Some results obtained in dynamical systems using a variational calculus theorem

Irina Meghea

Abstract. Among the main contributions of Brezis within the theory of nonlinear semigroups there exist two results owed to Brezis and Browder, developed further by Ekeland, which rely on a theorem of variational calculus and one of its corollaries. This paper adapts these two results of nonlinear semigroups to dynamical systems, firstly recovering the corollary of the variational theorem which is a prerequisite for these statements.

M.S.C. 2000: MSC2000: 37C99, 37L05, 49J40, 58E30, 58E50, 49S05.
Key words: dynamical system, nonlinear semi-group, pre-ordered set, contraction

Introduction

Nonlinear semigroup theory is not only of intrinsic interest, but is also important in the study of evolution problems. In recent years many developments have occurred, in particular, in the area of non-expansive semigroups in Banach spaces. As a rule, such semigroups are generated by accretive operators and can be viewed as nonlinear analogs of the classical linear contraction semigroups.

In the last forty years the theory of monotone and accretive operators has been intensively developed, and have provided many applications to nonlinear analysis and to optimization. This theory is closely connected with the general theory of nonlinear one-parameter semigroups of non-expansive mappings and with nonlinear evolution problems.

From mechanics to cybernetics, appears the general paradigm of regarding a dynamical system as an object - or process, for which one can define the concept of state as an instantaneous description; therefore, the temporal evolution of the system is given by the temporal evolution of the state.

Two important results in nonlinear semigroups have been obtained by H. Brezis and F. Browder [1], being based on a variational theorem.

BSG Proceedings 16. The Int. Conf. of Diff. Geom. and Dynamical Systems (DGDS-2008) and The V-th Int. Colloq. of Mathematics in Engineering and Numerical Physics (MENP-5) - math. sections, August 29 - September 2, 2008, Mangalia, Romania, pp. 91-98.

I. Ekeland in [2] develops an analogous subject, establishing links with his famous principle, and making also some interpretations in several applications of variants of the Ekeland principle.

Starting from the papers [1] and [2], the author completes the known results of variational calculus, placing them on the basis of two theorems from nonlinear semigroups theory, transposes them within the dynamical systems framework, and provides an intuitive interpretation.

1 Basic results

The main results of the article are Theorem 1 and its Corollary.

Theorem 1. *Let* $(X, \overset{\mathrm{B}}{\leq})$ *be a pre-ordered set, with the property that any decreasing sequence has a lower bound, and* $\varphi : X \to (-\infty, +\infty]$. *Assume that* φ *is increasing and admits a lower bounded. Then* $\forall\, x_0 \in X$, *there exists* $\nu \in X$, $\nu \leq x_0$, *such that*

$$x \in X, x \overset{\mathrm{B}}{\leq} \nu \Rightarrow \varphi(x) = \varphi(\nu). \tag{1.1}$$

Proof. We build by recurrence a decreasing sequence $(x_n)_{n\geq 0} \subset X$, having x_0 as first term. If x_n verifies (1.1), the construction stops, and the demonstration is finished. In the contrary case, $\exists\, x' \overset{\mathrm{B}}{<} x_n$ with $\varphi(x') < \varphi(x_n)$ and we take x_{n+1} such that

$$x_{n+1} \in I_n := \{y \in X : y \overset{\mathrm{B}}{\leq} x_n\} \tag{1.2}$$

and

$$\varphi(x_{n+1}) - \inf \varphi(I_n) \leq \frac{1}{2} \left[\varphi(x_n) - \inf \varphi(I_n)\right]. \tag{1.3}$$

The claim (1.3) is feasible, since $\varphi(x_n) - \inf \varphi(I_n) > 0$, as consequence of $x_n \in I_n$. Let $y_0 \in X$, such that $x_n \overset{\mathrm{B}}{\geq} y_0\ \forall\, n \geq 0$. It follows that the sequence $(\varphi(x_n))_{n\geq 0}$ is decreasing and lower bounded, therefore convergent, and

$$\varphi(y_0) \leq \lim_{n\to\infty} \varphi(x_n). \tag{1.4}$$

We remark that y_0 verifies (1.1). Indeed, suppose *par absurdum* that $\exists\, z_0 \in X, z_0 \overset{\mathrm{B}}{\leq} y_0$ and

$$\varphi(z_0) < \varphi(y_0), \tag{1.5}$$

which holds true since φ is monotone increasing. Then $z_0 \in I_n$ and hence, from (1.3),

$$\varphi(x_{n+1}) - \frac{1}{2}\varphi(x_n) \leq \frac{1}{2}\varphi(x_0)\ \forall\, n \geq 0.$$

By using (1.4) and taking the limit, we infer $\varphi(y_0) \leq \varphi(z_0)$, in contradiction with (1.5). $\square$

Corollary. *Let $(X, \stackrel{B}{\leq})$ be a pre-ordered set and let $\varphi : X \to \mathbb{R}$ be increasing, with the properties:*

i) for any decreasing sequence $(x_n)_{n\geq 1}$, if $(\varphi(x_n))_{n\geq 1}$ is lower bounded, then there exists y_0 so that $x_n \geq y_0 \; \forall \; n \geq 1$ and $\varphi(x_n) \to \varphi(y_0)$;

ii) $\forall \; x$ of X and $\forall \; \varepsilon > 0$, there exists $z \in X$ so that $x \stackrel{B}{\geq} z$ and $\varphi(x) > \varphi(z) \geq \varphi(x) - \varepsilon$.

Then, for any $x \in X$, the interval

$$I(x) := \{y \in X : y \stackrel{B}{\leq} x\}$$

is surjectively applied by φ onto the interval $(-\infty, \varphi(x)]$.

Proof. Let $x \in X$, and let $a \in (-\infty, \varphi(x))$. Consider the nonempty set

$$X_a := \{y \in I(x) : \varphi(y) \geq a\} \tag{1.6}$$

endowed with the pre-order relation $\stackrel{B}{\leq}$. X_a has the property requested by Theorem 1. Indeed, let $(x_n)_{n\geq 1}$ be a decreasing sequence in X_a. Then $\varphi(x_n) \geq a \; \forall \; n \geq 1$ and hence, according to i), $\exists \; y_0 \in X$ such that $x_n \stackrel{B}{\geq} y_0 \; \forall \; n \geq 1$ and $\varphi(x_n) \to \varphi(y_0)$. But $y_0 \in I(x)$ and $\varphi(y_0) \geq a$, and then, by taking the limit, it follows $y_0 \in X_a$. Since φ is lower bounded on X_a, one can apply Theorem 1; then $\exists \; \nu \in X_a$ with the property

$$z \in X_a, z \stackrel{B}{\leq} \nu \Rightarrow \varphi(z) = \varphi(\nu). \tag{1.7}$$

Hence we have $\varphi(\nu) \geq a$. We assume par absurdum that

$$\varphi(\nu) > a. \tag{1.8}$$

According to ii), for $\varepsilon := \varphi(\nu) - a$ and for ν, there exists $z \in X$ such that $z \stackrel{B}{\leq} \nu$ and $\varphi(\nu) > \varphi(z) \geq \varphi(\nu) - \varepsilon$. Hence $\varphi(z) \in [a, \varphi(\nu))$, and consequently we have $z \in X_a$, since $z \in I(x)$, and , as well, $\varphi(z) < \varphi(\nu)$, which is in contradiction with (1.7). □

Remark. This Corollary is the resumption of Corollary 5 from [2], which appears also in [1], where the condition ii) is incomplete ($\varphi(z) \geq \varphi(x) - \varepsilon$), and where the proof is erroneous.

The author presently achieves the completion of ii) with the correct condition: $\varphi(x) > \varphi(z) \geq \varphi(x) - \varepsilon$, and also provides an accurate proof. This is important since, by recovering the Corollary obtained by Brezis and Browder and developed by Ekeland, Theorem 2 remains valid.

2 Results in nonlinear semigroups

Let (X, d) be a metric space and let S be a *nonlinear semigroup* on X (for every $t \in [0, +\infty)$, there exists a map $S : X \to X$ having the properties:

$$S(0) = \text{id}_x \text{ (identical map of X)}, \tag{2.1}$$

$$S(t_1 + t_2) = S(t_1) \circ S(t_2)\ \forall\, t_1, t_2 \in [0, +\infty)]. \tag{2.2}$$

Suppose S is a *continuous semigroup of contractions*, i.e.,

$$\forall\, u \in X \quad \text{the map } t \to S(t)u \text{ is continuous on } [0, +\infty), \tag{2.3}$$

$$\forall\, t \geq 0, \forall\, u, \nu \in X\ \mathrm{d}(S(t)u, S(t)\nu) \leq \mathrm{d}(u, \nu). \tag{2.4}$$

According to [1], we associate to both S and $\rho \geq 0$, a binary relation $\underset{\approx}{<}$ in $X \times [0, +\infty)$, as follows:

$$(u, a) \underset{\approx}{>} (\nu, b) \overset{\text{def}}{\Leftrightarrow} b \geq a \text{ and } \mathrm{d}(S(b - a)u, \nu) \leq \rho(b - a). \tag{2.5}$$

Remark. We note that

$$(u, a) \underset{\approx}{>} (\nu, b) \text{ and } (u, a) \neq (\nu, b) \Rightarrow b > a, \tag{2.6}$$

Let V be a complete metric space, let S be a continuous semigroup of contractions on V and $\rho \geq 0$. Consider the function $\varphi : V \times [0, +\infty) \to \mathbb{R}$,

$$\varphi(u, a) = -a. \tag{2.7}$$

The mapping φ is increasing w.r.t. $\underset{\approx}{>}$. Moreover, φ satisfies the condition i) from the Corollary of Theorem 1, where $X = F \times [0, +\infty), F \subset V$ is closed and $\overset{\mathrm{B}}{\leq}$ is $\underset{\approx}{<}$ ([1, Lemma 1]).

Theorem 2. *Let V be a complete metric space, S a continuous semigroup of contractions on V, F a closed subset and $\rho \geq 0$ so that*

$$\underline{\lim}_{t \to 0} \frac{1}{t} \mathrm{dist}(S(t)u, F) \leq \rho\ \forall\, u \in F. \tag{2.8}$$

Then

$$\mathrm{dist}(S(t)u, F) \leq \rho t\ \forall\, u \in F,\ \forall\, t \geq 0.$$

Proof. Let α satisfy

$$\alpha > \rho. \tag{2.9}$$

We apply the Corollary of Theorem 1 with $X = F \times [0, +\infty)$ and $\underset{\approx}{<}$, with ρ replaced by α (cf. (2.5)) as $\overset{\mathrm{B}}{\leq}$, φ like in (2.7). As mentioned before, φ is increasing and i) is satisfied. We now consider ii); let $(u, a) \in X$ and let $\varepsilon > 0$. The relation (2.8) provides $t \in (0, \varepsilon)$ such that

$$\mathrm{dist}(S(t)u, F) < \alpha t. \tag{2.10}$$

Then, let $\nu \in F$ be such that $\mathrm{d}(S(t)u, \nu) < t$. Taking $b := a + t$ it follows that $(u, a) \overset{(2.5)}{\underset{\approx}{>}} (\nu, b)$ and $\varphi(u, a) > \varphi(\nu, b) \geq \varphi(u, a) - \varepsilon$, 2) is satisfied.

Therefore, let $u \in F$ and consider $t \geq 0$ arbitrary. Then $\varphi(u,t) = -t, \varphi(u,0) = 0$, and the surjectivity of φ relative to the interval $(-\infty, \varphi(u,0))$ provides $\nu \in F$ such that $(\nu, t) \underset{\approx}{<} (u,0)$, that is (cf. (1.2))

$$\mathrm{d}(S(t)u, \nu) \leq \alpha t. \tag{2.11}$$

From (2.11), it results $\mathrm{dist}(S(t)u, \nu) \leq \alpha t$, and taking into account (2.9), one obtains the conclusion. □

Remark. Taking $\rho = 0$ in Theorem 2, it follows that F is invariant relative to the semigroup S.

Theorem 3. *Let V, S, F be like in Theorem 2 and let*

$$c := \underline{\lim}_{t \to +\infty} \frac{1}{t} \mathrm{dist}(S(t)u, F). \tag{2.12}$$

Then c remains constant while u describes F and ([6], [1])

$$c = \sup_{u \in F} \inf_{t > 0} \frac{1}{t} \mathrm{dist}(S(t)u, F).$$

Proof. Let u, ν be arbitrary in F and let

$$c_u = \underline{\lim}_{t \to +\infty} \mathrm{dist}(S(t)u, F), \; c_\nu = \underline{\lim}_{t \to +\infty} \mathrm{dist}(S(t)\nu, F).$$

Assume par absurdum that $c_u \neq c_\nu$, for instance $c_u < c_\nu$. Then

$$|\mathrm{dist}(S(t)u, F) - \mathrm{dist}(S(t)\nu, F)| \leq \mathrm{d}(S(t)u, S(t)\nu) \overset{(2.4)}{\leq} \mathrm{d}(u, \nu),$$

hence

$$\lim_{t \to +\infty} \frac{1}{t} \left[\mathrm{dist}(S(t)u, F) - \mathrm{dist}(S(t)\nu, F)\right] = 0. \tag{2.13}$$

Let $r' > 0$ be such that

$$c_u < \inf_{t \in [r', +\infty)} \frac{1}{t} \mathrm{dist}(S(t)\nu, F) =: \alpha. \tag{2.14}$$

We consider $\beta \in (c_u, \alpha)$ and a sequence $(r_n)_{n \geq 1},\ r_n \geq r',\ r_n \to +\infty$. Since

$$\inf_{t \in [r_n, +\infty)} \frac{1}{t} \mathrm{dist}(S(t)u, F) \leq c_u < \beta < \alpha \; \forall\, n \geq 1,$$

we can find a sequence $(t_n)_{n \geq 1},\ t_n \in [r_n, +\infty), \frac{1}{t_n} \mathrm{dist}(S(t_n)u, F) \leq \beta < \alpha \; \forall\, n \geq 1$, and hence $\frac{1}{t_n} \mathrm{dist}(S(t_n)u, F) \leq \beta < \alpha \leq \frac{1}{t_n} \mathrm{dist}(S(t_n)\nu, F)$, $\frac{1}{t_n} \left[\mathrm{dist}(S(t_n)\nu, F) - \mathrm{dist}(S(t_n)u, F)\right] \geq \alpha - \beta \; \forall\, n \geq 1$, which is in contradiction with (2.13), since $t_n \to +\infty$. One concludes that $c_u = c_\nu$.

We have as well

$$\sup_{u\in F}\inf_{t>0}\frac{1}{t}\mathrm{dist}(S(t)u,F)\le c. \tag{2.15}$$

Indeed, assuming par absurdum the contrary, we find $u_1 \in F$ such that $c < \inf_{t>0}\frac{1}{t}\mathrm{dist}(S(t)u_1,F)$, which leads to a contradiction, since $c = \underline{\lim}_{t\to+\infty}\frac{1}{t}\mathrm{dist}(S(t)u_1,F)$.

We denote by l the first member of (2.15) and let ρ be arbitrary, satisfying

$$\rho > l. \tag{2.16}$$

We endow the set $X := F\times[0,+\infty)$ with the order relation $\underset{\approx}{<}$ (cf. (2.5)) and let φ be the function in (2.7). The condition (1.1) in Theorem 1 is not satisfied. Indeed, let $(u,a)\in X$. Then (2.6) yields $t>0$, such that $\mathrm{dist}(S(t)u,F)<\rho t$, hence $\exists\,\nu\in F$ with $\mathrm{d}(S(t)u,\nu)<\rho t$. Taking $b := a+t$ we infer $(\nu,b)\underset{\approx}{<}(u,a)$ and $\varphi(\nu,b) = -b = \varphi(u,a)-t<\varphi(u,a)$.

Consider for any $(u,a)\in X$ the interval $S(u,a) := (\leftarrow,(u,a))$ in X. But φ is not lower bounded on $S(u,a)$. Indeed, suppose par absurdum the contrary. This imposes that any decreasing sequence $(u_n,a_n)_{n\ge1}$ from $S(u,a)$ has a lower bound in $S(u,a)$, otherwise, using (1.3), one can find a subsequence $(u_{k_n},a_{k_n})_{n\ge1}$ with $a_{k_n}\to+\infty$, which yields $\lim_{n\to\infty}\varphi(u_{k_n},a_{k_n}) = -\infty$, a contradiction. Therefore, we can apply Theorem 1 to $S(u,a)$, $\exists\,(\nu,b)$ in $S(u,a)$, and hence in X, having the property (1.1), fact which has been already proven as being impossible (see above); φ is not bounded from below on any $S(u,a)$.

Then, let $u\in F$. By using the last claim, one can find a sequence $(u_n,t_n)_{n\ge1}$ in $S(u,0)$ with $t_n\to+\infty$ and $(u,0)\underset{\approx}{>}(u_n,t_n)\ \forall\, n\ge1$, that is $\mathrm{d}(S(t_n)u,u_n)\le\rho t_n\ \forall\, n\ge 1$; this imposes (cf. [5, vol. II, page 651, (11.9_1)], that $\underline{\lim}_{t\to+\infty}\frac{1}{t}\mathrm{dist}(S(t)u,F)\le\rho$ and by comparing with (2.6) one obtains, since ρ is arbitrary, that

$$\underline{\lim}_{t\to+\infty}\frac{1}{t}\mathrm{dist}(S(t)u,F)\le l.$$

By comparing this with (2.15) the proof ends (see also (2.12)). □

3 Results in dynamical systems

We further transfer these results to dynamical systems. Firstly, we introduce several notations and basics of nonlinear semigroups.

Let X be metric space. The continuous map $f : X\times\mathbb{R}\to X$, with the properties:

$I.\quad f(p,0) = p\ \forall\, p\in X,$

$II.\quad f(f(p,t_1),t_2) = f(p,t_1+t_2)\ \forall\, p\in X,\ \forall\, t_1,t_2\in\mathbb{R}$

is a *dynamical system* in X. We also introduce the notion of *contraction for dynamical systems*, i.e.

$$\forall\, t \in \mathbb{R},\ \forall\, p, q \in X,\ \mathrm{d}(f(p,t), f(q,t)) \leq \mathrm{d}(p,q).$$

The following statements hold true:

Theorem 4. *Let X be a complete metric space, f a dynamical system of contractions on X, F a closed subset of X and $\rho \geq 0$ so that*

$$\varlimsup_{t\to 0\pm} \frac{1}{|t|}\mathrm{dist}(f(p,t), F) \leq \rho,\ \forall\, p \in F.$$

Then

$$\mathrm{dist}(f(p,t), F) \leq \rho|t|\ \forall\, p \in F,\ \forall\, t \in \mathbb{R}.$$

Proof. We follow the idea from the proof of Theorem 2. The condition of continuity for the semigroup S is fulfilled by the stronger condition which is included in the definition of the dynamical system (f is just continuous, not only continuous relative to t as it was for S). S has $t \in [0, +\infty)$ and f in $\mathbb{R}$; hence we divide the problem into two similar problems, one for $t \in [0, +\infty)$ and the other for $t \in (-\infty, 0]$. For the second situation, instead of f with $t \in (-\infty, 0]$, one should consider the function g with $t \in [0, +\infty)$, $g(p,t) = f(p,-t)$, $\forall\ p \in X$, $\forall\ t \in (-\infty, 0]$ and replace t by $-t \in [0, +\infty)$. □

Theorem 5. *Let X, f, F be like in Theorem 4 and let*

$$c_1 := \varliminf_{t\to+\infty} \frac{1}{t}\mathrm{dist}(f(p,t), F),\ c_2 := \varliminf_{t\to-\infty} \frac{1}{|t|}\mathrm{dist}(f(p,t), F).$$

Then c_1 and c_2 remain constant while p describes F and

$$c_1 = \sup_{p\in F} \inf_{t>0} \frac{1}{t}\mathrm{dist}(f(p,t), F),\ c_2 = \sup_{p\in F} \inf_{t<0} \frac{1}{|t|}\mathrm{dist}(f(p,t), F).$$

Proof. We adapt the proof of Theorem 3 to $f \mid X \times [0, +\infty)$ and to $g : X \times [0, +\infty) \to X$, $g(p,t) = f(p,-t)$, which is also continuous, in particular, continuous relative to the second variable as a composition of continuous functions. In the second case, $c_2 := \varliminf_{t\to-\infty} \frac{1}{|t|}\mathrm{dist}(f(p,t), F)$ implies $c_2 := \varliminf_{t\to+\infty} \frac{1}{t}\mathrm{dist}(g(p,t), F)$, so $c_2 = \sup_{p\in F} \inf_{t>0} \frac{1}{t}\mathrm{dist}(g(p,t), F)$, which gives $c_2 = \sup_{p\in F} \inf_{t<0} \frac{1}{|t|}\mathrm{dist}(f(p,t), F)$. □

Theorems 4 and 5 are converse to each other in a certain sense. To have a better understanding of this fact, we give an intuitive description.

Imagine $f(p,t)$ as the position at time t of a particle escaping from a box F which has left at time 0 from the point $p \in$ bdy F.

The rate $\dfrac{d(f(p,t), F)}{t}$ will be called the mean escape speed; when $t \to 0+$ we get the normal exit speed, and when $t \to +\infty$ we get the asymptotic mean speed, which does not depend on the starting point anymore. Theorem 4 states that if all the normal exit speeds are less than C, then such are all the mean escape speeds (and hence the asymptotic mean speed). Theorem 5 states that if the asymptotic mean speed is c, for any $\varepsilon > 0$, then some trajectory can be found, along which the mean escape speed (and hence the normal exit speed) is always greater than $c - \varepsilon$. Hence the relationship $c \leq C$ is fulfilled. This remark could also apply to the second order dynamical systems from [7] and to the adjoint multidimensional acausal systems from ßso.

4 Conclusions

The Corollary of Theorem 1 has been recovered by modifying condition ii) and providing a correct proof. In this way Theorem 2 remains valid, and such remain the accompanying results. The novelty of the paper consists in adapting Theorem 2 and Theorem 3 to dynamical systems, thus obtaining the corresponding results Theorem 4 and Theorem 5. Several applications of these statements (Theorem 2 - Theorem 5), will be further developed a forecoming paper.

References

[1] H. Brezis and F. Browder, *A general ordering principle in nonlinear functional analysis*, Advances in Math. 21, (1976), 355-364.

[2] I. Ekeland, *Nonconvex minimization problems*, Bull. Amer. Math. Soc. 1, (1979), 443-474.

[3] I. Ekeland, *On the variational principle*, Cahiers de mathématique de la décision, No. 7217, Université Paris, 1972.

[4] I. Ekeland, *On the variational principle*, J. Math. Anal. and Appl. 47, (1974), 324-353.

[5] C. Meghea, and I. Meghea, *Differential Calculus and Integral Calculus*, vol. I, Technical House Eds., 1997; vol. II, Technical House Eds., Bucharest, 2000; vol. III, Printech Eds., Bucharest, 2002.

[6] I. Meghea, *Ekeland variational principle with generalizations and variants*, Old City Publishing, Philadelphia; Archives Contemporaines, Paris, 2009.

[7] V. Obadeanu, *Superfields associated to second order dynamical systems of some deformable continuous media*, Balkan Society of Geometers Proceedings (BGSP) 15 (2008), 137-147.

[8] V. Prepelita, *Adjoint multidimensional acausal systems*, Balkan Society of Geometers Proceedings (BGSP) 15 (2008), 176-190.

Author's address:

Irina Meghea
University POLITEHNICA of Bucharest,
Faculty of Applied Sciences,
Department of Mathematics II,
Splaiul Independenţei 313,
RO-060042, Bucharest, Romania.
E-mail: irina.meghea@upb.ro