6)* can be written in a similar way and we get the differential equations for λ_i, $i = 1, 2, 3$. Lyapunov exponents for the system *(2.12)*, with the initial conditions $x_1(0.1) = 0.1$, $x_2(0.1) = 0.2$ and $x_2(0.1) = 0.1$ are represented in the figures above.



The first Lyapunov exponent is positive, so it is chaotic, and its value is 3.9.

3 Synchronization of dynamical systems in $\mathbb{R}^2$ and $\mathbb{R}^3$ with the right side polynomial functions

The problem if two dynamical systems that perform in the same time could be forced to cover the same trajectory to the strange attractors, was of a great interest among researchers interested in chaotic behavior of the coupled dynamical systems. A beginning of this theory was given by Pecora and Carroll [1], and also by Ott [9]. Synchronization is an interesting theme, because of its practicability in electronics, information security, chemical reactions, biology, robotics, etc. [4].

Let us consider, in $\mathbb{R}^n$, the following dynamical system, with an initial condition:

$$\dot{x}(t) = X(x(t)), \quad x(0) = x_0 \tag{3.1}$$

and the dynamical system in $\mathbb{R}^n$

$$\dot{z}(t) = X(z(t)) + u(t), \quad z(0) = z_0, \tag{3.2}$$

where $u(t) = \varphi(e(t))$, $e(t) = x(t) - z(t)$. Dynamical systems (3.1) and (3.2) are called *synchronized with the control* $u(t)$ if the error vector, $e(t)$, satisfies the following relation:

$$\dot{e}(t) = E(x(t), e(t)), \tag{3.3}$$

where $E \in \mathcal{X}(\mathbb{R}^n)$ is a vector field on $\mathbb{R}^n$ and its solutions are asymptotically stable along the orbit of (3.1). The system (3.1) is called *drive system*, and the system (3.2) is called *response system with the control* $u(t)$.

We analyze synchronization of dynamical systems for which the vector field X has components polynomial functions of the form:

$$X_i(x(t)) = \sum_{j=1}^{n} a_{ij}x_j(t) + \sum_{j,k=1}^{n} b_{ijk}x_j(t)x_k(t) + \sum_{j,k,l=1}^{n} c_{ijkl}x_j(t)x_k(t)x_l(t), \tag{3.4}$$

where $i = 1, ...n$, and a_{ij}, b_{ijk}, c_{ijkl} are fixed real numbers. From (3.1) and (3.2), with $X(t) = (X_1(t), ..., X_n(t))$ given by (3.4), results the error dynamical system:

$$\dot{e}(t) = DX(x(t))e(t) - Q(x(t), e(t)) + U(e(t)) - u(t), \tag{3.5}$$

where

$$Q(x(t), e(t)) = (Q_1(x(t), e(t)), ..., Q_n(x(t), e(t)))^T, \quad U(e(t)) = (U_1(e(t)), ..., U_n(e(t)))^T,$$

(3.6)

$$Q_i(x(t), e(t)) = \sum_{k,l=1}^{n} q_{ikl}e_k(t)e_l(t), \quad U_i(e(t)) = \sum_{j,k,l=1}^{n} c_{ijkl}e_j(t)e_k(t)e_l(t),\ i = 1, 2, ..., n,$$

$$q_{ikl} = b_{ikl} + \sum_{j=1}^{n}(c_{ijkl} + c_{ikjl} + c_{iklj})x_j(t).$$

A way of choosing the control $u(t)$ is given in the following manner:

$$u(t) = Ke(t) + U(e(t)), \tag{3.7}$$

where $K(x(t))$ is a $(n, n)-$matrix, with its elements k_{ij}, $i, j = 1, ..., n$, depending on $x_1(t)$, $x_2(t)$, $x_3(t)$. From (3.5) and (3.7) results a new formulation for error dynamical system:

$$\dot{e}(t) = (DX(x(t)) - K(x(t)))e(t) - P(x(t), e(t)), \tag{3.8}$$

where

$$P(x(t), e(t)) = (e(t)^T P_1(x(t))e(t), ..., e(t)^T P_n(x(t))e(t))^T,$$

$$P_i(x(t)) = (p_{ikl}), \quad p_{ikl} = \frac{1}{2}(q_{ikl} + q_{ilk}). \tag{3.9}$$

Example 3.1. Let us consider a two dimensional drive dynamical system:

$$\begin{cases} \dot{x}_1(t) = x_2(t), \\ \dot{x}_2(t) = -x_1(t) - dx_2(t) + dx_1(t)^2 x_2(t), \end{cases} \tag{3.10}$$

and the response dynamical system:

$$\begin{cases} \dot{z}_1(t) = z_2(t), \\ \dot{z}_2(t) = -z_1(t) - dz_2(t) + dz_1(t)^2 z_2(t) + u(t). \end{cases} \tag{3.11}$$

If the control $u(t)$ is given by $u(t) = Ke(t) + \begin{bmatrix} 0 \\ de_1(t)^2 e_2(t) \end{bmatrix}$, where $K = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix}$ with K a $(2,2)-$matrix with k_{ij}, $i, j = 1, 2$, depending on $x_1(t)$, $x_2(t)$, $x_3(t)$, then the error dynamical system *(3.8)* is:

$$\begin{cases} \dot{e}_1(t) = -k_{11}e_1(t) + (1 - k_{12})e_2(t), \\ \dot{e}_2(t) = -(k_{21} - 2dx_1(t)x_2(t) + 1)e_1(t) - (k_{22} + d - dx_1(t)^2)e_2(t) - de_1(t)^2 x_2(t) \\ \qquad -2dx_1(t)e_1(t)e_2(t). \end{cases} \tag{3.12}$$

Example 3.2. In the Rabinovich-Fabrikant dynamical system case, the error system becomes:

$$\begin{cases} \dot{e}_1(t) = (\gamma + 2x_1(t)x_2(t) - k_{11})e_1(t) + (x_3(t) - 1 + x_1(t)^2 - k_{12})e_2(t) \\ \qquad +(x_2(t) - k_{13})e_3(t) - e_1(t)^2 x_2(t) - 2e_1(t)e_2(t)x_1(t), \\ \dot{e}_2(t) = (3x_3(t) + 1 - 3x_1(t)^2 - k_{21})e_1(t) + (\gamma - k_{22})e_2(t) + (3x_2(t) - k_{33})e_3(t) \\ \qquad +3x_1(t)e_1(t)^2 - 3e_1(t)e_3(t), \\ \dot{e}_3(t) = -(2x_2(t)x_3(t) + k_{13})e_1(t) - (2x_1(t)x_3(t) + k_{32})e_2(t) - (2\alpha + 2x_1(t)x_2(t) \\ \qquad +k_{33})e_3(t) + e_1(t)e_2(t)x_3(t) + e_1(t)e_3(t)x_2(t) + e_2(t)e_2(t)x_1(t). \end{cases} \tag{3.13}$$

3.1 Analysis of error dynamical system (3.8)

Let us consider the dynamical system (3.8) given by:

$$\dot{e}(t) = G(x(t))e(t) - P(x(t), e(t)), \tag{3.14}$$

where $G(x(t))$ is the matrix defined in the form $G(x(t)) = DX(x(t)) - K(x(t))$.

The dynamical system (3.14) has the stationary state $O(0, 0, 0)$.

Proposition 3.1. *If the functions $p_{ijk}(x(t))$ given by* (3.9) *satisfy relations*

$$p_{ijk} = 0,\ i = j, k, \quad p_{ijk} = 0,\ i \neq j \neq k, \quad p_{ikk} + p_{kik} + p_{kki} = 0, \tag{3.15}$$

and the matrix $G(x(t))$ is semi-negative defined along the orbit of the system (3.1), *then the function $L(e(t)) = \frac{1}{2} < e(t), e(t) >$ is a Lyapunov function for the dynamical system* (3.14).

Proof:

For the function $L : \mathbb{R}^n \to \mathbb{R}$, given by $L(e(t)) = \frac{1}{2} < e(t), e(t) >$ results that

$$\dot{L}e(t) = \frac{d}{dt}L(e(t)) =< e(t), \dot{e}(t) >= e(t)^T G(x(t))e(t) - \sum_{i,jk=1}^{n} p_{ikl}e_i(t)e_j(t)e_l(t).$$

From relation (3.15) and from $G(x(t))$ being semi-negative defined results that $L(e(t))$ is a Lyapunov function. □

Proposition 3.2. **(a)** *If the functions* $(k_{ij}(t))$, *of the matrix* K, *satisfy the following inequalities:*

$$\begin{aligned}
\Delta_1 &= k_{11} \geq 0, \\
\Delta_2 &= \begin{vmatrix} k_{11} & k_{12} - 1 \\ 1 - 2dx_1(t)x_2(t) + k_{21} & d - dx_1(t)^2 + k_{22} \end{vmatrix} \leq 0,
\end{aligned}$$

along the orbit of the dynamical system (3.10)*, then the stationary state* $O(0, 0, 0)$ *of the error dynamical system is asymptotically stable.*

(b) *If the elements of the matrix* K, $(k_{ij}(t))$, *satisfy:*

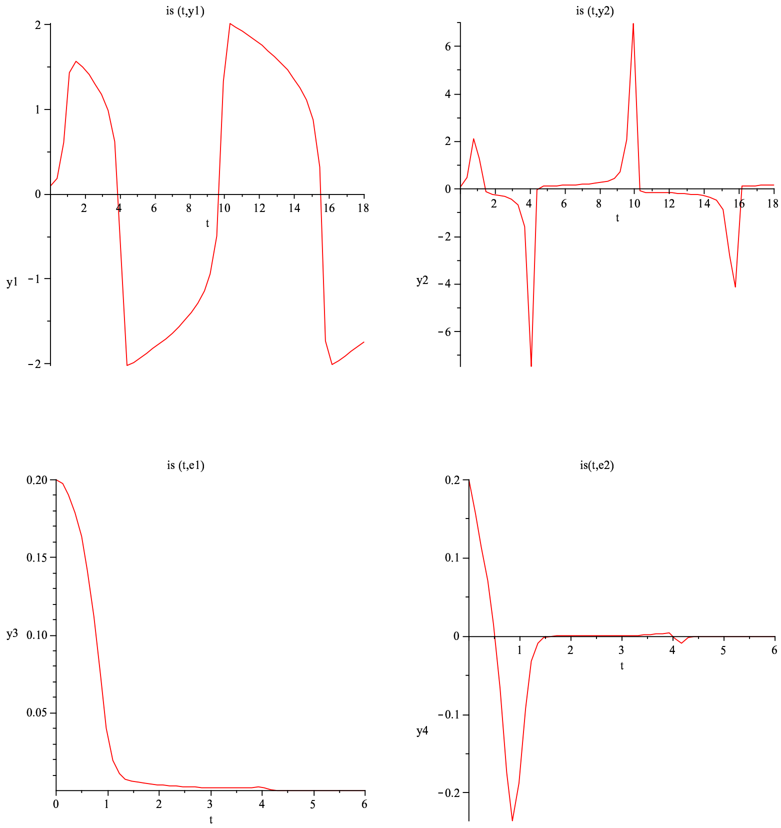
$$\begin{aligned}
\Delta_1 &= \alpha + 2x_1(t)x_2(t) - k_{11} \leq 0, \\
\Delta_2 &= \begin{vmatrix} \alpha + 2x_1(t)x_2(t) - k_{11} & x_3(t) - 1 - k_{12} \\ 3x_3(t) + 1 - 3x_1(t)^2 - k_{21} & \gamma - k_{22} \end{vmatrix} \leq 0, \\
\Delta_3 &= \begin{vmatrix} \alpha + 2x_1(t)x_2(t) - k_{11} & x_3(t) - 1 - k_{12} & x_2(t) - k_{13} \\ 3x_3(t) + 1 - 3x_1(t)^2 - k_{21} & \gamma - k_{22} & 3x_2(t) - k_{33} \\ 2x_2(t)x_3(t) + k_{31} & 2x_1(t)x_3(t) + k_{32} & 2\alpha + 2x_1(t)x_2(t) + k_{33} \end{vmatrix} \geq 0,
\end{aligned}$$

then, along the orbit of Rabinovich-Fabrikant dynamical system, stationary state $O(0, 0, 0)$ *of the error dynamical system* (3.13) *is asymptotically stable.*

□

We use Maple 11 software for visualizing the orbits of initial dynamical system and also of error's dynamical system.

Remark 3.1. If for the system *(3.10)* in $\mathbb{R}^2$, we consider $k_{11} = 1$, $k_{12} = k_{21} = 0$, $k_{22} = 6$ and $d = -5$, then the orbits of *(3.10)* and the orbits of the error system are represented in the following figures:



From the above figures, we can conclude that the error system is asymptotically stable along the orbits of the drive system. These techniques, for determining Lyapunov exponents and also for showing stability of the error system, can be applied to the whole class of Rabinovich systems, see [2], and not only. There are other ways for proving stability, for example using a Lyapunov function, see [11], but in this case, an explicit form is hard to find.

4 Cryptography using synchronized chaotic dynamical systems

An important domain where chaos synchronization can be found is in secure communication [10], practically in cryptography. The major concern in this field is that an encoded message is vulnerable for decryption by nonlinear dynamical systems, when it is hidden by the signal from the low-dimensional chaotic system.

In the following we will make the encryption using coupled chaotic dynamical systems and their synchronization. In this paper we gave two examples: one in $\mathbb{R}^2$ and another one in $\mathbb{R}^3$. We will work with the 2−dimensional one, in the other case the procedure is the same, as we have proved that the Rabinovich-Fabrikant dynamical system is chaotic and we gave necessary and sufficient conditions such that it can be synchronized.

Let us consider the dynamical system given in Example 3.1. Using the procedure presented in Section 2, we can say that the system (3.10) has a chaotic behavior. Next step is synchronization process of the considered coupled dynamical systems. We will use the drive system (3.10) and the response system (3.11). As we have shown, synchronization is achieved if and only if conditions from Proposition 3.2 (a) are fulfilled. Both dynamical systems have chaotic behavior and can be synchronized, so we can use them in communication using cryptographic encoding. The sender uses the dynamical system (3.10) and the receiver uses (3.11). They both choose values for variables $x(t)$ and $z(t)$ as secret keys, after a period of time, for example $t = 5$ in our case, that means after the synchronization between the considered dynamical systems took place [3].

The message that one part wants to send is called *plaintext* and its correspondent by decryption is called *ciphertext.* The plaintext and the ciphertext are represented by numbers, each letter from the alphabet is replaced by a corresponding number. So instead of 26 letters we will use numbers from 0 to 25. The formula for the encryption, ciphertext, is $c := p + k\,mod(26)$, and message decryption is done using the formula $p := c - k\,mod(26)$. A message does not always contain only letters, but also numbers. So, we can generalized the procedure presented above by attributing to letters from A to Z, numbers from 12 to 37 (in this case the formula corresponding for encryption, (respectively for decryption) is given by $c_i := p_i + k_i\,mod(38)$, $p_i := c_i - k_i\,mod(38)$, with k_i are secret keys that mask the message). The most general case is to consider small letters, upper letters and special characters. To each of it, a corresponding number is associated, in ASCII representation. For each letter or number we use a randomly generated key. For a complete message, secret keys are series of numbers $\{k_1, k_2, ..., k_n\}$. Actually, each key k_j hides the piece of message p_j.

We consider the example message "Hello Oscar". This is represented in the following table (the plaintex and the cipertext). Beginning with t_0, in our example equal with 5, both dynamical systems are synchronized, so t takes values greater than $t_0 = 5$, in the synchronized state. The first system sends the encoded message, letter by letter, and for each letter a key is randomly generated. The second system receives the encrypted system and also the key with which is was encrypted. In this example we work with the first components form the considered dynamical systems, $x_1(t)$ and with the keys k_i, $i = \overline{1, 10}$. Decryption is done using the first component of the second system $z_1(t)$, with $t \geq t_0$.

In this paper we have presented chaotic behavior of the dynamical system (3.10), synchronization between two coupled dynamical systems and we have presented a way of encryption and decryption a message.

References

[1] Carroll, T. L., Pecora, L. M., *Synchronization in chaotic systems*, Phys. Rev. Lett. 64 (1990), 821-824.

[2] Chiş, O., Puta, M., *The dynamics of Rabinovich system*, Differential Geometry-Dynamical Systems (DGDS) Electronic Journal 10, (2008) 91-98.

Table 1: Ciphertext and plaintext

Time t	$x_1(t)$	Key k	Plaintext p	Encryption $c = p + k\,mod(26)$	Decryption $p = c - k mod(26)$
5	-1.953...	996	h(8)	32	8(h)
6	-1.806...	427	e(5)	20	5(e)
7	-1.634...	730	l(12)	32	12(l)
8	-1.415...	730	l(12)	32	12(l)
9	-1.054...	944	o(15)	40	15(o)
10	1.774...	944	o(15)	40	15(o)
11	1.926...	736	s(19)	23	19(s)
12	1.775...	224	c(3)	24	3(c)
13	1.597...	395	a(1)	11	1(a)
14	1.362...	454	r(18)	27	18(r)

[3] Ghosh, D., Banerjee, S., Chowdhury, A.R., *Synchronization between variable time delayed systems and cryptography*, arXiv:0802.0768v1 [nlin.CD], 2008.

[4] Goriely, A., *Integrability and Nonintegrability of Dynamical Systems*, Advanced Series in Nonlinear Dynamics 19, World Scientific Publishing Co. Pte. Ltd., 2001.

[5] Habib, S., Rangarajan, G., Ryne, R.D., *Lyapunov exponents without rescaling and reorthogonalization*, Phys. Rev. Lett. 80 (1998), 3747-3750.

[6] Janaki, T.M., Rangarajan, G., *Lyapunov exponents for continuous-time dynamical systems*, Journal of the Indian Institute of Science 78, 267 (1998) 267-274.

[7] Lichtenberg, A.J., Lieberman, M.A., *Regular and Chaotic Dynamics*, Springer Verlag, New York, 1983.

[8] Marsden, J.E., Raţiu, T.S., *Introduction to Mechanics and Symmetry*, Springer-Verlag, Berlin, Heidelberg, New-York, Inc., 1994.

[9] Ott, E., Grebogi, C., Yorke,J.A., *Controlling chaos*, Phys. Rev. Lett. 64 (1990), 1196-1199.

[10] Sladewski, S.E., *Deterministic Chaos for Authentification and Secure Communication*, A Thesis Presented to The Faculty of Graduate Studies of The University of Guelph, 2001.

[11] Ţigan, G., *Stabilizing the chaotic dynamics of the L̈u system*, Applied Science (APPS) Electronic Journal 9 (2007), 174-180.

[12] Udrişte, C., *Geometric Dynamics*, Kluwer Academic Publishers, 2000.

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