The spectra of a class of convolution operators

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Abstract. The aim of this paper is to obtain results concerning the spectra of a class of convolution operators and to develop a numerical Matlab procedure to compute the spectra of these operators. The paper is a continuation of our previous works [4], [5]. We define the convolution operator $K_\mu : L^2(\mathbf{R}, H) \to L^2(\mathbf{R}, H)$ by $K_\mu x = \mu * x$, where μ is a bounded, Borelian, regular, $\mathcal{B}(H)$- valued measure on $\mathbf{R}$. In Theorem 2.1, it is proved that such a convolution operator K_μ and a multiplication operator $M_{\widehat{\mu}}$ are unitarily equivalent. In Theorem 2.2, we compute the spectra of the convolution operator K_μ, for a $\mathcal{B}(\mathbf{C}^n)$- valued measure μ.

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1 Introduction

In this paper we extend to continuous time- invariant systems, the results obtained in [6].

Let $\mathbf{R}$ be the aditive group of real numbers, let $(H, \|\cdot\|)$ be a complex Hilbert space and let $(\mathcal{B}(H), \|\cdot\|)$ be the Banach algebra of all linear bounded operators on H. Let $L^1(\mathbf{R}, H)$ and $L^2(\mathbf{R}, H)$ be the usual Hilbert spaces of H- valued, integrable (square integrable) functions on $\mathbf{R}$. Let $(\mathcal{M}(\mathbf{R}, \mathcal{B}(H)), \|\cdot\|_1)$ be the (non-commutative) Banach algebra of all bounded, Borelian, regular, $\mathcal{B}(H)$- valued measures on $\mathbf{R}$, with the total variation as norm and the convolution defined by

$$(\mu * \nu)(A)\xi = \int_{\mathbb{R}} \int_{\mathbb{R}} \chi_A(t+s)\xi d\nu(s)d\mu(t),$$

where χ_A is the characteristic function of the Borelian set A and $\xi \in H$. It is easy to observe that for every $\mu \in \mathcal{M}(\mathbf{R}, \mathcal{B}(H))$ and $x \in L^2(\mathbf{R}, H)$ the function

$$(\mu * x)(t) = \int_{\mathbf{R}} x(t-s)d\mu(s)$$

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is in $L^2(\mathbf{R}, H)$ because

$$\|\mu * x\|_2 \leq \|\mu\|_1 \|x\|_2.$$

Hence, the convolution system defined by

$$K_\mu \colon L^2(\mathbf{R}, H) \to L^2(\mathbf{R}, H), K_\mu x = \mu * x$$

is well defined and continuous.

Definition 1.1. ([8]) If $x \in L^1(\mathbf{R}, H) \cap L^2(\mathbf{R}, H)$, then its Fourier transform is

$$(\mathcal{F}x)(t) = \widehat{x}(t) = \int_{\mathbf{R}} \mathrm{e}^{-\mathrm{i}ts} x(s) ds$$

and the inverse of $\mathcal{F}$ is

$$(\mathcal{F}^{-1}x)(t) = \frac{1}{2\pi} \int_{\mathbf{R}} \mathrm{e}^{\mathrm{i}ts} x(s) ds$$

Definition 1.2. ([8]) The Fourier transform of a measure $\mu \in \mathcal{M}(\mathbf{R}, \mathcal{B}(H))$ is

$$\widehat{\mu} = \mathcal{F}(\mu) \colon \mathbf{R} \to \mathcal{B}(H)$$

defined by

$$\widehat{\mu}(t)\xi = \int_{\mathbf{R}} \mathrm{e}^{-\mathrm{i}ts} \xi d\mu(s), \;\; \forall t \in \mathbf{R}, \xi \in H.$$

Let λ be the Lebesgue measure on $\mathbf{R}$, let $\mathrm{Bor}(\mathbf{R})$ be the family of Borelian sets on $\mathbf{R}$ and let $\phi \colon \mathbf{R} \to \mathcal{B}(H)$ be a measurable function; the essential supremum of ϕ is

$$\|\phi\|_\infty = \inf \big\{ \sup_{t \in \mathbf{R} \setminus A} \|\phi(t)\|; A \in \mathrm{Bor}(\mathbf{R}), \lambda(A) = 0 \big\}.$$

We denote by $L^\infty(\mathbf{R}, \mathcal{B}(H))$ the Banach algebra of essentially bounded functions, i.e. $\|\phi\|_\infty < \infty$.

Definition 1.3. ([1]) For every $\phi \in L^\infty(\mathbf{R}, \mathcal{B}(H))$, the multiplication operator

$$M_\phi \colon L^2(\mathbf{R}, H) \to L^2(\mathbf{R}, H)$$

is defined by

$$M_\phi x = \phi x,$$

where $(\phi x)(t) = \phi(t) x(t)$ for all $t \in \mathbf{R}$.

2 Main results

Using some of the results in [4], we prove the following theorems:

Theorem 2.1. *For every $\mu \in \mathcal{M}(\mathbf{R}, \mathcal{B}(H))$, the operators K_μ and $M_{\widehat{\mu}}$ are unitarily equivalent and $\mathcal{F} K_\mu \mathcal{F}^{-1} = M_{\widehat{\mu}}$.*

Proof. First we show that the function $\widehat{\mu}$ is uniformly continuous. For every $t, s \in \mathbf{R}$, we have

$$\|\widehat{\mu}(t) - \widehat{\mu}(s)\| = \sup_{\|\xi\|=1} \|(\widehat{\mu}(t) - \widehat{\mu}(s))\xi\| \leq \int_{\mathbf{R}} |1 - \mathrm{e}^{-\mathrm{i}(t-s)u}| d\|\mu\|_1(u).$$

The measure $\|\mu\|_1$ is regular, hence for every $\varepsilon > 0$ there is a compact $A = A(\varepsilon) \subset \mathbf{R}$ such that $\|\mu\|_1(\mathbf{R} \setminus A) < \varepsilon$. Because A is compact, there is $B = B(A, \varepsilon) \subset \mathbf{R}$ such that if $t, s \in B$, then $|1 - \mathrm{e}^{-\mathrm{i}(t-s)u}| < \varepsilon, \forall u \in A$. It results that if $t, s \in B$, we have

$$\|\widehat{\mu}(t) - \widehat{\mu}(s)\| \leq \int_A |1 - \mathrm{e}^{-\mathrm{i}(t-s)u}| d\|\mu\|_1(u) + \int_{\mathbf{R}\setminus A} |1 - \mathrm{e}^{-\mathrm{i}(t-s)u}| d\|\mu\|_1(u) \leq$$

$$\leq \varepsilon(\|\mu\|_1(A) + 2) \longrightarrow 0 \text{ if } \varepsilon \longrightarrow 0$$

For every $\mu \in \mathcal{M}(\mathbf{R}, \mathcal{B}(H))$ and $x \in L^2(\mathbf{R}, H)$, we prove that $(\widehat{\mu * x})(t) = \widehat{\mu}(t)\widehat{x}(t), \forall t \in \mathbf{R}$. Let $\mu \in \mathcal{M}(\mathbf{R}, \mathcal{B}(H)), \xi \in H, t \in \mathbf{R}, x \in L^1(\mathbf{R}, H) \cap L^2(\mathbf{R}, H)$. We obtain :

$$\left\langle (\widehat{\mu * x})(u), \xi \right\rangle = \left\langle \int_{\mathbf{R}} \mathrm{e}^{-\mathrm{i}tu} (\mu * x)(t) dt, \xi \right\rangle =$$

$$= \left\langle \int_{\mathbf{R}} \left(\int_{\mathbf{R}} \mathrm{e}^{-\mathrm{i}tu} x(t-s) dt \right) d\mu(s), \xi \right\rangle =$$

$$= \left\langle \int_{\mathbf{R}} \mathrm{e}^{-\mathrm{i}su} \widehat{x}(u) d\mu(s), \xi \right\rangle = \left\langle \widehat{\mu}(u)\widehat{x}(u), \xi \right\rangle.$$

Since $L^1(\mathbf{R}, H) \cap L^2(\mathbf{R}, H)$ is dense in $L^2(\mathbf{R}, H)$, the equality holds for every $x \in L^2(\mathbf{R}, H)$. Consequently, the operators K_μ and $M_{\widehat{\mu}}$ are unitarily equivalent. □

Theorem 2.2. *Let $\mu \in \mathcal{M}(\mathbf{R}, \mathcal{B}(\mathbf{C}^n))$. Then*

$$\sigma(K_\mu) = \sigma(M_{\widehat{\mu}}) = \bigcup_{T \in \overline{\widehat{\mu}(\mathbf{R})}} \sigma(T),$$

where σ denotes the spectrum.

Proof. By Theorem 2.1, since the operators K_μ and $M_{\widehat{\mu}}$ are unitarily equivalent, it results that they have the same spectrum and the same eigenvalues.

We leave to the reader to check that $\overline{\widehat{\mu}(\mathbf{R})}$ is compact (the dimension of H is finite); we must prove the equivalence : $M_{\widehat{\mu}}$ is invertible if and only if T is invertible for every $T \in \overline{\widehat{\mu}(\mathbf{R})}$. Let us first suppose that all the operators $T \in \overline{\widehat{\mu}(\mathbf{R})}$ are invertible and let $\psi(t) = (\widehat{\mu}(t))^{-1}, \forall t \in \mathbf{R}$. We define $G \colon \widehat{\mu}(\mathbf{R}) \to \psi(\mathbf{R}), G(T) = T^{-1}$. Then $\psi(\mathbf{R}) \subseteq G(\overline{\widehat{\mu}(\mathbf{R})})$. The last set is compact because G is continuous and $\overline{\widehat{\mu}(\mathbf{R})}$ is compact. It results that $\psi \in L^\infty(\mathbf{R}, \mathcal{B}(H))$, hence $M_{\widehat{\mu}}$ is invertible and its inverse is M_ψ.

Conversely, if $M_{\widehat{\mu}}$ is invertible results that $\widehat{\mu}$ is invertible in $L^\infty(\mathbf{R}, \mathcal{B}(\mathbf{C}^n))$, hence every operator $T \in \widehat{\mu}(\mathbf{R})$ is invertible. Let $T \in \overline{\widehat{\mu}(\mathbf{R})}$ and let $T_n \in \widehat{\mu}(\mathbf{R})$ such that $T_n \longrightarrow T$. The sequence $S_n = T_n^{-1}$ is bounded (because $\widehat{\mu}$ is invertible) and is contained in the compact set $\overline{\widehat{\mu}^{-1}(\mathbf{R})}$. It results that there is a convergent subsequence $S_{k_n} \subseteq S_n$. Let $S = \lim S_{k_n}$; then from the equalities $T_{k_n} S_{k_n} = S_{k_n} T_{k_n} = I$, it results that S is the inverse of T, hence T is invertible. □

3 The algorithm

As a consequence of Theorems 2.1 and 2.2 the following algorithm for computing the spectra of convolution operators holds :

1. Read $\mu(t), t \in \mathbf{R}$;
2. Compute the Fourier transform $\widehat{\mu}(w), w \in \mathbf{R}$;
3. For every $w \in \mathbf{R}$, compute the spectrum (eigenvalues) of the matrix $\widehat{\mu}(w)$;
4. The spectrum, $\sigma(K_\mu) = \bigcup\limits_{w \in \mathbf{R}} \sigma(\widehat{\mu}(w))$;
5. Draw the graphics of the eigenvalues.

The numerical procedure is a MatLab 7.5 application whose details we skip (we just mention that the function fspec was used) .

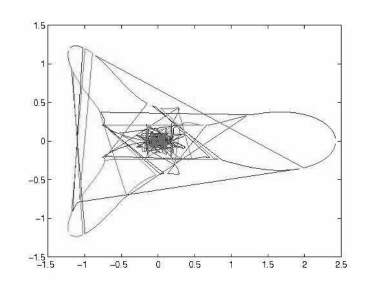
4 Examples

Example 4.1. Let

$$\mu_1(t) = \begin{bmatrix} 0 & \chi_{[0,1]} & \chi_{[0,1]} & 0 \\ 0 & 0 & \chi_{[-1,0]} & \chi_{[-1,0]} \\ \chi_{[-1,1]} & 0 & 0 & \chi_{[-1,1]} \\ \chi_{[0,1]} & \chi_{[0,1]} & 0 & 0 \end{bmatrix}$$

With the algorithm in the previous section we obtain the Fourier transform:

$$\widehat{\mu_1}(w) = \begin{bmatrix} 0 & \frac{\mathrm{i}(\mathrm{e}^{-\mathrm{i}w}-1)}{w} & \frac{\mathrm{i}(\mathrm{e}^{-\mathrm{i}w}-1)}{w} & 0 \\ 0 & 0 & \frac{\mathrm{i}(1-\mathrm{e}^{-\mathrm{i}w})}{w} & \frac{\mathrm{i}(1-\mathrm{e}^{-\mathrm{i}w})}{w} \\ \frac{2\sin w}{w} & 0 & 0 & \frac{2\sin w}{w} \\ \frac{\mathrm{i}(\mathrm{e}^{-\mathrm{i}w}-1)}{w} & \frac{\mathrm{i}(\mathrm{e}^{-\mathrm{i}w}-1)}{w} & 0 & 0 \end{bmatrix}$$



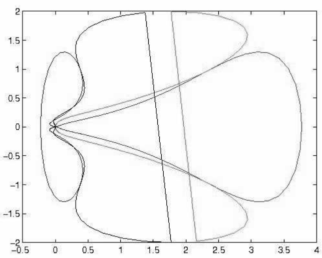
The graphic of the spectrum of the Fourier transform $\widehat{\mu_1} = (w)$

Example 4.2. Let

$$\mu_2(t) = \begin{bmatrix} e^{-t^2} & e^{-|t|} & 0 & te^{-|t|} \\ te^{-|t|} & e^{-t^2} & e^{-|t|} & 0 \\ 0 & te^{-|t|} & e^{-t^2} & e^{-|t|} \\ e^{-|t|} & 0 & te^{-|t|} & e^{-t^2} \end{bmatrix}$$

Then the Fourier transform is:

$$\widehat{\mu_2}(w) = \begin{bmatrix} \sqrt{\pi}e^{-\frac{w^2}{4}} & \frac{2}{1+w^2} & 0 & -\frac{4iw}{(1+w^2)^2} \\ -\frac{4iw}{(1+w^2)^2} & \sqrt{\pi}e^{-\frac{w^2}{4}} & \frac{2}{1+w^2} & 0 \\ 0 & -\frac{4iw}{(1+w^2)^2} & \sqrt{\pi}e^{-\frac{w^2}{4}} & \frac{2}{1+w^2} \\ \frac{2}{1+w^2} & 0 & -\frac{4iw}{(1+w^2)^2} & \sqrt{\pi}e^{-\frac{w^2}{4}} \end{bmatrix}$$



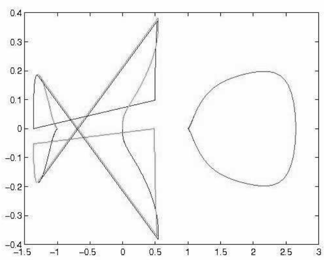
The graphic of the spectrum of the Fourier transform $\widehat{\mu_2} = (w)$

Example 4.3. Let

$$\mu_3(t) = \begin{bmatrix} e^{-t^2} & \delta_t & 0 \\ e^{-|t|} & 0 & \delta_t \\ te^{-|t|} & \delta_t & 0 \end{bmatrix}$$

Then the Fourier transform is:

$$\widehat{\mu_3}(w) = \begin{bmatrix} \sqrt{\pi}e^{-\frac{w^2}{4}} & 1 & 0 \\ \frac{2}{1+w^2} & 0 & 1 \\ -\frac{4iw}{(1+w^2)^2} & 1 & 0 \end{bmatrix}$$



The graphic of the spectrum of the Fourier transform $\widehat{\mu_3} = (w)$

References

[1] A. V. Balakrishnan, *Applied Functional Analysis*, Springer- Verlag, 1975.

[2] Mircea Cirnu, *The continuity of linear physical systems*, Proceedings of the 4-th International Colloquium of "Mathematics in Engineering and Numerical Physics" (MENP-4), October 6-8, 2006, University Politehnica of Bucharest, Bucharest, Romania, BSG Proceedings 14, Geometry Balkan Press 2007, 46-51.

[3] J. M. Mihaila, O. Olteanu and C. Udriste, *La construction de quelques fonctions par des moments donnees*, Balkan Journal of Geometry and Its Applications (BJGA), 13, 1 (2008), 77-86.

[4] M. Olteanu, *Continuous Time - Invariant Infinite - Dimensional Systems*, IFAC Conf. On System Structure and Control, Bucharest, Oct. (1997) 479- 485.

[5] M. Olteanu, T.-L. Costache, M. Zamfir, *An algorithm for computing the spectra of multiplication systems*, Proceedings of the 4-th International Colloquium of "Mathematics in Engineering and Numerical Physics" (MENP-4), October 6-8, 2006, University Politehnica of Bucharest, Bucharest, Romania, BSG Proceedings 14, Geometry Balkan Press, Bucharest 2007, 126-130.

[6] M. Olteanu, T.-L. Costache, M. Zamfir, *A MathCAD application for computing the spectra of a class of convolution operators* , Proceedings of The International Conference "Trends and Challenges in Applied Mathematics" - ICTCAM 2007, Technical University of Civil Engineering Bucharest, Romania, June 2007, 374-377.

[7] V. Prepelita, *Adjoint multidimensional acausal systems*, Proceedings of The International Conference "Differential Geometry and Dynamical Systems" (DGDS-2007), October 5-7, 2007 Bucharest, Romania, BSG Proceedings 15, Geometry Balkan Press, Bucharest 2008, 176-190.

[8] W. Rudin, *Fourier Analysis Groups*, Interscience, 1963.

[9] A. Zlatescu and L. Lemnete-Ninulescu, *L-moment problems*, Proceedings of The International Conference "Differential Geometry and Dynamical Systems" (DGDS-2007), October 5-7, 2007 Bucharest, Romania, BSG Proceedings 15, Geometry Balkan Press, Bucharest 2008, 241-245.

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