On a stability analysis method of dynamical systems with numerical tests

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Abstract. The research is focused on the study of the stability in sense of Liapunov for evolution of the dynamical systems that depend of parameters. An original hypothesis referred on the stable and unstable zones, in the plane of chosen principal parameters of the system, is formulated. A method for identification of the stable and unstable regions in the plane of principal parameters of the mathematical model, using our hypothesis, is related in the paper. We study the dependence of the motion stability by some parameters of the mathematical model using dimensional values of the parameters.

M.S.C. 2000: 34D20.
Key words: Liapunov stability, dynamical system, parametric stability.

1 Introduction

Many authors as Floquet [2], Hayashi [3], Mathieu [5], Minorski [11], Poincaré [15], Timosenko [16], has studied models of mechanical systems for which has analyzed the motion stability.

Evolution of the dynamical system that depends of parameters is studied through the stability properties of them [1], [4], [7], [8], [14], [17]. An algorithm for identify the stability regions, in the plane of chosen principal parameters of the system, using our hypothesis on the possibility of separation between the stability and instability zones, is described.

BSG Proceedings 16. The Int. Conf. of Diff. Geom. and Dynamical Systems (DGDS-2008) and The V-th Int. Colloq. of Mathematics in Engineering and Numerical Physics (MENP-5) - math. sections, August 29 - September 2, 2008, Mangalia, Romania, pp. 99-107.

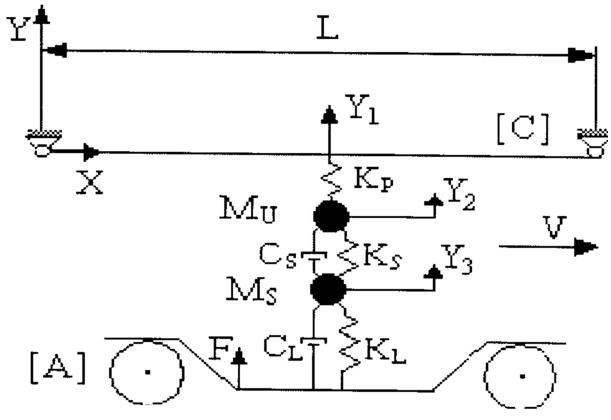

Fig.1 Physical model

The method is applied on the mathematical model attached to physical model (fig.1) of the pantograph-contact wire dynamical system of the electrical locomotive [8], [9].

The concrete physical model consist from a vehicle [A] in a uniform linear motion which compresses with a force F an oscillating system, compound of two sprung concentrated masses, on the wire [C]. The oscillating system is moving with a constant speed in the same time with the vehicle [A].

We study the motion stability of the system, using modified values of the system parameters of the mathematical model as horizontal tension in the wire, the speed specified in the model, the stiffness or damping elements of the system. The dimensional values of the parameters are used in the numerical tests that permit to appreciate if the values of the parameters or of the results are realistic.

2 Mathematical model of pantograph

The physical model presented in fig.1 is described with the following equations of motion (see also [1], [4], [8], [14]):

$$\begin{aligned} &M_s\ddot{y}_3 + c_s(\dot{y}_3 - \dot{y}_2) + k_s(y_3 - y_2) + k_L y_3 + c_L\dot{y}_3 = 0 \\ &M_u\ddot{y}_2 + c_s(\dot{y}_2 - \dot{y}_3) + k_s(y_2 - y_3) + k_p(y_2 - y_1) = 0 \\ &EI\frac{\partial^4 y}{\partial x^4} - T\frac{\partial^2 y}{\partial x^2} + \beta\frac{\partial y}{\partial t} + m\frac{\partial^2 y}{\partial t^2} = f(t)w(x,t) \end{aligned} \tag{2.1}$$

The system of equations (2.1) can be substituted by the equivalent system as below:

$$\begin{aligned} &M_s\ddot{y}_3 + c_s(\dot{y}_3 - \dot{y}_2) + k_s(y_3 - y_2) + k_L y_3 + c_L\dot{y}_3 = 0 \\ &M_u\ddot{y}_2 + M_s\ddot{y}_3 + k_L y_3 + c_L\dot{y}_3 + k_p(y_2 - y_1) = 0 \\ &EI\frac{\partial^4 y}{\partial x^4} - T\frac{\partial^2 y}{\partial x^2} + \beta\frac{\partial y}{\partial t} + m\frac{\partial^2 y}{\partial t^2} = f(t)w(x,t) \end{aligned} \tag{2.2}$$

where y_1, y_2, y_3 are respectively, the deflections of the wire compressed by the oscillating system, the deflections of the centroid of the masses M_u and M_s from the equilibrium position, $y(x,t)$ is the deflection of the wire, EI is the bending stiffness of the wire, T is horizontal tension in the wire, β is viscous damping of the wire, m is the mass per unit length of the wire, c_s and c_L are damping coefficients, k_s, k_L and k_p are stiffness elements of the system, $y_0 = y(x,0)$ is the micro-irregularity in x, d is the length of the contact zone between pantograph and contact wire, L is the length of the span for the wire, g is gravity acceleration, t is the time and

$$f(t) = (M_s + M_u)g - (F + M_u\ddot{y}_2 + M_s\ddot{y}_3 + k_L y_3 + c_L\dot{y}_3)$$

$$w(x,t) = \begin{cases} \dfrac{1}{d} & \text{for} \quad vt - \dfrac{d}{2} \le x \le vt + \dfrac{d}{2} \\ 0 & \text{in the rest} \end{cases}$$

$$y_1(t) = \int_0^L [y_0(x) + y(x,t)]w(x,t)dx = \frac{1}{d}\int_{vt-\frac{d}{2}}^{vt+\frac{d}{2}} [y_0(x) + y(x,t)]dx$$

We introduce dimensionless variables and parameters (see also [1]) (where the fundamental natural frequency of the wire span is ω_1, the maximum displacement of a simply supported wire when a unit load is applied in its midpoint is y_{st}), as follows:

$$\begin{aligned} &\xi = \frac{\pi x}{L},\ \tau = \frac{\pi v t}{L},\ \tilde{v} = \sqrt{\frac{m}{T}}\,v = \tilde{v}_T,\ \tilde{M} = \frac{M_s + M_u}{mL},\ \tilde{g} = \frac{48}{\pi^4 \tilde{M}\tilde{v}^2} \\ &y_{st} = \frac{gL^3}{48EI},\ \tilde{y} = \frac{y}{y_{st}},\ \mu = \frac{M_u}{M_s + M_u},\ \tilde{\omega}_{ns} = \frac{L}{\pi v}\sqrt{\frac{k_s}{M_s + M_u}}, \\ &\tilde{\omega}_{nL} = \frac{L}{\pi v}\sqrt{\frac{k_L}{M_s + M_u}},\ \tilde{\Omega}_n = \frac{L}{\pi v}\sqrt{\frac{k_p}{M_s + M_u}},\ \tilde{v}_\beta = \frac{m\pi}{\beta L}v, \\ &\zeta_s = \frac{c_s}{2\sqrt{k_s(M_s + M_u)}},\ \zeta_L = \frac{c_L}{2\sqrt{k_L(M_s + M_u)}},\ \Delta = \frac{\pi d}{L},\ \tilde{v}_{EI} = \frac{L}{\pi}\sqrt{\frac{m}{EI}}\ v \end{aligned} \tag{2.3}$$

Equations (2.2) have now the following form, using (2.3),:

$$\begin{aligned} &(1-\mu)\ddot{\tilde{y}}_3 + 2\zeta_s\tilde{\omega}_{sn}(\dot{\tilde{y}}_3 - \dot{\tilde{y}}_2) + \tilde{\omega}_{sn}^2(\tilde{y}_3 - \tilde{y}_2) + \tilde{\omega}_{nL}^2\tilde{y}_3 + 2\zeta_L\tilde{\omega}_{nL}\dot{\tilde{y}}_3 = 0 \\ &\mu\ddot{\tilde{y}}_2 + (1-\mu)\ddot{\tilde{y}}_3 + \tilde{\Omega}_n^2(\tilde{y}_2 - \tilde{y}_1) + \tilde{\Omega}_{nL}^2\tilde{y}_3 + 2\zeta_L\tilde{\omega}_{nL}\dot{\tilde{y}}_3 = 0 \\ &\frac{1.}{\tilde{v}_{EI}^2}\frac{\partial^4\tilde{y}}{\partial\xi^4} - \frac{1.}{\tilde{v}_T^2}\frac{\partial^2\tilde{y}}{\partial\xi^2} + \frac{1.}{\tilde{v}_\beta^2}\frac{\partial\tilde{y}}{\partial\tau} + \frac{\partial^2\tilde{y}}{\partial\tau^2} = \tilde{f}(\tau)\tilde{w}(\xi,\tau). \end{aligned} \tag{2.4}$$

The following notations are used in equation (2.4):

$$\begin{aligned} &\tilde{f}(\tau) = \tilde{M}\pi(\tilde{g} - \mu\ddot{\tilde{y}}_2 - (1-\mu)\ddot{\tilde{y}}_3 - \tilde{\omega}_{nL}^2\tilde{y}_3 - 2\zeta_L\tilde{\omega}_{nL}\dot{\tilde{y}}_3) \\ &\tilde{w}(\xi,t) = \begin{cases} \dfrac{1.}{\Delta} & \text{for} \quad \tau - \dfrac{\Delta}{2} \le \xi \le \tau + \dfrac{\Delta}{2} \\ 0 & \text{in the rest} \end{cases} \\ &\tilde{y}_1 = \frac{1}{\Delta}\int_{\tau-\frac{\Delta}{2}}^{\tau+\frac{\Delta}{2}} [\tilde{y}(\xi,\tau) + \tilde{y}_0(\xi)]d\xi \end{aligned} \tag{2.5}$$

We study the third equation of (2.4), where is neglected the constant force of the right hand for analysis the motion stability of the system, as follows:

$$\begin{aligned}\frac{1.}{\tilde{v}_{EI}^2}\frac{\partial^4\tilde{y}}{\partial\xi^4}-\frac{1.}{\tilde{v}_T^2}\frac{\partial^2\tilde{y}}{\partial\xi^2}+\frac{1.}{\tilde{v}_\beta}\frac{\partial\tilde{y}}{\partial\tau}+\frac{\partial^2\tilde{y}}{\partial\tau^2}=\\ =-\tilde{M}\pi(\mu\ddot{\tilde{y}}_2+(1-\mu)\ddot{\tilde{y}}_3+\tilde{\omega}_{nL}^2\tilde{y}_3+2\zeta_L\tilde{\omega}_{nL}\dot{\tilde{y}}_3)\tilde{w}(\xi,\tau)\end{aligned} \tag{2.6}$$

The solution of the equation (2.6) is expanded after its natural modes $\sin j\,\xi$, $j=1,2,\dots$ (corresponding to a simply supported wire) and after the same system of functions is developed the function $\tilde{w}(\xi,\tau)$. Thus we have the following expression of them:

$$\begin{aligned}\tilde{y}(\xi,\tau)&=\sum_{j=1}^{\infty}T_j(\tau)\sin j\xi\\ \tilde{w}(\xi,\tau)&=\frac{2.}{\pi}\sum_{j=1}^{\infty}\frac{\sin j\Delta}{j\Delta}\sin j\xi\sin j\tau\end{aligned} \tag{2.7}$$

The functions $T_j(\tau)$ are defined by the following differential equations:

$$\begin{aligned}\frac{d^2T_j}{d\tau^2}+\frac{1.}{\tilde{v}_\beta}\frac{dT_j}{d\tau}+\left(\frac{j^4}{\tilde{v}_{EI}^2}+\frac{j^2}{\tilde{v}_T^2}\right)=\\ =-\tilde{M}\pi(\mu\ddot{\tilde{y}}_2+(1-\mu)\ddot{\tilde{y}}_3+\tilde{\omega}_{nL}^2\tilde{y}_3+2\zeta_L\tilde{\omega}_{nL}\dot{\tilde{y}}_3)\frac{2.}{\pi}\frac{\sin j\Delta}{j\Delta}\sin j\tau\end{aligned} \tag{2.8}$$

The displacement $\tilde{y}_1(\tau)$ has the expression:

$$\tilde{y}_1(\tau)=\sum_{j=1}^{\infty}Tj(\tau)\sin j\tau \tag{2.9}$$

In equation (2.9) we consider an approximation with five terms.

Finally, the dimensionless system of equations that specifies the state problem for the dynamic system, described by fig.1, is:

$$\begin{aligned}&(1-\mu)\ddot{\tilde{y}}_3+2\zeta_s\tilde{\omega}_{sn}(\dot{\tilde{y}}_3-\dot{\tilde{y}}_2)+\tilde{\omega}_{nL}^2(\tilde{y}_3-\tilde{y}_2)+\tilde{\omega}_{nL}^2\tilde{y}_3+2\zeta_L\tilde{\omega}_{nL}\dot{\tilde{y}}_3=0\\ &\mu\ddot{\tilde{y}}_2+(1-\mu)\ddot{\tilde{y}}_3+\tilde{\Omega}_n^2\left(\tilde{y}_2-\sum_{j=1}^{\infty}[Tj(\tau)+w_j]\frac{\sin j\Delta}{j\Delta}\sin j\tau\right)+\\ &+\tilde{\omega}_{nL}^2\tilde{y}_3+2\zeta_L\tilde{\omega}_{nL}\dot{\tilde{y}}_3=0\\ &\frac{d^2T_j}{d\tau^2}+\frac{1.}{\tilde{v}_\beta}\frac{dT_j}{d\tau}+\left(\frac{j^4}{\tilde{v}_{EI}^2}+\frac{j^2}{\tilde{v}_T^2}\right)T_j=-2\tilde{M}(\mu\ddot{\tilde{y}}_2+(1-\mu)\ddot{\tilde{y}}_3+\tilde{\omega}_{nL}^2\tilde{y}_3+\\ &+2\zeta_L\tilde{\omega}_{nL}\dot{\tilde{y}}_3)\sin j\tau,\quad j=1,2,\dots\end{aligned} \tag{2.10}$$

In equation system (2.10) we use the notations:

$$\tilde{v}_{EI}^2=\frac{mL^2}{EI\pi^2}v^2,\quad \tilde{v}_T^2=\frac{m}{T}v^2,\quad \tilde{v}_\beta=\frac{m\pi}{\beta L}v.$$

The initial conditions for the system of equations (2.10), considered known, that assure the uniqueness of the solution, are:

$$\tilde{y}_3(0) = \tilde{y}_{03},\ \dot{\tilde{y}}_3(0) = \dot{\tilde{y}}_{03},\ \tilde{y}_2(0) = \tilde{y}_{02},$$
$$\dot{\tilde{y}}_2(0) = \dot{\tilde{y}}_{02},\ T_i(0) = T_{0i},\ \dot{T}_i(0) = \dot{T}_{0i},\ i = 1, 2, \ldots$$

3 Hypothesis of separation

We formulate the following hypothesis, supposed to be respected of the stable and unstable solutions of the dynamical system:

If $y : I \to R^n$ is a stable solution of the system $y' = Ay$, with matrix A of continuous components, defined by parameters, for fixed parameters, we suppose that there exist a neighborhood of fixed parameters where the solution y is also stable. For an unstable solution of the system we suppose that we can formulate analogue property.

An algorithm for identify the stable and unstable regions, in the plane of principal parameters of the dynamical system, will be defined by the authors, in the case that our hypothesis referred to concrete dynamical system is true. We use this hypothesis analogue with hypothesis of separability of variables in solution of differential equations.

Our algorithm for separation of stable and unstable zones, in the plane of chosen two principal parameters of the dynamical system, consist in covering of the fixed domain for analysis, in the parameters plane, with a sufficient fine mesh and to study the evolution of the specified solution of the dynamical system, in the mesh points. The separation between stable and unstable solutions, in the plane of two principal parameters, is realized by curves that define periodical solutions. In the neighborhood of the periodic points of the parameters plane one can use a refined mesh.

For other theoretical recent researches in the area can see also [6], [12], [13].

4 Numerical tests

The above theoretical exposure is analysed in this paragraph for concrete physical model described by fig.1. The values of the contact wire parameters, used here for all the tests, are defined by the span length $L = 60[m]$, by the mass of the wire per unit length $m = 0.63[kg/m]$ and by viscous damping $\beta = 0.1[Ns/m]$. The values of the pantograph dynamical system parameters, used here for all the tests, are defined by the concentrated masses $M_u = 2.2[kg]$, $M_s = 19.8[kg]$, the value of the rapport $\mu = M_u/(M_s + M_u) = 0.1$ and $\zeta_s = 0.3$, $\zeta_L = 0.3$. The physical model of pantograph chosen here is obtained by decomposition of pantograph mass in two concentrated masses with hypothetical values of the parameters and with a hypothetical design. The stiffness $k_p = 2250M/m$ is also used in the mathematical model calculation.

4.1 Identification of the stability and instability zones

For the test, we consider also the following fixed dimensionless values of the parameters:

$$\tilde{\Omega}_n = 3.185, \quad \tilde{v}_\beta - 19.8, \quad \tilde{\omega}_{nL} = 0.48, \quad \tilde{v}_{EI} = 90.96. \tag{4.1}$$

The corresponding real values of the parameters are:

$$v = \frac{L}{\pi\tilde{\Omega}_n}\sqrt{\frac{k_p}{M_s + M_u}} = 60.64[m/s], \quad \beta = \frac{m\pi}{\tilde{v}_\beta L}v = 0.1[Ns/m],$$

$$\sqrt{k_L} = \frac{\pi v \tilde{\omega}_{nL}}{L}\sqrt{M_s + M_u} = 7.148[N/m], \quad \sqrt{EI} = \frac{L}{\pi\tilde{v}_{EI}}\sqrt{m} = 10.1[Nm^2],$$

$$EI = 102[Nm^2].$$

The free parameters in the plane of parameters are in this case dimensionless variables $\tilde{\omega}_{ns}$ and $\tilde{v}_T$ for which correspond, respectively, stiffness element k_s of the system and horizontal tension in the wire T, real parameters of the dynamical system.

We analyze the stability of motion for the dimensionless displacement $\tilde{y}_2$ of the concentrated mass M_u in the free vibration of the system.

In fig.2 is plotted with continuous line the domain of periodic solutions of $\tilde{y}_2$ in the two chosen parameters plane defined by the variables $\tilde{\omega}_{ns}$ and $\tilde{v}_T$, in the case of free vibrations of the system.

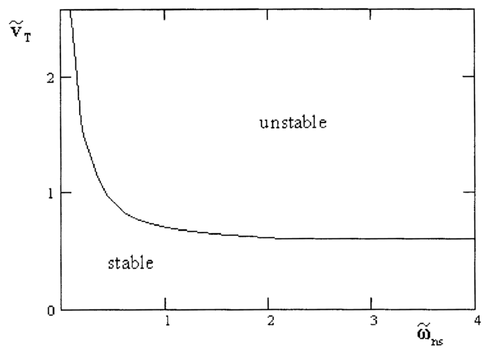

Fig.2 Stable and unstable zones in the plane of parameters

On the figure are specified the stable and unstable zones of the displacement $\tilde{y}_2$. The algorithm, specified in section 3, has been applied for perform the stability regions from fig.2.

4.2 Influence of pantograph speed on the system stability

In this section we study influence of the pantograph speed on the stability of the dynamical system described by equations (2.10), where $j = 1, 2, \ldots, 5$. Evolution of the displacement $\tilde{y}_2$, of the concentrated mass M_u, is used for analyze the motion stability of the physical model from fig.1.

The last real values of the parameters, that define the mathematical model of dynamical system tested here, are $k_s = 40N/m$, $T = 1500N$ and $v = 25m/s$, $30m/s$, $35m/s$.

In fig.3 are plotted evolutions of the displacement $\tilde{y}_2$ versus dimensionless variable τ that correspond to real variable of time. For fixed values of the parameters of the dynamical system, with exception of the speed v, we have identified existence of the critical value of the speed v of the dynamical system, looking the curves drown in fig.3.

By choosing of other free parameter of the system, as tension T in the wire, we can identify other critical values of the dynamical system described by physical model from fig.1.

The values of the free parameter, chosen in the neighborhood of the critical value, permit us to find influence of the all other parameters of the dynamical system pantograph – contact wire on its evolution.

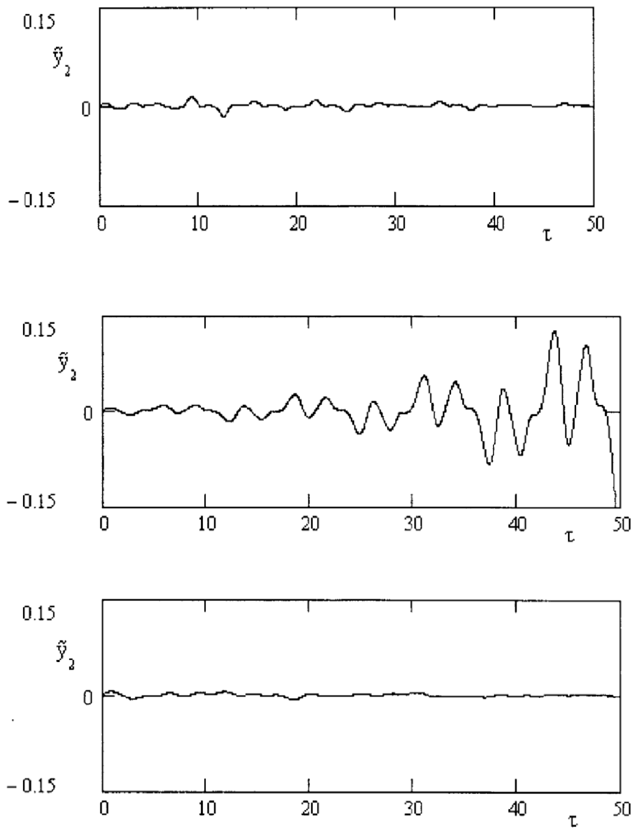

Fig.3 Influence of speed $v = 25m/s$, $30m/s$, $35m/s$, on mass M_u evolution

The results of this section underline possibility to study evolution of our concrete dynamical system by the mathematical model attached to physical model.

5 Conclusions

A hypothesis of separation of the stable and unstable solutions, in the plane of two principal parameters of the dynamical system is formulated by the authors. This hypothesis is applied for identification of the stable and unstable zones such as hypothesis of separation of variables in solution of differential equations. An algorithm for identification of the stable and unstable zones of the dynamical system, using our hypothesis, is defined. The method proposed is applied to dynamical system pantograph-contact wire with physical model and mathematical model defined using real and dimensionless values of the parameters. A design of pantograph approximated by two sprung superposed masses, with concrete values of the parameters, may be substituted by other model with other stability structure and other critical values of the parameters of the dynamical system.

For fixed values of the parameters of the dynamical system, with exception of one free parameter, we can perform critical value of this parameter.

The values of the free parameter, chosen in the neighborhood of the critical value, permit us to discover influence of the all other parameters of the dynamical system on its evolution.

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