Fractional differential systems associated to Toda lattice

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Abstract. The main purpose of this paper is to investigate the fractional differential systems associated to classical Toda-type systems in terms of fractional Caputo derivatives.

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1 Introduction

Fractional differential equations are equations that contains derivatives of non-integer order [12, 1, 7].

In the last three decades, the fractional calculus has played an impotant role in various branches of mathematics, engineering, chaotic dynamics, classical and quantum mechanics, field theory and optimal control, biology and economics [6, 7, 10, 12, 14, 15]. Also, the fractional calculus has been used successfully to the description of classical dynamical systems, especially non-conservative dynamical systems [8, 9, 11, 14, 15].

Various applications of fractional calculus are based on replacing the time derivative in an evolution equation with a derivative of fractional order.

The paper is structured as follows. In Section 2, some preliminaries concerning the fractional Leibniz structure defined on a manifold are presented. In Section 3 we define the concept of fractional Leibniz realization of a fractional dynamical system. Also is defined the $n-$ dimensional fractional Toda lattice and its fractional Poisson realization is constructed. In Sections 4 and 5, the fractional metriplectic system (4.8) associated to the $2-$ dimensional fractional Toda lattice and its numerical integration are investigated.

2 Fractional Leibniz manifolds

We start this section with some basic formulas of the fractional calculus.

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Let $f \in C^{\infty}(R)$ and $\alpha \in R, \alpha > 0$. The fractional differential operator is described by

$$D_t^{\alpha} f(t) = J^{m-\alpha} f^{(m)}(t),\ \ \alpha > 0,$$

where $m \in N^*$ such that $m-1 \leq \alpha \leq m,\ f^{(m)}(t)$ is the general $m-$ order derivative, and J^{β} is the $\beta-$ order Riemann - Liouville integral operator which is expressed as follows

$$J^{\beta} f(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} f(s) ds, \quad \beta > 0,$$

where Γ is the Euler gamma function.

The operator D_*^{α} is called the $\alpha-$ *order Caputo differential operator* (for details see [1, 7]).

It is well known that, if $\alpha \to 1$ then $D_t^{\alpha} f(t) = \frac{df}{dt}$.

In this paper we suppose that $\alpha \in (0,1)$.

The $\alpha-$ order Caputo differential operator has the following properties:

(i) If $f(t) = c, (\forall) t \in [a,b]$ then $D_t^{\alpha} f(t) = 0$;

(ii) If $f(t) = t^{\beta}, (\forall) t \in [a,b]$ then $D_t^{\alpha} f(t) = \frac{\Gamma(1+\beta)}{\Gamma(1+\beta-\alpha)} t^{\beta-\alpha}$.

Let M be a $n-$ dimensional smooth manifold, $U \subset M$ a local chart and $(x^i), i = \overline{1,n}$ a system of local coordinates on U. For $f \in C^{\infty}(U)$, we denote with $D_{x^i}^{\alpha} f$ the Caputo partial derivatives defined by

$$D_{x^i}^{\alpha} f(x) = \frac{1}{\Gamma(1-\alpha)} \int_0^{x^i} \frac{\partial f(x^1, ..., x^{i-1}, s, x^{i+1}, ..., x^n)}{\partial x^i} \frac{1}{(x^i - s)^{\alpha}} ds,\ i = \overline{1,n},$$

where $(\frac{\partial}{\partial x^i}), i = \overline{1,n}$ is the canonical basis of vector fields on U.

We denote by $\mathcal{X}^{\alpha}(U)$ be the module of the fractional vector fields generated by the operators $D_{x^i}^{\alpha}, i = \overline{1,n}$. A fractional vector field $\overset{\alpha}{X} \in \mathcal{X}^{\alpha}(U)$ has the form (see [4]):

$$\overset{\alpha}{X} = \overset{\alpha}{X}{}^i D_{x^i}^{\alpha}, \qquad \overset{\alpha}{X}{}^i \in C^{\infty}(U), i = \overline{1,n}.$$

Let $\mathcal{D}^{\alpha}(U)$ the module generated by 1 - forms $d(x^i)^{\alpha}, i = \overline{1,n}$ on U. The fractional exterior derivative $d^{\alpha} : C^{\infty}(U) \to \mathcal{D}^{\alpha}(U)$, is defined by (see [2, 4]):

$$d^{\alpha}(f) = d(x^i)^{\alpha} D_{x^i}^{\alpha}(f), \qquad f \in C^{\infty}(U).$$

Let us consider a fractional 2− tensor field $\overset{\alpha}{P} \in \mathcal{X}^{\alpha}(M) \times \mathcal{X}^{\alpha}(M)$ and $d^{\alpha} f, d^{\alpha} g \in \mathcal{D}^{\alpha}(M)$. The bilinear map $[\cdot,\cdot]^{\alpha} : C^{\infty}(M) \times C^{\infty}(M) \to C^{\infty}(M)$ defined by

$$(2.1) \qquad [f,h]^{\alpha} = \overset{\alpha}{P}(d^{\alpha} f, d^{\alpha} h), \quad (\forall) f, h \in C^{\infty}(M),$$

is called the *fractional Leibniz bracket*.

The pair $(M, [\cdot,\cdot]^{\alpha})$ is called *fractional Leibniz manifold.* If the bracket $[\cdot,\cdot]^{\alpha}$ is skew-symmetric (equivalently, $\overset{\alpha}{P}$ is skew-symmetric), we say that $(M, [\cdot,\cdot]^{\alpha})$ is a *fractional almost Poisson manifold.* If $\alpha \to 1$, then one obtain the concept from [13].

For $h \in C^{\infty}(M)$, where M is a fractional Leibniz manifold, the fractional vector field $\overset{\alpha}{X}_h$ defined by

$$(2.2) \qquad \overset{\alpha}{X}_h(f) = [f,h]^{\alpha}, \ (\forall) f \in C^{\infty}(M),$$

is called the *fractional Leibniz vector field associated to* h and its associated dynamical system is the *fractional Leibniz system.*

Locally, if the fractional 2− tensor field $\overset{\alpha}{P}$ on M is generated by the matrix $\overset{\alpha}{P} = (\overset{\alpha}{P}{}^{ij})$, then the fractional Leibniz system is given by

$$(2.3) \qquad D_t^{\alpha} x^i(t) = [x^i(t), h(t)]^{\alpha}, \quad \text{where} \quad [x^i, h]^{\alpha} = \overset{\alpha}{P}{}^{ij} \cdot D_{x^j}^{\alpha} h.$$

If $\overset{\alpha}{P}$ is skew-symmetric, then (2.3) is called the *fractional almost Poisson system associated to* $\overset{\alpha}{P}$ *with the Hamiltonian* $h \in C^{\infty}(M)$.

For more details concerning the classical Leibniz bracket and the fractional Leibniz structures, the reader can be consult the papers [8,9].

3 Fractional Leibniz realization of fractional dynamical systems on R^n

Definition 3.1. A *fractional dynamical system* on R^n, is a system of fractional differential equations of the following type:

$$(3.1) \qquad D_t^{\alpha} x^i(t) = f^i(x^1(t), \ldots, x^n(t)), \quad f^i \in C^{\infty}(R^n, R), \ i = \overline{1,n}. \qquad \Box$$

Let $[\cdot,\cdot]^{\alpha}$ be a Leibniz structure on R^n with $\overset{\alpha}{P} = (\overset{\alpha}{P}{}^{ij})$ the associated matrix, i.e. $\overset{\alpha}{P}{}^{ij} = [x^i, x^j]^{\alpha}, \ i,j = \overline{1,n}$.

Definition 3.2. We say that a *fractional dynamical system* on R^n of the form (3.1) has a *fractional Leibniz realization*, if there exist an almost Leibniz structure $[\cdot,\cdot]^{\alpha}$ on R^n generated by the matrix $\overset{\alpha}{P} = (\overset{\alpha}{P}{}^{ij})$ and a Hamiltonian function $\overset{\alpha}{H} \in C^{\infty}(R^n, R)$ such that the fractional dynamics (3.1) can be written in the following form:

$$(3.2) \qquad D_t^{\alpha} x(t) = \overset{\alpha}{P}(x(t)) \cdot \nabla^{\alpha} \overset{\alpha}{H}(x(t)),$$

where $x(t) = (x^1(t), \ldots, x^n(t))^T$ and $\nabla^{\alpha} \overset{\alpha}{H} = (D_{x^1}^{\alpha} \overset{\alpha}{H}, \ldots, D_{x^n}^{\alpha} \overset{\alpha}{H})^T$. $\Box$

A fractional Leibniz realization is denoted by $(R^n, [\cdot,\cdot]^{\alpha}, \overset{\alpha}{H})$ or $(R^n, \overset{\alpha}{P}, \overset{\alpha}{H})$.

If the matrix $\overset{\alpha}{P} = (\overset{\alpha}{P}{}^{ij})$ is skew-symmetric, then $(R^n, \overset{\alpha}{P}, \overset{\alpha}{H})$ is called the *fractional almost Poisson realization* of the system (3.1).

Definition 3.3. A *fractional Casimir function* of the configuration $(R^n, [\cdot,\cdot]^\alpha, \overset{\alpha}{H})$ is a function $\overset{\alpha}{C} \in C^\infty(R^n, R)$ with the property that:

$$(3.3) \qquad [\overset{\alpha}{C}, f]^\alpha = 0, \quad (\forall) f \in C^\infty(R^n, R). \qquad \square$$

Remark 3.1. *A function $\overset{\alpha}{C} \in C^\infty(R^n, R)$ is a Casimir of the configuration $(R^n, [\cdot,\cdot]^\alpha, \overset{\alpha}{H})$ if and only if the following relation holds:*

$$(3.4) \qquad \overset{\alpha}{P}(x(t)) \cdot \nabla^\alpha \overset{\alpha}{C}(x(t)) = 0 \qquad \square$$

As example we discuss the fractional Leibniz realization of the fractional dynamical system associated to n- dimensional Toda lattice.

For definitions and results about the classical dynamical system of the Toda- type, see the paper [5].

The phase space consists of variables x^i, y^j for $1 \le i \le n-1$ and $1 \le j \le n$.

The fractional differential equations

$$(3.5) \qquad \begin{cases} D_t^\alpha x^i &= x^i(y^{i+1} - y^i), \quad i = \overline{1, n-1} \\ D_t^\alpha y^j &= 2[(x^j)^2 - (x^{j-1})^2], \quad j = \overline{1, n} \end{cases}$$

where $x^0 = x^n = 0$, is called the *n-dimensional fractional Toda lattice.* $\square$

Let $\overset{\alpha}{P}$ be a skew-symmetric fractional $2-$ tensor field on R^{2n-1} generated by the matrix $P = (P^{rs})$, $r, s = \overline{1, 2n-1}$ where

$$(3.6) \qquad P^{rs} = -P^{rs} = \begin{cases} -x^i & \text{for} \quad r = i, s = n+i-1, i = \overline{1, n-1} \\ x^i & \text{for} \quad r = i, s = n+i, \ i = \overline{1, n-1} \\ 0 & \quad \text{in the rest.} \end{cases}$$

Proposition 3.1. *(i) The n-dimensional fractional Toda lattice* (3.5) *has a fractional almost Poisson realization $(R^{2n-1}, \overset{\alpha}{P}, \overset{\alpha}{H})$, where $\overset{\alpha}{P} = P$ given by* (3.6) *and $\overset{\alpha}{H}$ is defined by*

$$(3.7) \quad \overset{\alpha}{H}(x^1, \ldots, x^{n-1}, y^1, \ldots, y^n) = \frac{1}{\Gamma(2+\alpha)}[2\sum_{i=1}^{n-1}(x^i)^{1+\alpha} + \sum_{j=1}^{n}(y^j)^{1+\alpha}].$$

(ii) A fractional Casimir function $\overset{\alpha}{C} \in C^\infty(R^{2n-1}, R)$ of the configuration $(R^{2n-1}, P, \overset{\alpha}{H})$, is given by

$$(3.8) \qquad \overset{\alpha}{C}(x^1, \ldots, x^{n-1}, y^1, \ldots, y^n) = \frac{1}{\Gamma(1+\alpha)}\sum_{j=1}^{n}(y^j)^\alpha.$$

Proof. It must to prove that the system (3.5) can be written in the form (3.2). For $\nabla^\alpha \overset{\alpha}{H} = (\nabla^\alpha D^\alpha_{x^1} \overset{\alpha}{H}, \ldots, \nabla^\alpha D^\alpha_{x^{n-1}} \overset{\alpha}{H}, \nabla^\alpha D^\alpha_{y^1} \overset{\alpha}{H}, \ldots, \nabla^\alpha D^\alpha_{y^n} \overset{\alpha}{H})^T$ we have $D^\alpha_{x^i} \overset{\alpha}{H} = 2x^i$, $i = \overline{1, n-1}$ and $D^\alpha_{y^j} \overset{\alpha}{H} = y^j$, $j = \overline{1, n}$.

The line of rank i for $i=\overline{1,n-1}$ of the matrix $\overset{\alpha}{P}\cdot\nabla^{\alpha}\overset{\alpha}{H}$ is

$$\sum_{k=1}^{n-1}P^{ik}D^{\alpha}_{x^k}\overset{\alpha}{H}+\sum_{\ell=n}^{2n-1}P^{i\ell}D^{\alpha}_{y^{\ell-n+1}}\overset{\alpha}{H}=P^{i,n-1+i}D^{\alpha}_{y^i}\overset{\alpha}{H}+P^{i,n+i}D^{\alpha}_{y^{i+1}}\overset{\alpha}{H}=$$
$$=-x^iy^i+x^iy^{i+1}=x^i(y^{i+1}-y^i).$$

Since $x^i(y^{i+1}-y^i)=D^{\alpha}_t x^i$ for $i=\overline{1,n-1}$, we obtain that the lines $1,2,\ldots,n-1$ of the relation (3.2) are verified.

The line of rank $n-1+j$ for $j=\overline{1,n}$ of the matrix $\overset{\alpha}{P}\cdot\nabla^{\alpha}\overset{\alpha}{H}$ is

$$\sum_{k=1}^{n-1}P^{n-1+j,k}D^{\alpha}_{x^k}\overset{\alpha}{H}+\sum_{\ell=n}^{2n-1}P^{n-1+j,\ell}D^{\alpha}_{y^{\ell-n+1}}\overset{\alpha}{H}=P^{n-1+j,j-1}D^{\alpha}_{x^{j-1}}\overset{\alpha}{H}+$$
$$+P^{n-1+j,j}D^{\alpha}_{x^j}\overset{\alpha}{H}=-x^{j-1}\cdot 2x^{j-1}+-x^j\cdot 2x^j=2[(x^j)^2-(x^{j-1})^2].$$

Since $2[(x^j)^2-(x^{j-1})^2]=D^{\alpha}_t y^j$ for $j=\overline{1,n}$, we obtain that the lines $n,n+1,\ldots,2n-1$ of the relation (3.2) are verified.

(ii) For $\nabla^{\alpha}\overset{\alpha}{C}=(\nabla^{\alpha}D^{\alpha}_{x^1}\overset{\alpha}{C},\ldots,\nabla^{\alpha}D^{\alpha}_{x^{n-1}}\overset{\alpha}{C},\nabla^{\alpha}D^{\alpha}_{y^1}\overset{\alpha}{C},\ldots,\nabla^{\alpha}D^{\alpha}_{y^n}\overset{\alpha}{C})^T$ we have $D^{\alpha}_{x^i}\overset{\alpha}{C}=0,\ i=\overline{1,n-1}$ and $D^{\alpha}_{y^j}\overset{\alpha}{C}=1,\ \ j=\overline{1,n}.$

The line of rank i for $i=\overline{1,n-1}$ of the matrix $\overset{\alpha}{P}\cdot\nabla^{\alpha}\overset{\alpha}{C}$ is

$$\sum_{k=1}^{n-1}P^{ik}D^{\alpha}_{x^k}\overset{\alpha}{C}+\sum_{\ell=n}^{2n-1}P^{i\ell}D^{\alpha}_{y^{\ell-n+1}}\overset{\alpha}{C}=P^{i,n-1+i}D^{\alpha}_{y^i}\overset{\alpha}{C}+P^{i,n+i}D^{\alpha}_{y^{i+1}}\overset{\alpha}{C}=$$
$$=-x^i\cdot 1+x^i\cdot 1=0.$$

The line of rank $n-1+j$ for $j=\overline{1,n}$ of the matrix $\overset{\alpha}{P}\cdot\nabla^{\alpha}\overset{\alpha}{C}$ is

$$\sum_{k=1}^{n-1}P^{n-1+j,k}D^{\alpha}_{x^k}\overset{\alpha}{C}+\sum_{\ell=n}^{2n-1}P^{n-1+j,\ell}D^{\alpha}_{y^{\ell-n+1}}\overset{\alpha}{C}=P^{n-1+j,j-1}D^{\alpha}_{x^{j-1}}\overset{\alpha}{C}+$$
$$+P^{n-1+j,j}D^{\alpha}_{x^j}\overset{\alpha}{C}=P^{n-1+j,j-1}\cdot 0+P^{n-1+j,j}\cdot 0=0.$$

Hence, the relation (3.4) holds. □

Let $[\cdot,\cdot]^{\alpha}$ be a Leibniz structure on R^n generated by the matrix $\overset{\alpha}{P}=(\overset{\alpha}{P}{}^{ij})$ and $\overset{\alpha}{C}_1,\ldots,\overset{\alpha}{C}_k\in C^{\infty}(R^n,R)$ a complete set of functionally independent Casimir functions. Let $\overset{\alpha}{G}$ be a smooth function from R^n to the vector space of symmetric matrices of type $n\times n$.

Definition 3.4. An *fractional almost metriplectic system* on R^n is a system of fractional differential equations of the following type:

$$D^{\alpha}_t x(t)=\overset{\alpha}{P}(x(t))\cdot\nabla^{\alpha}\overset{\alpha}{H}(x(t))+\overset{\alpha}{G}(x(t))\cdot\nabla^{\alpha}\varphi(\overset{\alpha}{C}_1,\ldots,\overset{\alpha}{C}_k)(x(t)),\tag{3.9}$$

where $\overset{\alpha}{H}\in C^{\infty}(R^n,R)$ and $\varphi\in C^{\infty}(R^k,R)$ such that the following conditions hold:

$(i) \qquad \overset{\alpha}{P}(x) \cdot \nabla^{\alpha} \overset{\alpha}{C}_i(x) = 0, \quad i = \overline{1,k},$

i.e. for each $i = \overline{1,k}$, the function $\overset{\alpha}{C}_i$ is a Casimir of the configuration $(R^n, \overset{\alpha}{P}, \overset{\alpha}{H})$;

$(ii) \qquad \overset{\alpha}{G}(x) \cdot \nabla^{\alpha} \overset{\alpha}{H}(x) = 0;$

$(iii) \qquad (\nabla^{\alpha} \varphi(\overset{\alpha}{C}_1, \ldots, \overset{\alpha}{C}_k)(x))^T \cdot \overset{\alpha}{G}(x) \cdot \nabla^{\alpha} \varphi(\overset{\alpha}{C}_1, \ldots, \overset{\alpha}{C}_k)(x) \leq 0.$ □

If the matrix $\overset{\alpha}{P} = (\overset{\alpha}{P}{}^{ij})$ is skew-symmetric, then the system (3.9) is called the *fractional metriplectic system* on R^n.

Remark 3.2. The fractional (almost) metriplectic system (3.9) can be regarded as a perturbation of the fractional Leibniz system

$$D_t^{\alpha} x(t) = \overset{\alpha}{P}(x(t)) \cdot \nabla^{\alpha} \overset{\alpha}{H}(x(t)) \tag{3.10}$$

with the fractional dissipation therm $\overset{\alpha}{G}(x) \cdot \nabla^{\alpha} \varphi(\overset{\alpha}{C}_1, \ldots, \overset{\alpha}{C}_k)(x)$.

The function $\varphi(\overset{\alpha}{C}_1, \ldots, \overset{\alpha}{C}_k)$ is called the *fractional control function* of the fractional dynamics (3.10). □

Remark 3.3. If in the fractional metriplectic system (3.9) we consider $\alpha \to 1$, then we obtain the concept of metriplectic system on R^n discussed in the paper [3].□

4 Fractional metriplectic system associated to 2− dimensional fractional Toda lattice

In this section we associate a fractional metriplectic system for the fractional Toda lattice in the particular case $n = 2$.

The 2− dimensional fractional Toda lattice is described by:

$$D_t^{\alpha} x^1 = -x^1 y^1 + x^1 y^2, \quad D_t^{\alpha} y^1 = 2(x^1)^2, \quad D_t^{\alpha} y^2 = -2(x^1)^2. \tag{4.1}$$

Using the transformations $x^1 = x^1,\ y^1 = x^2,\ y^2 = x^3$ the fractional dynamical system (4.1) becomes:

$$\begin{cases} D_t^{\alpha} x^1 &= x^1(-x^2 + x^3) \\ D_t^{\alpha} x^2 &= 2(x^1)^2 \\ D_t^{\alpha} x^3 &= -2(x^1)^2 \end{cases} \tag{4.2}$$

Applying Proposition 3.1, the 2− dimensional fractional Toda lattice (4.2) has a fractional almost Poisson realization $(R^3, \overset{\alpha}{P} = P_{TL}, \overset{\alpha}{H} = \overset{\alpha}{H}_{TL})$ with the Casimir function $\overset{\alpha}{C} = \overset{\alpha}{C}_{TL} \in C^{\infty}(R^3, R)$, where

$$P_{TL} = \begin{pmatrix} 0 & -x^1 & x^1 \\ x^1 & 0 & 0 \\ -x^1 & 0 & 0 \end{pmatrix}, \tag{4.3}$$

$$(4.4) \qquad \overset{\alpha}{H}_{TL}(x^1,x^2,x^3) = \frac{1}{\Gamma(2+\alpha)}[2(x^1)^{1+\alpha} + (x^2)^{1+\alpha} + (x^3)^{1+\alpha})],$$

$$(4.5) \qquad \overset{\alpha}{C}_{TL}(x^1,x^2,x^3) = \frac{1}{\Gamma(1+\alpha)}((x^2)^{\alpha} + (x^3)^{\alpha}).$$

For the fractional Hamiltonian $\overset{\alpha}{h}_1 = \overset{\alpha}{H}_{TL}$, we have
$D^{\alpha}_{x^1}\overset{\alpha}{h}_1 = 2x^1, \quad D^{\alpha}_{x^2}\overset{\alpha}{h}_1 = x^2, \quad D^{\alpha}_{x^3}\overset{\alpha}{h}_1 = x^3.$

We determine now the symmetric matrix $\overset{\alpha}{G} = (\overset{\alpha}{g}{}^{ij})$, where

$$(4.6) \qquad \overset{\alpha}{g}{}^{ii} = -\sum_{k=1,k\neq i}^{k=3}(D^{\alpha}_{x^k}\overset{\alpha}{h}_1)^2, \quad \overset{\alpha}{g}{}^{ij} = \overset{\alpha}{g}{}^{ji} = D^{\alpha}_{x^i}\overset{\alpha}{h}_1 \cdot D^{\alpha}_{x^j}\overset{\alpha}{h}_1 \quad \text{for} \quad i \neq j.$$

Using the relations (4.6) we find the matrix $\overset{\alpha}{G} = G$, where

$$(4.7) \qquad G = \begin{pmatrix} -(x^2)^2-(x^3)^2 & 2x^1x^2 & 2x^1x^3 \\ 2x^1x^2 & -4(x^1)^2-(x^3)^2 & x^2x^3 \\ 2x^1x^3 & x^2x^3 & -4(x^1)^2-(x^2)^2 \end{pmatrix}.$$

We consider now the function $\overset{\alpha}{h}_2 = a\overset{\alpha}{C}_{TL}$ with $a \in R$.

For the skew-symmetric matrix $\overset{\alpha}{P} = P_{TL}$, the fractional Hamiltonian $\overset{\alpha}{h}_1 = \overset{\alpha}{H}_{TL}$, the symmetric matrix $\overset{\alpha}{G} = G$ given by (4.7) and the function $\varphi(\overset{\alpha}{C}) = a\overset{\alpha}{C}_{TL}$, the fractional system (3.9) becomes:

$$D^{\alpha}_t x(t) = \overset{\alpha}{P}(x(t)) \cdot \nabla^{\alpha}\overset{\alpha}{h}_1(x(t)) + G(x(t)) \cdot \nabla^{\alpha}\overset{\alpha}{h}_2(x(t)),$$

or equivalently

$$(4.8) \qquad \begin{cases} D^{\alpha}_t x^1 = x^1[(2a-1)x^2 + (2a+1)x^3] \\ D^{\alpha}_t x^2 = 2(1-2a)(x^1)^2 - ax^3(x^3-x^2) \\ D^{\alpha}_t x^3 = -2(1+2a)(x^1)^2 + ax^2(x^3-x^2). \end{cases}$$

The fractional differential system (4.8) is called the *fractional perturbed system* of the 2-dimensional fractional Toda lattice.

If in (4.8) we consider $a = 1$ we obtain the *fractional revised system* of the 2-dimensional fractional Toda lattice.

Proposition 4.1. *The fractional dynamical system* (4.8) *is a fractional metriplectic system on* R^3.

Proof. We shall verify the conditions $(i)-(iii)$ from Definition 3.4. From Proposition 3.1, follows that the condition (i) is satisfied.

We have

$$\overset{\alpha}{G}\cdot\nabla^{\alpha}\overset{\alpha}{h}_1 = \begin{pmatrix} -(x^2)^2-(x^3)^2 & 2x^1x^2 & 2x^1x^3 \\ 2x^1x^2 & -4(x^1)^2-(x^3)^2 & x^2x^3 \\ 2x^1x^3 & x^2x^3 & -4(x^1)^2-(x^2)^2 \end{pmatrix} \begin{pmatrix} 2x^1 \\ x^2 \\ x^3 \end{pmatrix} =$$

$= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$. Hence the condition (ii) holds.

Since $D^{\alpha}_{x^1} \overset{\alpha}{h}_2 = 0$, $D^{\alpha}_{x^2} \overset{\alpha}{h}_2 = a$, $D^{\alpha}_{x^3} \overset{\alpha}{h}_2 = a$, we have $(\nabla^{\alpha} \overset{\alpha}{h}_2)^T \cdot \overset{\alpha}{G} \cdot \nabla^{\alpha} \overset{\alpha}{h}_2 =$

$$= \begin{pmatrix} 0 & a & a \end{pmatrix} \begin{pmatrix} -(x^2)^2-(x^3)^2 & 2x^1x^2 & 2x^1x^3 \\ 2x^1x^2 & -4(x^1)^2-(x^3)^2 & x^2x^3 \\ 2x^1x^3 & x^2x^3 & -4(x^1)^2-(x^2)^2 \end{pmatrix} \begin{pmatrix} 0 \\ a \\ a \end{pmatrix} =$$

$= a^2[-8(x^1)^2 - (x^2 - x^3)^2] \leq 0$. Hence the condition (iii) holds. □

5 Numerical integration of the fractional metriplectic system (4.8)

We consider the fractional differential equations given by

$$\text{(5.1)} \qquad \begin{cases} D^{\alpha}_t x^1(t) &= \ell^1(x^1(t), x^2(t), x^3(t)) \\ D^{\alpha}_t x^2(t) &= \ell^2(x^1(t), x^2(t), x^3(t)) \\ D^{\alpha}_t x^3(t) &= \ell^3(x^1(t), x^2(t), x^3(t)) \end{cases},$$

with the initial values $x^i(0) = x^i_0, i = \overline{1,3}$, $\alpha \in (0, \infty)$ and $0 \leq t \leq T$.

The fractional dynamical system (5.1) is equivalent to the Volterra integral equation (see [7]):

$$\text{(5.2)} \qquad x^i(t) = x^i(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \ell^i(x^1(s), x^2(s), x^3(s)) ds.$$

We can integrate numerically the set of fractional differential equations (5.1). For this, let $h = \dfrac{T}{N}$, $t_n = nh$, for $n = 0, 1, \ldots, N$.

We shall use the following notations:

$A = \dfrac{h^{\alpha}}{\Gamma(\alpha+2)}$, $B = \dfrac{h^{\alpha}}{\alpha\Gamma(\alpha)}$, $b_1 = 2^{\alpha+1} - 1$, $c_1 = 2^{\alpha+1} - 2$,

$x^i[n] = x^i(nh)$, $\ell^i[n] = \ell^i(x^1[n], x^2[n], x^3[n])$, for $i = \overline{1,3}$ and

$\ell^i_p[n] = \ell^i(x^1_p[n], x^2_p[n], x^3_p[n])$, for $i = \overline{1,3}$.

Discretizing the Volterra integral equation (5.2) as above yields:

$$\text{(5.3)} \qquad \begin{cases} x^i_p[n+1] = x^i[0] + B(\sum\limits_{k=0}^{n-1} b[k, n+1]\ell^i[k] + b_1 \ell^i[n]), \ i = \overline{1,3} \\ x^i[n+1] = x^i[0] + A(\sum\limits_{k=0}^{n-1} a[k, n+1]\ell^i[k] + c_1 \ell^i[n] + x^i_p[n+1]), \ i = \overline{1,3}, \end{cases}$$

where:

$$(5.4)\quad \begin{cases} a[0,n+1] &= n^{\alpha+1}-(n-\alpha)(n+1)^{\alpha} \\ a[k,n+1] &= (n-k+2)^{\alpha+1}+(n-k)^{\alpha+1}-2(n-k+1)^{\alpha+1} \\ b[0,n+1] &= (n+1)^{\alpha}-n^{\alpha} \\ b[k,n+1] &= (n+1-k)^{\alpha}-(n-k)^{\alpha} \end{cases}$$

for $k=\overline{1,n-1}$.

The above scheme given by the relations (5.2) and (5.3) is called the *predictor-corrector Moulton- Adams algorithm for fractional differential system* (5.1) (see for details [6]).

The error estimate for the algorithm described by (5.2) and (5.3) is:

$$(5.5)\quad \max\{\max_{0\le k\le N}(x^i[k]-x^i_p[k]) \mid i=\overline{1,3}\}=O(h^{\alpha+1}).$$

In what follows, using the above algorithm and the software Maple 11 we obtain the orbits of the system (4.8) for: $a=0$, $\alpha=0.8$, $\alpha=1$ and $a=1$, $\alpha=0.8$, $\alpha=1$.

Fig1.(x1[j],x2[j],x3[j]),alpha=0.8,a=0.

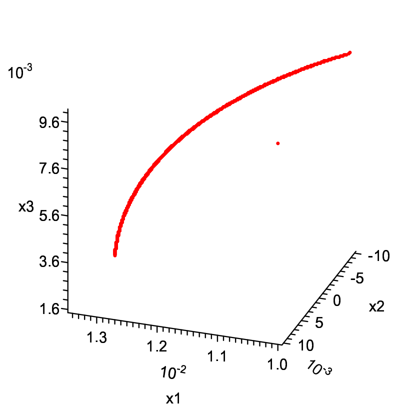


Fig2.(x1[j],x2[j],x3[j]),alpha=1,a=0

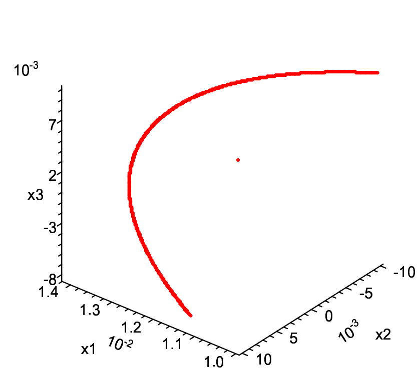


Fig3.(x1[j],x2[j],x3[j]),alpha=0.8,a=1.

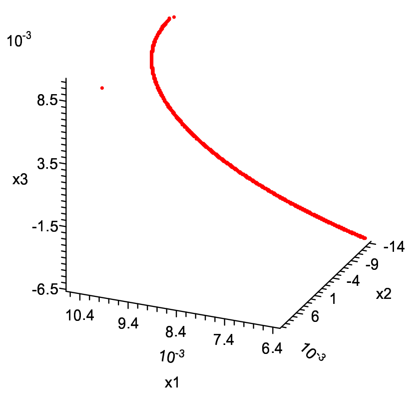
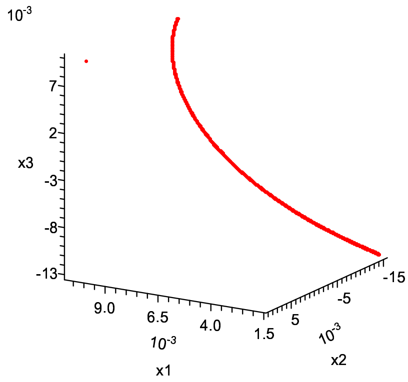


Fig4.(x1[j],x2[j],x3[j]),alpha=1,a=1



In the Figure 1 respectively Figure 2 is represented the orbit of the numerical solution of the $2-$ dimensional fractional Toda lattice [i.e. the system (4.2)] respectively the orbit of the numerical solution of the $2-$ dimensional Toda lattice [i.e. the system (4.8) with $\alpha = 1$ and $a = 0$].

In the Figure 3, respectively Figure 4 is represented the orbit of the numerical solution of the fractional revised system of the $2-$ dimensional fractional Toda lattice [i.e. the system (4.8) with $\alpha = 0.8$ and $a = 1$], respectively the orbit of the numerical solution of the revised system of the $2-$ dimensional Toda lattice [i.e. the system (4.8) with $\alpha = 1$ and $a = 1$].

Remark. Applying the algorithm given by (5.2) and (5.3) and the software Maple 11, for other values of $\alpha > 0$ and $a \in R$ we obtain the orbits of the solutions for the fractional metriplectic system (4.8). $\square$

Related results can be found in [10, 2].

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