Numerical study for two point boundary problems in nonlinear fluid mechanics

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Abstract. We study a nonlinear second order differential equation for two point boundary problems. We present a numerical algorithm in order to approximate the solution. We generalize the method for a system of two second order differential equations. We discuss upon accuracy and precision of the algorithm. In case of heat transfer problem for a third grade fluid flow, we obtain the approximate solution, and compare it with those obtained by V. Ţigoiu [7]. We conclude that the numerical technique presented can be used for new two point boundary problems, where the exact solution is unknown.

M.S.C. 2000: 76A05, 65G50.
Key words: nonlinear fluid mechanics, two point boundary value problem, numerical methods, error analysis.

1 Introduction

A lot of problems in nonlinear fluid mechanics are modelled by second or higher order differential equations. For example, for heat transfer problem of a third grade fluid flow between two heated parallel plates (see Figure 1) were obtained the following dimensionless two point boundary value problems:

$$
\begin{aligned}
&\psi''(\eta) + af(\eta)\psi'(\eta) - 2af'(\eta)\psi(\eta) + af''^2(\eta) = 0,\\
&\psi(0) = \psi(1) = 0.
\end{aligned}
\tag{1.1}
$$

and

$$
\begin{aligned}
&\varphi''(\eta) + af(\eta)\varphi'(\eta) + 2\psi(\eta) + 4af'^2(\eta) = 0,\\
&\varphi(0) = 0; \varphi(1) = b = (\theta_1 - \theta_0)/M; M = ct.,
\end{aligned}
\tag{1.2}
$$

BSG Proceedings 16. The Int. Conf. of Diff. Geom. and Dynamical Systems (DGDS-2008) and The V-th Int. Colloq. of Mathematics in Engineering and Numerical Physics (MENP-5) - math. sections, August 29 - September 2, 2008, Mangalia, Romania, pp. 57-64.

where $\psi(\eta)$ and $\varphi(\eta)$, with $\eta = y/d$, define the temperature of the fluid (see V. Ţigoiu [7]) by

$$\theta(\bar{x}, \eta) - \theta_0 = M(\varphi(\eta) + \bar{x}^2\psi(\eta)); \quad \theta(\bar{x}, 0) = \theta_0, \theta(\bar{x}, 1) = \theta_1.$$

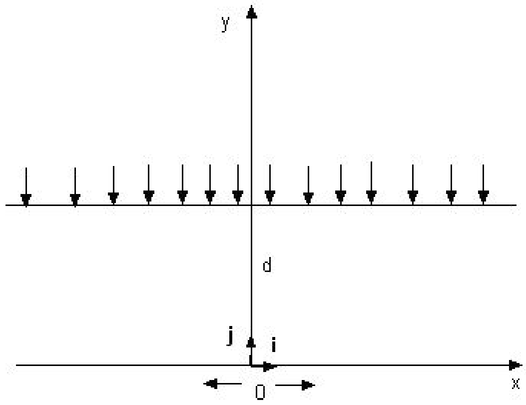

Figure 1: Geometric representation of the flow domain.

The function f, related to the velocity field: $u = cxf'(\eta),\ v = -cdf(\eta)$, is determined from the equations of motion, solving a fourth nonlinear differential equation with boundary conditions on the plates (see E.C. Cipu, C. Pricina [1]):

$$\begin{aligned} &f^{IV} = Re(f' f'' - f f''') + \bar{\alpha}_1 Re(f f^V - f' f^{IV}), \\ &f(0) = f'(1) = 0, f'(0) = 1, f(1) = v_0/cd = K. \end{aligned} \tag{1.3}$$

2 Numerical study of the bilocal problem for one ODE

2.1 General problem. Notations

In order to solve the particular problems (1.1) and (1.2) we shall study a more general one. Find the function $y \in C^2\,[0, 1]$, solution of the problem:

$$y''(x) + F(x)y' + G(x)y = k(x); \quad y(0) = d_0,\ y(1) = d_f, \tag{2.1}$$

with d_0 and d_f known constants. There exist one solution of (2.1). We approximate the solution using the finite difference method. The corresponding problem of (2.1) in differences is:

$$\begin{aligned} &-y_{i+1}(F_i h/2 + 1) + (G_i h^2 - 2)y_i + (F_i h/2 - 1)y_{i-1} + \\ &\qquad\qquad\qquad\qquad + (C_1 y_i + F_i h C_2 y_i) = -k_i h^2,\ i \in \overline{2, n}, \\ &y_1 = d_0, y_N = d_f. \end{aligned} \tag{2.2}$$

We shall consider common notations, as follows:

- N=n+1 is the number of the points $x_i = (i-1)h$ of the division of interval, $I = [0,1]$, $h = 1/n$, $y_i = y(x_i), F_i = F(x_i), G_i = G(x_i), k_i = k(x_i)$
- Difference operators:

  D = differentiation operator; $E = e^{hD}$, $\Delta = 1 + E$, $\delta = E^{1/2} - E^{-1/2} = 2sh(hD/2)$, $\mu = E^{1/2} + E^{-1/2} = 2ch(hD/2)$;
- $C_1 = -\frac{1}{12}\delta^4 + \frac{1}{90}\delta^6 - \ldots$; $C_2 = -\frac{1}{6}\mu\delta^3 + \frac{1}{30}\delta^5 - \ldots$.

Remark 2.1.
For $i \in \overline{2,n}$, the following statements hold true:

1. $y(x + i \cdot h) = E^i y(x)$;
2. $\delta y(x) = y(x + h/2) - y(x - h/2)$; and $\delta y(x_i + h/2) = y_{i+1} - y_{i-1}$
3. $\mu y(x) = y(x + h/2) + y(x - h/2)$ and $\mu y(x_i + h/2) = y_{i+1} + y_{i-1}$,
4. $C_1 y_i = -\frac{1}{12}\delta^4 y_i + \frac{1}{90}\delta^6 y_i - \ldots$; $C_2 y_i = -\frac{1}{6}\mu\delta^3 y_i + \frac{1}{30}\delta^5 y_i - \ldots$

2.2 Approximation of the solution

We shall find approximations for the problem in differences (2.2) with a given error. We shall make a **first approximation of the solution**, that will be done by neglecting the term $C_1 y_i + F_i h C_2 y_i$ in (2.1), called *correction*. The system (2.2) becomes:

$$AY = B, \tag{2.3}$$

with

$$Y = (y_2, y_3, \ldots, y_n)^t,$$

$$B = -\left(k_2 h^2 + d_0\left(F_i h/2 - 1\right), k_3 h^2, \ldots, k_{n-1} h^2, k_n h^2 - d_f (F_i h/2 + 1)\right),$$

A is tridiagonal matrix with nonzero components: $A_{i,i} = 2 - h^2 G_i,\ i \in \overline{1, n-1}$; $A_{i-1,i} = \dfrac{h}{2} F_i - 1,\ i \in \overline{2, n-1}$; $A_{i,i+1} = -\dfrac{h}{2} F_i - 1,\ i \in \overline{1, n-2}$.

Remark 2.2.
For n large enough (meaning $h \to 0$), we have $A \approx A_{n-1}$, for

$$A_n = \begin{pmatrix} 2 & -1 & 0 & 0 & \ldots & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & \ldots & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & \ldots & 0 & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & 0 & \ldots & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & \ldots & 0 & -1 & 2 \end{pmatrix} \in M_{n,n}.$$

with the determinant $a_n = det(A_{n+1}) = -(n+2)$, $n = 1, 2, \ldots$.

It results that for n large enough, the matrix A^{-1} exists and the solution of (2.3) could be written as: $Y = A^{-1}B$.

Approximation of the solution using correction in differences

The approximation will be made starting with $n = 4$, meaning $h = 1/4$, with unknowns $(y_2^{(1)}, y_3^{(1)}, y_4^{(1)})$, in points $(1/4, 1/2, 3/4)$. For this, the matrix A of the system (2.3) is:

$$A = \begin{pmatrix} 2 - G_2/16 & -F_2/8 - 1 & 0 \\ F_3/8 - 1 & 2 - G_3/16 & -F_3/8 - 1 \\ 0 & F_4/8 - 1 & 2 - G_4/16 \end{pmatrix}.$$

Hypothesis:

For functions F, G and k we consider the following hypotheses:
$F \in C(I)$ growing on $I = [0, 1]$,with $F(0) = 0$, $G \in C(I)$ is a non positive function.

Remark 2.3. Since $det(A) \neq 0$ we have $Y^{(1)} = A^{-1}B$.

Proof. a. If $G \equiv 0$ then $det(A) = 4 + F_3\left(F_4 + F_2\right)/32 + \left(F_4 - F_2\right)/8 > 0$.
b. If $F \equiv 0$ then $det(A) = 4 - G_2G_3G_4/4096+$

$$+\left(G_2G_4 + G_2G_3 + G_3G_4\right)/128 - 3\left(G_2 - 4G_3 + G_4\right)/16 \;> 0$$

c. In the complementary case, we get

$$det(A) = 4 - G_2G_3G_4/4096 + \left(G_2G_4 + G_2G_3 + G_3G_4\right)/128-$$

$$-3\left(G_2 - 4G_3 + G_4\right)/16 - F_2F_3G_4/1024 - F_3F_4G_2/1024+$$

$$+(F_2G_4 - F_3G_4)/128 + (F_2F_3 + F_3F_4)/32 + (F_4 - F_2)/4. \Rightarrow det(A) > 0.$$

□

In order to obtain the error of neglected term $Cy_i = C_1y_i + F_ihC_2y_i$ we shall construct the Table 1. In this table we also use the values of $y^* = y^{(1)}$ at external points $(-1/2, -1/4, 5/4, 3/2)$ obtained from equation (2.2), using $i = 1, 0;\ 5,$ and 6

$$y_0^{(1)} = (k_1h^2 + (G_1h^2 - 2)y_1^{(1)} - (1 + F_1h/2)y_2^{(1)})/(1 - F_1h/2),$$

$$y_{-1}^{(1)} = (k_0h^2 - (1 + F_0h/2)y_1^{(1)} + (G_0h^2 - 2)y_0^{(1)})/(1 - F_0h/2),$$

$$y_6^{(1)} = (k_5h^2 + (G_5h^2 - 2)y_5^{(1)} + (F_5h/2 - 1)y_4^{(1)})/(1 + F_5h/2),$$

$$y_7^{(1)} = (k_6h^2 + (G_6h^2 - 2)y_6^{(1)} + (F_6h/2 - 1)y_5^{(1)})/(1 + F_6h/2),$$

and the following notations:
$\delta_{11} = y_0^{(1)} - y_{-1}^{(1)}$, $\delta_{21} = d_0 - y_{-1}^{(1)}$, $\delta_{31} = y_2^{(1)} - d_0$, $\delta_{41} = y_3^{(1)} - y_2^{(1)}$,
$\delta_{51} = y_4^{(1)} - y_3^{(1)}$, $\delta_{61} = y_5^{(1)} - y_4^{(1)}$, $\delta_{71} = y_6^{(1)} - y_5^{(1)}$, $\delta_{81} = y_7^{(1)} - y_6^{(1)}$,
$\delta_{i,j} = \delta_{i+1,j-1} - \delta_{i,j-1}$, $i \in \overline{1,7}$; $j \in \overline{2,6}$; $\mu\delta_{i,j} = \delta_{i+1,j} - \delta_{i,j}$.

The procedure is as follows:

- step1: calcul of $C^{i(1)} = -\delta_{i+1,4}/12 + \delta_{i,6}/90 - (\mu\delta_{i+1,3} - \mu\delta_{i,5}/5)F_i/24$, $i = 1, 2, 3,$

Table 1.

x	y^*	δy^*	$\delta^2 y^*$	$\delta^3 y^*$	$\delta^4 y^*$	$\delta^5 y^*$	$\delta^6 y^*$	$\mu\delta^3 y^*$	$\mu\delta^5 y^*$	Cy^*
-1/2	$y_{-1}^{(1)}$									
		δ_{11}								
-1/4	$y_0^{(1)}$		δ_{12}							
		δ_{21}		δ_{13}						
0	d_0		δ_{22}		δ_{14}			$\mu\delta_{13}$		
		δ_{31}		δ_{23}		δ_{15}				
1/4	$y_2^{(1)}$		δ_{32}		δ_{24}		δ_{16}	$\mu\delta_{23}$	$\mu\delta_{15}$	C^{1*}
		δ_{41}		δ_{33}		δ_{25}				
1/2	$y_3^{(1)}$		δ_{42}		δ_{34}		δ_{26}	$\mu\delta_{33}$	$\mu\delta_{25}$	C^{2*}
		δ_{51}		δ_{43}		δ_{35}				
3/4	$y_4^{(1)}$		δ_{52}		δ_{44}		δ_{36}	$\mu\delta_{43}$	$\mu\delta_{35}$	C^{3*}
		δ_{61}		δ_{53}		δ_{45}				
1	d_f		δ_{62}		δ_{54}			$\mu\delta_{53}$		
		δ_{71}		δ_{63}						
5/4	$y_6^{(1)}$		δ_{72}							
		δ_{81}								
3/2	$y_7^{(1)}$									

- step 2: compare $C^{i(1)}$ with the error imposed ϵ,
- step 3: if $C^{i(1)} \leq \epsilon$ the algorithm stops, else: make a correction to the first approximation, a better approximation of the solution will be $Y^{(2)} = Y^{(1)} + Z$. The problem reduces to finding Z that has zero null values on the boundary and satisfies:

$$\begin{aligned} &Z = A^{-1}B^{(1)}, \\ (2.4) \qquad &B^{(1)} = -((C_1 + F_2hC_2)y_2^{(1)},\, (C_1 + F_3hC_3)y_3^{(1)},\, (C_1 + F_4hC_4)y_4^{(1)})^t, \\ &Z = (z_2, z_3, z_4)^t. \end{aligned}$$

- step 4: reconstruct the Table 1 for increment Z after finding Z, at the external points. Go to step 2.

The solution will be $Y^{(n)}$.

Remark 2. 4. For $h = 1/4k,\ k \geq 2$, we can calculte the solution $Y_k^{(m_k)}$ after m_k iterations, with error ϵ. The algorithm stops when $\left\| Y_{k+1}^{(m_{k+1})} - Y_k^{(m_k)} \right\| \leq err$, and $\left\| Y_k^{(m_k)}((i-1)/4) - Y^{(n)}((i-1)/4) \right\| \leq err$, for a given error.

Remark 2.5. Other processes for correction

1. Use the differential equation for approximation $Y^{(1)}$:

$$\begin{aligned} &{Y^{(1)}}''(x) + F(x){Y^{(1)}}' + G(x)Y^{(1)} = k(x) + e(x);\ e(x) \text{ residual error}, \\ &Y^{(1)}(0) = d_0,\, Y^{(1)}(1) = d_f. \end{aligned}$$

and for the correction Z: $Z''(x) + F(x)Z' + G(x)Z = -e(x);\ Z(0) = Z(1) = 0.$

2. Iterative processes:

   (a) evaluate a sequence of functions $Y_m, m \in \overline{0,n}$ that satisfies:

   $Y_{m+1}{}''(x) + G(x)Y_{m+1} = k(x) - F(x)Y_m{}';\ x \in [0,1]\,,\ Y_m(0) = d_0,\ Y_m(1) = d_f,$ until $\|Y_{n+1} - Y_n\| \leq err$; process generraly used when $G(x) \equiv 0$.

   In this case the problem in differences obtained is: $AY_m = V$, with

   $$\begin{aligned} &Y_m = (y_{m,2}, y_{m,3}, \ldots, y_{m,n})^t\,; V = (v_{m,2}, v_{m,3}, \ldots, v_{m,n})^t\,;\\ &v_{m,i} = F_i(y_{m-1,i+1} - y_{m-1,i-1})h/2 - k_i h^2/2;\\ &A_{i,i} = 2 - h^2 G_i,\ i \in \overline{1,n};\, A_{i-1,i} = -1,\ i \in \overline{2,n};\ A_{i,i+1} = -1,\ i \in \overline{1,n-1}. \end{aligned}$$

   LEMA: A is a symmetric and positive defined tensor.

   Consequently, there exists a solution, $Y_m = A^{-1}V$ (see I. Roşca [6]).

   (b) for $G(x) \neq 0, \forall x \in [0,1]$, evaluate a sequence of functions $Y_m, m \in \overline{0,n}$ which satisfy:

   $Y_{m+1} = k(x)/G(x) - Y_m{}''(x) - F(x)/G(x)Y_m{}';\ x \in [0,1],$

   $Y_m(0) = d_0$, $Y_m(1) = d_f$, until $\|Y_{n+1} - Y_n\| \leq err$.

The choice of the iterative process must be made taking into account the dominant term of the differential equation.

2.3 Extension for two differential equations of order two

We consider two equations of type (2.1) as:

$$\begin{gathered} y''(x) + F_1(x)y' + G_1(x)y + H_1(x)z' + R_1(x)y = k_1(x);\\ z''(x) + F_2(x)z' + G_1(x)z + H_2(x)y' + R_2(x)z = k_2(x);\\ y(0) = d_0,\ y(1) = d_f; \quad z(0) = g_0,\ y(1) = g_f. \end{gathered} \tag{2.5}$$

The corresponding problem in differences is:

$$\begin{aligned} &y_{i+1}(F_{1i}h/2 + 1) + (G_{1i}h^2 - 2)y_i + (1 - F_{1i}h/2)y_{i-1} +\\ &\qquad + H_{1i}z_{i+11}h/2 + R_{1i}z_i h^2 - hH_{1i}z_{i-1}/2 + (K_i^1) = k_{1i}h^2,\ i \in \overline{2,n},\\ &z_{i+1}(F_{2i}h/2 + 1) + (G_{2i}h^2 - 2)z_i + (1 - F_{2i}h/2)z_{i-1} +\\ &\qquad + H_{2i}z_{i+1}h/2 + R_{2i}y_i h^2 - hH_{2i}y_{i-1}/2 + (K_i^2) = k_{2i}h^2,\ i \in \overline{2,n},\\ &y_1 = d_0, y_N = d_f, \quad z_1 = g_0, z_N = g_f, . \end{aligned} \tag{2.6}$$

By neglecting (K_i^1) and (K_i^2) we find: $AU = B$, $A = \begin{pmatrix} A^1 & A^2 \\ A^3 & A^4 \end{pmatrix}$, $U = (y_2, y_3, \ldots, y_n, z_2, z_3, \ldots, z_n)^t$, $A^k \in M_{n-1,n-1}$ tridiagonal matrix of type A defined by (2.3). Therefore problem (2.6) admits a solution which approximates the solution of (2.5).

2.4 Accuracy and precision

A better accuracy is obtained by using the differential equations in differences with smaller coefficients or the coefficient of the last difference as small as possible (see L. Fox, [3]). A better precision is given by smaller values for the errors imposed: ϵ, err.

3 Conclusions

3.1 Application to some problems from nonlinear fluid mechanics

In Figure 2 is presented the solution of (1.1) without correction, for different values of n. Then the iterative process presented in Remark 2.4 is applied. For a better accuracy we have considered the following errors: $\epsilon = 10^{-15}$ and $err = 10^{-4}$. The results obtained for solutions of the problems (1.1) and (1.2) are presented in Figure 3. The solution of the temperature of the fluid $\theta(\bar{x}, \eta)$ is expressed in Figure 4. We observe small differences on the shape of the solution in comparision with those obtained by V. Tigoiu [7], using asymptotic development techniques and functional analysis.

3.2 Final remarks

The study of non-linear fluid mechanics problems leads, often, to solving Cauchy problems for differential equations (see M.O. Daramola et al. [2]) or partial differential equations (see for instant A.M. Mirza [5], and R.T. Matoog et al. [4]). We have considered a numerical study of a general two point boundary problem that appears in modelling the behaviour of non-Newtonian fluids. We were interested also on accuracy and precision of the algorithm, that are expressed by two types of established errors.

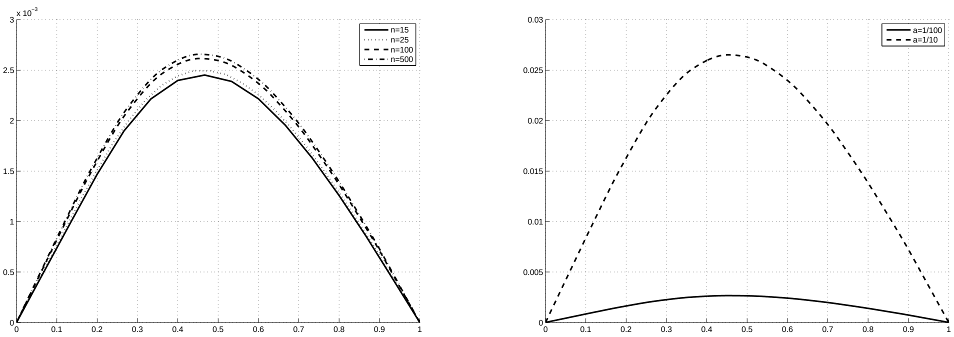

Figure 2. Function ψ for $b = -0.4$, $a = 0.01$, solution without correction;ψ for $a = 0.1; 0.01$

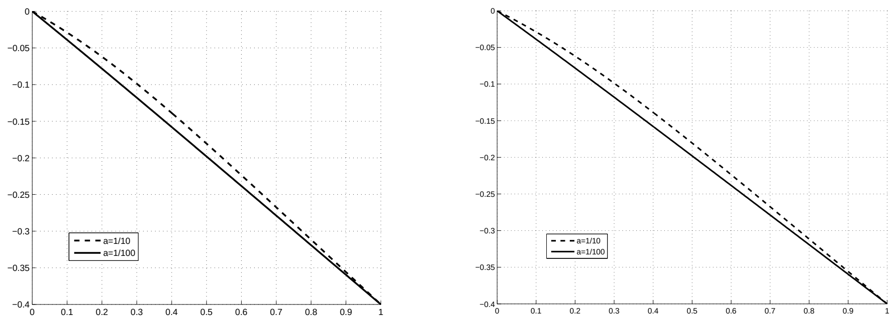

Figure 3. a) Function ϕ; b) Function $\theta(\bar{x}, \eta)$, $\bar{x} = 0.7$; for $b = -0.4$, and $a = 0.1; 0.01$

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