Solving high order nonholonomic systems using Gibbs-Appell method

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Abstract. In this paper we present a new formulation, based on Gibbs-Appell method, for solving a large group of high order nonholonomic constrained systems. High order nonholonomic constraints (HONC) are a portion of constraints that happen in variety of dynamics applications such as robotics and control. The majority of the methods, which have been introduced in nonholonomic mechanics, deal with first order nonholonomic systems, and use a group of new variables called "Lagrange multipliers". Using Lagrange multipliers results in higher amount of needed calculation. The presented method can solve a vast group of high order nonholonomic systems without using Lagrange multipliers. The introduced method makes dealing with high order nonholonomic systems very easy.

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1 Introduction

Nonholonomic systems were discovered by Euler when he was studying the rolling of the rigid bodies, and the term "nonholonomic system" was introduced by Hertz in 1894 [3]. He also expressed the exact distinction between holonomic and nonholonomic systems [5]. After that, at the end of 19th and the beginning of the 20th century it was discovered that there are also some other dynamic systems whose equations of motion cannot be found by Lagrange method [14]; even at that time a new tendency of leaving Lagrange method began [9]. But some scientists started to find a way for solving nonholonomic systems by using classical mechanics methods. They expanded the capability of the Lagrange and Hamilton methods to meet nonholonomic systems [3], [13], [4] and [16].

Most of the methods, derived from classical methods of Analytical dynamics, deal with linear first order nonholonomic systems, and use a group of new variables called

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Lagrange multipliers. The number of these new variables is equal to the number of nonholonomic constraints, which results in higher amount of needed calculation, considering the fact that the equations of motion are usually coupled second order differential equations. Especially in problems with high degrees of freedom it will be very complicated to eliminate Lagrange multipliers. Methods introduced on Gibbs-Appell formulation can analyze linear second order constraints [9], while there are many HONCs in robotic and control systems. That is the reason why studying high order nonholonomic systems is important.

The aim of this paper is to propose a method for solving high order nonholonomic systems. We introduce a way which is based on Gibbs-Appell formulation to solve systems with HONC. The presented method can analyze a vast branch of HONCs and it has no limit in the order of constraints. By using this method there no longer will be a need for Lagrange multipliers and this will result in lower amount of computation. The proposed formulation makes dealing with high order nonholonomic systems much easier.

The paper is organized in 5 sections. In section 2, we describe constraints, their sources and the definition of material and non-material constraints. We also illustrate the classifications of constraints and present the branch of HONCs which we deal with in this paper. In section 3, we derive the new formulation based on developing Gibbs-Appell equations. We also present the principle that is used in derivation of the new method too. In section 4, we express the application of the introduced method in dynamics and control. We show the ease and advantages of our method by an example.

2 Holonomic and nonholonomic constraints

Based on the sources of constraints, dynamic constraints can be classified as either material or non-material constraints. The conditions imposed to the system by nature or environment, are called material constraints. An example of material constrains is the motion of a particle on a surface, they are also called passive constraints [1]. Sometimes the designer puts some restrictions on the system, for example on velocities or accelerations or any other feature of the system. These restrictions and conditions are called non-material constraints, another term for non-material constraints is servo constraints [1]. These constraints can have a variety of forms and usually include high derivatives of the generalized coordinates [10].

Suppose there is a system with N degrees of freedom in which we must select a set of M generalized coordinates $M > N$ to represent it. In this system there are M-N constraint equations between generalized coordinates. If the constraints may be written in the following form

$$f_i(q_1, q_2, ..., q_M, t) = 0, \qquad i = 1, 2, ..., k \quad (k = M - N), \tag{2.1}$$

they are called holonomic constraints. These constraints are also referred to as configuration constraints [6]. Sometimes it is not achievable to write the constraints as holonomic constraints since the time derivatives of generalized coordinates appear in the constraints. In this case the constraints are nonholonomic. If the nonholonomic constraint includes only first time derivatives of generalized coordinates it is called first order nonholonomic constraint or velocity constraint [6]. As follows

$$g(q_1, q_2, ..., q_M, \dot{q}_1, \dot{q}_2, ..., \dot{q}_M, t) = 0. \tag{2.2}$$

Sometimes the velocity constraint is linear and we can write it in the following form [6]

$$\sum_{j=1}^{M} a_j(q_1, q_2, ..., q_M, t)\dot{q}_j + b(q_1, q_2, ..., q_M, t) = 0. \tag{2.3}$$

A very popular example of linear first order nonholonomic constraint is the rolling of rigid body. Holonomic and first order nonholonomic constraints are usually material constraints.

There are many instances in applications of dynamics in which the nonholonomic constraints involve high order derivatives of generalized coordinates. The general form of these constraints is as follows

$$h(q_1, q_2, ..., q_M, \dot{q}_1, \dot{q}_2, ..., \dot{q}_M, q_1^{(s)}, q_2^{(s)}, ..., q_M^{(s)}, t) = 0. \tag{2.4}$$

where the power *(s)* means the s^{th} time derivative of the quantity. These constraints are called high order nonholonomic constraints (HONC). HONCs usually appear in non-material constraints but there are some cases in material constraints which contain high order derivatives of generalized coordinates. HONCs arise in many engineering cases such as underactuated robots which are examples of second order nonholonomic systems. For more examples see [1], [8], [17] and [12].

In this paper the authors deal with a group of HONCs which may be transformed into the following form

$$q_j^{(s_j)} = f_j(q_1, q_2, ..., q_M, \dot{q}_1, \dot{q}_2, ..., \dot{q}_M, q_{k+1}^{(s_j)}, ..., q_M^{(s_j)}, t), \quad j = 1, 2, ..., k, \tag{2.5}$$

where k is the number of HONCs, and s_j is the order of the constraint. It is not necessary that the order of constraints be equal to each other. It's possible to convert nonlinear constraints to the form of (2.5) by differentiating the constraint with respect to time.

In the following section we deal with Gibbs-Appell formulation and derive a method for solving systems with constraints in form of (2.5).

3 Deriving the new method from Gibbs-Appell formulation

The basic structure of the Gibbs-Appell equations was introduced by Gibbs in 1879 and was developed further by Appell in 1900 [11]. This method defines a new function called Gibbs-Appell function or the energy of acceleration [6] and uses quasicoordinates to find the equations of motion. The prefix quasi means that we need only the time derivatives of the quantities have physical meaning, and it is not necessary that a quasicoordinate describe a position in the system [6]. For information about quasicoordinates see [6] and [2].

Using Gibbs-Appell method starts with defining the generalized coordinates and quasicoordinates. Then we use kinematic of the system to find relations between generalized velocities and quasivelocities ($\dot{\gamma}_j$). The relations are as follows

$$\dot{q}_i = \sum_{j=1}^{M} v_{ij}(q_k, t)\,\dot{\gamma}_j + h_i(q_k, t), \qquad i = 1, 2, ..., M. \tag{3.1}$$

In this equations M is the number of generalized coordinates and quasicoordinates.

For a system of P particles, in three dimensional space with Cartesian coordinates of $x_l\,(l = 1, 2, ..., 3P)$ we use kinematic to write the relation between Cartesian velocities and quasivelocities in the following form

$$\dot{x}_l = \sum_{j=1}^{M} c_{lj}(q_k, t)\,\dot{\gamma}_j + d_l(q_k, t), \qquad l = 1, 2, ..., 3P. \tag{3.2}$$

And by differentiating the above equation, the acceleration components are

$$\ddot{x}_l = \sum_{j=1}^{M} \left[c_{lj}\ddot{\gamma}_j + \left(\sum_{k=1}^{M} \frac{\partial c_{lj}}{\partial q_k}\dot{q}_k + \frac{\partial c_{lj}}{\partial t} \right) \dot{\gamma}_j \right] + \sum_{k=1}^{M} \frac{\partial d_l}{\partial q_k}\,\dot{q}_k + \frac{\partial d_l}{\partial d}. \tag{3.3}$$

Now, we use D'Alembert's principle of virtual work to derive Gibbs-Appell equations. First we define virtual work as follows

$$\delta w = \sum_{l=1}^{3P} f_l \delta x_l = \sum_{l=1}^{3P} m_l \ddot{x}_l \delta x_l = \sum_{l=1}^{3P} \sum_{j=1}^{M} m_l \ddot{x}_l c_{lj}\,\delta\gamma_j. \tag{3.4}$$

In deriving the above relation we used (3.2) to find δx_l. The definition of the generalized forces corresponding to the system is

$$\delta w = \sum_{j=1}^{M} \Gamma_j\,\delta\gamma_j, \tag{3.5}$$

Therefore we have

$$\sum_{j=1}^{M} \Gamma_j\,\delta\gamma_j = \sum_{j=1}^{M} \sum_{l=1}^{3P} m_l \ddot{x}_l c_{lj}\,\delta\gamma_j. \tag{3.6}$$

Now, we need the following calculations

$$c_{lj} = \frac{\partial \ddot{x}_l}{\partial \ddot{\gamma}_j} \Rightarrow \ddot{x}_l c_{lj} = \ddot{x}_l \frac{\partial \ddot{x}_l}{\partial \ddot{\gamma}_j} = \frac{\partial}{\partial \ddot{\gamma}_j}\left(\frac{1}{2}\ddot{x}_l^2\right), \tag{3.7}$$

and we define Gibbs-Appell function for system of P particles as follows

$$S = \frac{1}{2}\sum_{l=1}^{3P} m_l \ddot{x}_l^2. \tag{3.8}$$

Now, (3.6) can be written in the following form

$$\sum_{j=1}^{M}\left(\frac{\partial S}{\partial \ddot{\gamma}_j} - \Gamma_j\right)\delta\gamma_j = 0. \tag{3.9}$$

If $\delta\gamma_j$ parameters be independent we can write (3.9) as Gibbs-Appell equations. The $\delta\gamma_j$ parameters can be independent in following cases:

- The number of degrees of freedom N is equal to M
- All constraints are holonomic or
- The system is first order nonholonomic and it is possible to convert quasicoordinates into unconstrained quasicoordinates, for information about unconstrained quasicoordinates see [6].

$$\frac{\partial S}{\partial \ddot{\gamma}_j} - \Gamma_j = 0, \qquad j = 1, 2, ..., M. \tag{3.10}$$

This is Gibbs-Appell equations and some scientists believe that "they probably are the simplest and most comprehensive form of motion equation ever discovered" [15].

Gibbs-Appell equations for rigid bodies are the same as (3.10) but the following equation defines the Gibbs-Appell function.

$$S = \frac{1}{2}m\vec{a}_A \cdot \vec{a}_A + \frac{1}{2}\vec{\alpha} \cdot \frac{\delta \vec{H}_A}{\delta t} + \vec{\alpha} \cdot (\vec{\omega} \times \vec{H}_A), \tag{3.11}$$

where A is the mass center of the body, or a fixed point in a purely rotating body [6]. As follows we derive a method for solving high order nonholonomic systems.

For $\vec{r} = \vec{r}(q_1, q_2, ..., q_M, t)$, the virtual displacement $\delta\vec{r}$ is defined by following formulation [7]

$$\delta\vec{r} = \frac{\partial \vec{r}}{\partial q_1}\delta q_1 + \frac{\partial \vec{r}}{\partial q_2}\delta q_2 + ... + \frac{\partial \vec{r}}{\partial q_M}\delta q_M = \sum_{k=1}^{M}\frac{\partial \vec{r}}{\partial q_k}\delta q_k. \tag{3.12}$$

And the first time derivative of $\vec{r}$ is

$$\dot{\vec{r}} = \frac{\partial \vec{r}}{\partial q_1}\dot{q}_1 + \frac{\partial \vec{r}}{\partial q_2}\dot{q}_2 + ... + \frac{\partial \vec{r}}{\partial q_M}\dot{q}_M + \frac{\partial \vec{r}}{\partial t} = \sum_{k=1}^{M} \frac{\partial \vec{r}}{\partial q_k}\dot{q}_k + \frac{\partial \vec{r}}{\partial t}. \tag{3.13}$$

It is obvious that $\frac{\partial \dot{\vec{r}}}{\partial \dot{q}_k} = \frac{\partial \vec{r}}{\partial q_k}$, thus we can write (3.12) as

$$\delta \vec{r} = \frac{\partial \dot{\vec{r}}}{\partial \dot{q}_1}\delta q_1 + \frac{\partial \dot{\vec{r}}}{\partial \dot{q}_2}\delta q_2 + ... + \frac{\partial \dot{\vec{r}}}{\partial \dot{q}_M}\delta q_M = \sum_{k=1}^{M} \frac{\partial \dot{\vec{r}}}{\partial \dot{q}_k}\delta q_k. \tag{3.14}$$

By continuing differentiating $\vec{r}$ we find that

$$\frac{\partial \vec{r}}{\partial q_k} = \frac{\partial \dot{\vec{r}}}{\partial \dot{q}_k} = ... = \frac{\partial \vec{r}^{\,(n)}}{\partial q_k^{(n)}}, \tag{3.15}$$

Therefore we can write the virtual displacement $\delta \vec{r}$ as follows

$$\delta \vec{r} = \sum_{k=1}^{M} \frac{\partial \vec{r}^{\,(n)}}{\partial q_k^{(n)}}\delta q_k, \qquad n = 1, 2, 3, ... \tag{3.16}$$

We use this equation to find a new formulation based on Gibbs-Appell method for analyzing high order nonholonomic systems in which the HONCs can be written in the form of (2.5).

In sequences of derivation of the Gibbs-Appell equations we achieved (3.9) and derived Gibbs-Appell equations for conditions in which $\delta\gamma_j$ parameters were independent. But in systems with HONCs we cannot find a way to make quasivelocities unconstrained, and it is not possible to write the Gibbs-Appell equations for high order nonholonomic systems as (3.10). Now, we use (3.16) to find a new formulation for solving high order nonholonomic systems.

In a system with k HONCs, without losing generality of the problem, we assume that the first k quasicoordinates are written in the form of (2.5). By means of (3.16) we find that

$$\begin{aligned} &\left(\frac{\partial S}{\partial \ddot{\gamma}_1} - \Gamma_1\right) \sum_{j=k+1}^{M} \frac{\partial \gamma_1^{(s_1)}}{\partial \gamma_j^{(s_1)}}\delta\gamma_j + ... + \left(\frac{\partial S}{\partial \ddot{\gamma}_k} - \Gamma_k\right) \sum_{j=k+1}^{M} \frac{\partial \gamma_k^{(s_k)}}{\partial \gamma_j^{(s_k)}}\delta\gamma_j \\ &\qquad + \sum_{j=k+1}^{M} \left(\frac{\partial S}{\partial \ddot{\gamma}_j} - \Gamma_j\right)\delta\gamma_j = 0. \end{aligned} \tag{3.17}$$

Now, in this equation all $\delta\gamma_j$ parameters are independent and we can derive the following equations for high order nonholonomic systems in which the HONCs can be written in the form of (2.5)

$$\left(\frac{\partial S}{\partial \ddot{\gamma}_j} - \Gamma_j\right) + \sum_{i=1}^{k}\left(\frac{\partial S}{\partial \ddot{\gamma}_i} - \Gamma_i\right)\frac{\partial \gamma_i^{(s_i)}}{\partial \gamma_j^{(s_i)}} = 0, \qquad j = k = 1, ..., M. \tag{3.18}$$

The presented equations can analyze a vast part of high order nonholonomic systems very easily and there is no need to Lagrange multipliers. In the following section we illustrate this method with an example of high order nonholonomic system.

4 Example

There are many motives for studying high order nonholonomic systems. Numerous systems can be found in robotics and control which include HONCs. In this section we illustrate the introduced method by solving the Appell-Hamel problem.

The mechanism of the problem is composed of a massless frame with two legs, which slide without friction on x-y plane, and a wheel. The frame consists of two massless pulleys with radius b to support a string. A particle of mass m attached to the end of this string and restricted to move along the vertical bar of the frame. The other end of string is wounded around a drum with radius b which is fixed to the wheel. The total mass of the drum and wheel is M and the mass moment of inertia of this complex about its main axis is I_y and about any axis normal to the main axis is I_x [1]. The figure 1 depicts the mechanism.

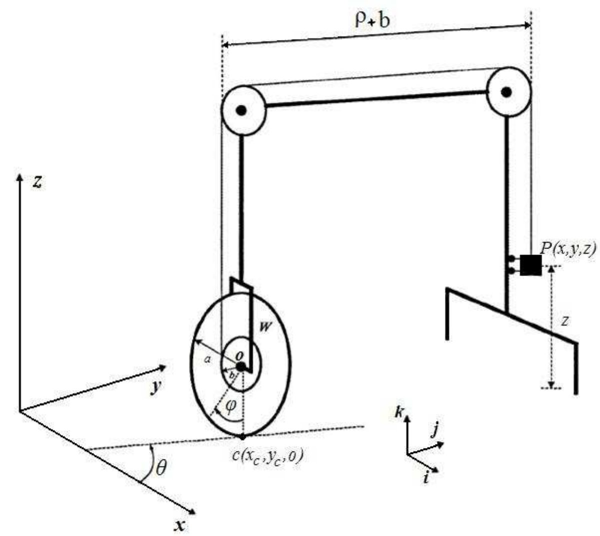

Figure 1: Schematic of the Appell-Hamel mechanism [1]

We choose five generalized coordinates to analyze this problem. Three coordinates x, y, z in the inertial Cartesian coordinate xyz indicate the position of the particle P. We choose φ to specify the rotation of the wheel about its main axis and θ to identify the angle between direction of the wheel and the x axis. The velocity of the center of the wheel can be calculated by two methods, as follows

$$\dot{x}_o = a\dot{\varphi}\cos\theta, \qquad \dot{y}_o = a\dot{\varphi}\sin\theta, \tag{4.1}$$

$$\dot{x}_o = \dot{x} + \rho\dot{\theta}\sin\theta, \qquad \dot{y}_o = \dot{y} - \rho\dot{\theta}\cos\theta. \tag{4.2}$$

We can write (4.1) as

$$\dot{x}_o^2 + \dot{y}_o^2 = a^2\dot{\varphi}^2, \qquad \dot{x}_o\sin\theta - \dot{y}_o\cos\theta = 0, \tag{4.3}$$

and substituting (4.2) into (4.3) results

$$\begin{gathered}(\dot{x} + \rho\dot{\theta}\sin\theta)^2 + (\dot{y} - \rho\dot{\theta}\cos\theta)^2 = a^2\dot{\varphi}^2, \\ \dot{x}\sin\theta - \dot{y}\cos\theta + \rho\dot{\theta} = 0,\end{gathered} \tag{4.4}$$

There is also a holonomic constrains on motion as follows

$$z = -b\varphi + z_0 \tag{4.5}$$

where z_0 is a constant.

Now, we use the generalized Gibbs-Appell equations to solve Appell-Hamel problem. As shown, the mechanism has two degrees of freedom. We can choose many sets of quasivelocities, but for simplicity we select quasivelocities as: $\dot{\gamma}_1 = \dot{x}, \dot{\gamma}_2 = \dot{y}, \dot{\gamma}_3 = \dot{z}, \dot{\gamma}_4 = \dot{\varphi}, \dot{\gamma}_5 = \dot{\theta}$. For replacing quasivelocities in constraints and then writing them in the form of (2.5), whereas one of the nonholonomic constraints is nonlinear, it's required to differentiate equations. (4.4).

After these calculations finally $\ddot{x}$ and $\ddot{y}$ are derived as functions of all generalized coordinates, their first derivative and second derivative of θ and φ.

$$\begin{aligned}\ddot{x} &= -\frac{\sin\theta\,\dot{y}^2\dot{\theta} + \rho\sin^2\theta\,\dot{y}\ddot{\theta} + \cos\theta\,\dot{x}(\dot{y}\dot{\theta} + \rho\sin\theta\,\ddot{\theta}) - a^2\cos\theta\,\dot{\varphi}\ddot{\varphi}}{\cos\theta\dot{x} + \sin\theta\dot{y}}, \\ \ddot{y} &= \frac{\cos\theta\,\dot{x}^2\dot{\theta} + \dot{x}(\sin\theta\,\dot{y}\dot{\theta} + \rho\cos^2\theta\ddot{\theta}) + \sin\theta(\rho\cos\theta\,\dot{y}\ddot{\theta} + a^2\dot{\varphi}\ddot{\varphi})}{\cos\theta\dot{x} + \sin\theta\dot{y}}\end{aligned} \tag{4.6}$$

The Gibbs-Appell function or the acceleration energy for the Appell-Hamel mechanism is as follows

$$\begin{aligned}S = \frac{1}{2}[(-2I_x + 3I_y + a^2M)\dot{\gamma}_4^2\dot{\gamma}_5^2 + m(\ddot{\gamma}_1^2 + \ddot{\gamma}_2^2 + \ddot{\gamma}_3^2) \\ + (I_y + a^2M)\ddot{\gamma}_4^2 + I_x\ddot{\gamma}_5^2].\end{aligned} \tag{4.7}$$

We use (3.18) to find equations of motion of this mechanism, they can be obtained from following equations

$$\begin{aligned}
&\frac{\partial S}{\partial \ddot{\gamma}_4} - \Gamma_4 + \sum_{i=1}^{3} \left(\frac{\partial S}{\partial \ddot{\gamma}_i} - \Gamma_i \right) \frac{\partial \gamma_i^{(s_i)}}{\partial \gamma_4^{(s_i)}} = 0, \\
&\frac{\partial S}{\partial \ddot{\gamma}_5} - \Gamma_5 + \sum_{i=1}^{3} \left(\frac{\partial S}{\partial \ddot{\gamma}_i} - \Gamma_i \right) \frac{\partial \gamma_i^{(s_i)}}{\partial \gamma_5^{(s_i)}} = 0,
\end{aligned} \tag{4.8}$$

Using the above equations and the constraints (4.5) and (4.6), equations of motion of the Appell-Hamel mechanism are

$$\begin{aligned}
&I_x \ddot{\theta} + m\rho \cos\theta \ddot{y} - m\rho \sin\theta \ddot{x} = 0, \\
&a^2 m\dot{\varphi}(\cos\theta \ddot{x} + \sin\theta \ddot{y}) + (\cos\theta \dot{x} + \sin\theta \dot{y})(bm[g - \ddot{z}] + [I_y + a^2 M]\ddot{\varphi}) = 0, \\
&\ddot{x} + \frac{\sin\theta\, \dot{y}^2 \dot{\theta} + \rho \sin^2\theta\, \dot{y}\ddot{\theta} + \cos\theta\, \dot{x}(\dot{y}\dot{\theta} + \rho\sin\theta\, \ddot{\theta}) - a^2 \cos\theta\, \dot{\varphi}\ddot{\varphi}}{\cos\theta \dot{x} + \sin\theta \dot{y}} = 0, \\
&\ddot{y} - \frac{\cos\theta\, \dot{x}^2 \dot{\theta} + \dot{x}(\sin\theta\, \dot{y}\dot{\theta} + \rho\cos^2\theta \ddot{\theta}) + \sin\theta(\rho\cos\theta\, \dot{y}\ddot{\theta} + a^2 \dot{\varphi}\ddot{\varphi})}{\cos\theta \dot{x} + \sin\theta \dot{y}} = 0, \\
&\ddot{z} + b\ddot{\varphi} = 0.
\end{aligned} \tag{4.9}$$

The result is the same as the results obtained by other methods, while in the presented method deriving equations of motion is much easier and, whereas in this method we don't use Lagrange multipliers, lower amount of calculation is needed.

5 Conclusion

The study of nonholonomic systems started in the end of 19th century but because of its numerous applications, it is still continued. Most of the methods introduced so far, deal with first order nonholonomic systems, and/or use Lagrange multipliers and require higher much more calculations. In this paper a new formulation based on Gibbs-Appell method has been derived. This method can analyze a large group of high order nonholonomic systems very easily, and it doesn't use Lagrange multipliers or any new variables. Also, the order of constraints is not restricted in the presented method. Finally, the introduced method is illustrated with an example.

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