Moment problems on unbounded subsets of $\mathbf{R}^n$, optimization and some applications

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Abstract. In the first Section we recall some recent results on moment problem and approximation theory. Section 2 contains a solution of a two dimensional moment problem. The same result, with a similar proof, works for any dimension. A constructive - type result is also proved. In the first part of Section 3, we apply some of the results of Sections 1 and 2, to some moment problems on unbounded curves and surfaces, respectively. Then we solve an optimization problem related to a real function of two real variables, on an unbounded subset. The next result solves an optimization problem, this time for a complex function involving the exponential, on the closed unit disc. In Theorem 3.5, one proves some equalities based on the well known formula for the area of $f(U)$, (f holomorphic and injective on the unit open disc U). Section 4 is devoted to an operator - valued version of a two - dimensional moment problem.

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1 Introduction

The present work is related to some earlier results, from [1], [4], [3], [5], [6], [11],[12], [13], [41], [17], [34], [24], [43], [38], [27] - [29], [36], [42], [37], [10], as well as to some more recent results, from [30] - [32], [21], [22], [35], [23], [2], [40], [33], [15], [16], [18], [19], [25], [26], [44], [20], [8], [9]. Theorem 2.2 of the present work is a "constructive" result. Similar statements on the construction of some functions by using their moments have been published in [18] and [20].

We recall the following results on the approximation of some continuous functions by polynomials, on some unbounded subsets of $\mathbf{R}^n$ (see [19], [25]).

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Lemma 1.1 *Let* $x : [0,\infty[\to \mathbf{R}_+$ *be a continuous function such that there exists* $\lim\limits_{t\to\infty} x(t) \in \mathbf{R}_+$. *Then there exits a sequence* $(e_k)_{k\in\mathbf{Z}_+}$, $(e_k)_k \in Sp\{\exp(-kt); k \in \mathbf{Z}_+\}$ *such that* $e_k(t) > x(t)$, *for all nonnegative* t, $\forall k \in \mathbf{Z}_+$, *and* $\lim\limits_k e_k = x$ *uniformly in* $[0,\infty[$. *It follows that there exists a sequence of polynomials* $(p_k)_{k\in\mathbf{Z}_+}$, $p_k > e_k > x$, $\forall k \in \mathbf{Z}_+$ (*on* $[0,\infty[$), $p_k \to x$ *uniformly on compact subsets of* $[0,\infty[$. *Note that the sequence* (p_k) *is not monotone.*

Lemma 1.2 *Let* ν *be a determinate positive regular Borel measure on* $[0,\infty[$ *with finite moments of all natural orders. If* x, $(p_k)_{k\in\mathbf{Z}_+}$ *are as in Lemma 1.1, then there exists a subsequence* $(p_{k_m})_{m\in\mathbf{Z}_+}$, *such that* (p_{k_m}) *converges to* x *in* $L^1_\nu([0,\infty[)$. *In particular, it follows that for such measures, the positive polynomials are dense in* $L^1_\nu([0,\infty[)_+$.

For the proof of these results see [19, Lemmas 1.4 and 1.5(a)]. In the same paper [19], the following one dimensional moment problem on $[0,\infty[$ is solved, by using Lemma 1.2 stated above.

Theorem 1.1 *Let* $d\nu = \rho_\nu(t)dt$, *where* ρ_ν *is a continuous positive function on* $[0,\infty[$ *such that* $\int_0^\infty t^j d\nu < \infty$, $\forall j \in \mathbf{Z}_+$. *Assume that* ν *is M-determinate. Let* $\{y_j; j \in \mathbf{Z}_+\} \subset \mathbf{R}$. *The following statements are equivalent*:

(a) *there exists a Borel function* h *on* $[0,\infty[$, *such that*

$$(1.1)\quad \int_0^\infty t^j h(t)d\nu = y_j,\ \forall j \in \mathbf{Z}_+ \text{ and}$$

$$(1.2)\quad 0 \le h(t) \le 1,\ \nu - a.e.;$$

(b) *for any finite subset* $J_0 \subset \mathbf{Z}_+$ *and any* $\{\lambda_j; j \in J_0\} \subset \mathbf{R}$, *we have*:

$$(1.3.1)\quad 0 \le \sum_{i,j\in J_0} \lambda_i\lambda_j y_{i+j} \le \sum_{i,j\in J_0} \lambda_i\lambda_j \int_0^\infty t^{i+j} d\nu;$$

$$(1.3.2)\quad 0 \le \sum_{i,j\in J_0} \lambda_i\lambda_j y_{i+j+1} \le \sum_{i,j\in J_0} \lambda_i\lambda_j \int_0^\infty t^{i+j+1} d\nu;$$

The following more general result concerning approximation on any closed subset of $\mathbf{R}^n$ is proved in [25].

Lemma 1.3 (Lemma 1.4 [25]). *Let* $A \subset \mathbf{R}^n$ *be a closed subset of* $\mathbf{R}^n$. *Let* ν *be a determinate positive regular Borel measure on* A, *with finite moments of all orders. Then for any* $x \in (C_0(A))_+$ *such that* $x \le p$ *for some* $p \in P_+$ (P *is the subspace of all polynomial functions on* A), *there exists a sequence* $(p_m)_{m\in\mathbf{Z}_+}$, $p_m \in P_+$, $p_m \ge x$, $p_m \to x$ *in* $L^1_\nu(A)$. *In particular, we have* $\lim\limits_m \int_A p_m(t)d\nu = \int_A x(t)d\nu$, P_+ *is dense in* $(L^1_\nu(A))_+$ *and* P *is dense in* $L^1_\nu(A)$. *The sequence* $(p_m)_m$ *is not always monotone.*

2 Moment problems on $[0, \infty[^2$

Lemma 1.3 stated above and other approximation results, such as Lemma 1.3 [25], (d), allow solving the following multidimensional moment problem on $[0, \infty[^n$. We state and prove this result for $n = 2$, but the same idea works for any natural n.

Theorem 2.1 *Let A be a closed subset of $\mathbf{R}^2$. Let ν be a determined positive Borel regular measure on A, with finite moments of all natural orders. For any given sequence of real numbers $(y_{(m,n)})_{(m,n)}$, $(m,n) \in \mathbf{N}^2$, the following statements are equivalent:*

(a) *there exists h in $L_\nu^\infty(A)$, such that $\iint_A h(t_1,t_2) t_1^m t_1^n d\nu = y_{(m,n)}$, for all $(m,n) \in \mathbf{N}^2$ and $0 \le h(t_1,t_2) \le 1$ ν – a.e. in A;*

(b) *for any $(m,n) \in \mathbf{N}^2$, and any $\{a_{(k,l)}; k = 0,1,\ldots,m,\ l = 0,1,\ldots,n\} \subset \mathbf{R}$, we have:*

$$\sum_{(k,l) \in \{0 \ldots m\} \times \{0 \ldots n\}} a_{(k,l)} t_1^k t_2^l \ge 0 \quad \text{for all} \quad (t_1,t_2) \in A \Rightarrow$$

$$0 \le \sum_{(k,l) \in \{0,\ldots,m\} \times \{0,\ldots,n\}} a_{(k,l)} y_{(k,l)} \le \sum_{(k,l) \in \{0,\ldots,m\} \times \{0,\ldots,n\}} a_{(k,l)} \iint_A t_1^k t_2^l d\nu.$$

Proof. Obviously, (a) implies (b). To prove that (b) implies (a), we define the linear form F_0 on the subspace P of all polynomials by:

$$F_0 \left[\sum_{k,l} a_{(k,l)} x_{(k,l)} \right] = \sum_{k,l} a_{(k,l)} y_{(k,l)},\ a_{(k,l)} \in \mathbf{R},\ x_{(k,l)}(t_1,t_2) = t_1^k t_2^l,\ (t_1,t_2) \in A,$$

finite sums. Then the implication from (b) can be rewritten as: $0 \le F_0(p) \le \iint_A p d\nu$, for all $p \in P_+$, i.e. for all nonnegative polynomials.

Next we apply Theorem 4 [27], the implication (b) implies (a), to the spaces $X = C_0(A)$, $Y = \mathbf{R}$, $f_2(x) = \iint_A x d\nu$, $f_1(x) = 0$, $x \in X$. Let $p = u_2 - u_1$, $p \in P$, $u_1, u_2 \in X_+$. From Lemma 1.3 of Section 1, there exists a sequence (p_m) of polynomials, $p_m > u_2 \ge 0$ on A for all m, $\lim p_m = u_2$ in $L_\nu^1(A)$. From this and also using the preceding arguments, one obtains:

$$p \le u_2 \le p_m \Rightarrow F_0(p) \le F_0(p_m) \le \iint_A p_m d\nu = f_2(p_m),$$

for all m. Passing to the limit over m, we obtain $F_0(p) \le f_2(u_2) = f_2(u_2) - f_1(u_1)$. By [27, Theorem 4], there exists a linear form F on $C_0(A)$, such that $F(x_{(m,n)}) = y_{(m,n)}$ for all $(m,n) \in \mathbf{N}^2$, which verifies

$$0 \le F(u) \le f_2(u), \quad \text{for all} \quad u \in X_+.$$

Then: $|F(x)| \le \iint_A |x| d\nu = ||x||$ (in $L_\nu^1(A)$), for all $x \in X = C_0(A)$. This shows that F is continuous on the subspace X of $L_\nu^1(A)$, with respect to the L^1 - norm, while

X is dense in $L^1_\nu(A)$. Thus F has a unique linear continuous and positive extension of norm smaller or equal to 1, to the whole space $L^1_\nu(A)$. Using the representation of such linear continuous positive forms, the conclusion follows.

□

Corollary 2.1 *Let* $p > 0$, $q > 0$ *be fixed constants and let* $(y_{(m,n)})_{(m,n)}$ *be a sequence of real numbers. The following statements are equivalent*:

(a) *there exists* $h \in L^\infty_{e^{-pt_1-qt_2}dt_1dt_2}([0,\infty[^2)$, *such that*

$$\iint_{[0,\infty[^2} t_1^m t_2^n h(t_1,t_2) e^{-pt_1-qt_2} dt_1 dt_2 = y_{(m,n)},$$

for all $(m,n) \in \mathbf{N}^2$, *and such that* $0 \le h(t_1,t_2) \le 1$, dt_1dt_2 *- a.e. in* $[0,\infty[^2$;

(b) *for any* $(m,n) \in \mathbf{N}^2$ *and any* $\{a_{(k,l)}; k = 0,1,\ldots,m,\ l = 0,1,\ldots,n\} \subset \mathbf{R}$, *we have*:

$$0 \le \sum_{k,l} a_{(k,l)} t_1^k t_2^l \text{ for all } (t_1,t_2) \in [0,\infty[^2 \Rightarrow$$

$$0 \le \sum_{k,l} a_{(k,l)} y_{(k,l)} \le \sum_{k,l} a_{(k,l)} \frac{k!}{p^{k+1}} \frac{l!}{q^{l+1}},$$

where the sums are over $(k,l) \in \{0,\ldots,m\} \times \{0,\ldots,n\}$.

Proof. One applies Theorem 2.1 to $A = [0,\infty[^2$, endowed with the measure $d\nu = e^{-pt_1-qt_2}dt_1dt_2$. This measure is determined on $[0,\infty[^2$, by Carleman's criterion and also using [3, Theorem 3.4]. □

The next result gives the construction of the solution h from (a), Theorem 1.1, in the particular case $d\nu = e^{-pt^2}dt$, on $[0,\infty[$, using only the moments y_n, $n \in \mathbf{N}$ (see below). Such type problems were recently considered in [18] and [20], in a different context and using other methods in their proofs.

Theorem 2.2 *Let* h *be a Borel measurable function,* $0 \le h(t) \le 1$ dt *- a.e in* $[0,\infty[$, $h(t) = 0$ *for* $t < 0$, *and let* $f(x) := h(x)e^{-px^2} / \int_{[0,\infty[} h(t)e^{-pt^2}dt$. *Let us denote by* y_n *the moments* $y_n = \int_{[0,\infty[} t^n h(t) e^{-pt^2} dt$, $n \in \mathbf{N}$. *Then we have*:

$$h(x) = \frac{e^{px^2}}{2\pi} \int_{\mathbf{R}} e^{-itx} \left(\sum_{n=0}^{\infty} i^n t^n y_n / n! \right) dt,\ x \ge 0,\ h(x) = 0 \text{ for all } x < 0.$$

Proof. Let u be the characteristic function associated to a random variable which has as p.d.f. the function f defined in the statement of the theorem. Obviously, f is an element of $L^2(\mathbf{R})$, and so is u.

It is also easy to see that u is analytic on $\mathbf{R}$, because of the Lebesgue dominated convergence theorem. By the Fourier inversion formula, we obtain:

$$h(x)e^{-px^2}/I = \frac{1}{2\pi} \int_{\mathbf{R}} e^{-itx} u(t) dt.$$

On the other hand, the analyticity of u leads to:

$$u(t) = \sum_{n=0}^{\infty} \frac{i^n t^n y_n}{n!}/I, \text{ where } I = \int_{[0,\infty[} h(x)e^{-px^2}dx.$$

Multiplying by e^{px^2}, the conclusion follows. The convergence deduced from the Fourier inversion formula holds in L^2 $([0, \infty[)$. $\square$

3 Applications to varieties. Optimization and applications of complex analysis

The first two results of this Section are applications of theorems 1.1., respectively 2.1., to some special moment problems on unbounded curves, respectively surfaces.

Theorem 3.1 *Let T be the curve in $\mathbf{R}^2$, given by: $y = x^{2m}$, $x \geq 0$, m in $\mathbf{N}$. The orientation on T is chosen such that x (and y) are increasing from 0 to ∞. The following statements are equivalent*:

(a) *there exists a Borel measurable function $h_1 = h_1(x, y)$ defined on T, such that*:

$$\int_T x^{2k} y^l h_1(x, y)e^{-px^2}dx = y_{k+lm}$$

for all $(k, l) \in \mathbf{N}^2$, and $0 \leq h_1(x, x^{2m}) \leq 2x$, for all $x \geq 0$;

(b) *for any $n \in \mathbf{N}$, and any $\{a_0, a_1, \ldots, a_n\} \in \mathbf{R}$, we have: $\sum a_j u^j \geq 0$ for all $u \geq 0$, implies $0 \leq \sum a_j y_j \leq \sum \frac{a_j j!}{p^{j+1}}$, where all sums are over j from 0 to n;*

(c) *for any $n \in \mathbf{N}$, and any $\{a_0, a_1, \ldots, a_n\} \in \mathbf{R}$, we have*:

$$0 \leq \sum a_i a_j y_{i+j} \leq \sum \frac{a_i a_j (i + j)!}{p^{i+j+1}}$$

and

$$0 \leq \sum a_i a_j y_{i+j+1} \leq \sum \frac{a_i a_j (i + j + 1)!}{p^{i+j+2}},$$

where i, j run over $\{0, 1, \ldots, n\}$.

Proof. Consider the natural parametrization of $T : x = t$, $y = t^{2m}$, $t \geq 0$. Then (a) is equivalent to

$$\int_{[0,\infty[} t^{2k} t^{2ml} h_1(t, t^{2m})e^{-pt^2}dt = y_{k+lm},$$

i.e.

$$\int_{[0,\infty[} u^{k+lm} h_2(u^{1/2})/(2u^{1/2}))e^{-pu}du = y_{k+lm},$$

where $u := t^2$, $t \geq 0$, $h_2(u^{1/2}) = h_2(t) := h_1(t, t^{2m})$. By the second relation (a) we have: $0 \leq h_2(u^{1/2})/(2u^{1/2})) \leq 1$ for all $u \geq 0$. We denote: $h(u) = h_2(u^{1/2})/(2u^{1/2}))$, $u \geq 0$. The conclusion (a) is equivalent to (b) by the one dimensional version of Corollary 2.1, while (a) is equivalent to (c) by Theorem 1.1. $\square$

Theorem 3.2 *Let S be the surface defined by $S = \{(x, y, e^{-px-qy}); (x, y) \in [0, \infty[^2\}$, where $p, q > 0$ are given numbers. Let $(y_{(m,n)})_{(m,n)}$ be a sequence of real numbers. The following statements are equivalent*:

(a) there exists a Borel measurable function $h_3 : S \to [0,1]$, such that

$$\iint_S x^m y^n h_3(x,y,z)dydz = y_{(m,n)} \quad \text{for all} \quad (m,n) \in \mathbf{N}^2$$

(*the orientation of* S *is taken such that all the components of the normal are positive*);

(b) *for any* $(m,n) \in \mathbf{N}^2$*, and any* $\{a_{(k,l)}; k = 0,\ldots,m,\ l = 0,\ldots,n\} \subset \mathbf{R}$ *we have*:

$$\sum_{k,l} a_{(k,l)} u^k v^l \geq 0 \quad \text{for all} \quad (u,v) \in [0,\infty[^2 \Rightarrow$$

$$0 \leq \sum_{k,l} a_{(k,l)} y_{(k,l)} \leq \sum_{k,l} a_{(k,l)} \frac{k!}{p^k} \frac{l!}{q^{l+1}}.$$

Proof. Using the parametrization: $S : x = u,\ y = v,\ z = e^{-pu-qv}$, $(u,v) \in [0,\infty[^2$, the normal vector has the components $A = pe^{-pu-qv}$, $B = qe^{-pu-qv}$, $C = 1$. Thus the moment conditions at (a) can be rewritten as:

$$\begin{aligned} y_{(m,n)} &= p \iint_{[0,\infty[\times[0,\infty[} u^m v^n h_3(u,v,e^{-pu-qv}) e^{-pu-qv} dudv \\ &= p \iint_{[0,\infty[\times[0,\infty[} u^m v^n h(u,v) e^{-pu-qv} dudv, \end{aligned}$$

where $h(u,v) := h_3(u,v,e^{-pu-qv})$ is in $[0,1]$ for all $(u,v) \in [0,\infty[^2$. Now the conclusion follows from Corollary 2.1. □

Next we consider an optimization problem related to fiability, on an unbounded subset of $\mathbf{R}^2$.

Theorem 3.3 *Let* $g(x,y) := \left(\frac{1}{y+1}\right)^{px}$*, where* $p > 0$ *is a constant. Then we have:*

$$M(n_1) := \max\{g(x,y); 0 < n_1 \leq x,\ y \geq 1/x\} = \left(\frac{n_1}{n_1+1}\right)^{pn_1}$$

where $n_1 > 0$ *is a real number. In particular, we have* $M(n_1) > e^{-p}$ *for all* $n_1 > 0$.

Proof. We write $g(x,y)$ as $e^{px(-\ln(1+y))}$. It is easy to see that both first order partial derivatives of g are negative on our set. Hence, the maximum cannot be attained at any point which is in the interior of this set. It is also easy to see that sup g is actually attained at a point of the given set. It follows that this point must be on the boundary of the set, i.e. on the curve $y = 1/x$, $x \geq n_1$. Replacing y by $1/x$, our problem becomes a one dimensional maximization problem on $[n_1,\infty[$, of the function $h(x) = -x\ln(1+1/x)$. Computation of the first order derivative h', leads (via the inequality: $\ln(1+1/x) \geq 1/(1+x)$, $x > 0$), to the fact that h is decreasing in the interval $[n_1,\infty[$, so that

$$\max\{h(x); x \geq n_1\} = h(n_1) = -n_1 \ln(1+1/n_1).$$

This yields:

$$\max\{g(x,y); x \geq n_1,\ y \geq 1/x\} = \exp[ph(n_1)] = \left(\frac{n_1}{1+n_1}\right)^{pn_1} = M(n_1) < 1.$$

On the other hand, the last expression $M(n_1)$ is an decreasing function of n_1, so that $M(n_1) > \lim\limits_{n_1 \to \infty} M(n_1) = e^{-p}$, $\forall n_1 > 0$. $\square$

Remark. Complementary results can be found in [8], [9].

The next result is an application of a consequence of Schwarz's Lemma (see [38, Section 12.5]).

Theorem 3.4 *For any natural number* $p \geq 1$, *and any complex number* z, *with* $|z| \leq 1$, *we have*:

$$|e^{z-1} z^{p-1}(p+z)| \leq \frac{1 - |z^p e^{z-1}|^2}{1 - |z|^2},$$

and equality holds if and only if $z = 1$. In particular, for any x in $[0, 1]$ and any natural number n, we have:

$$e^{x-1}[(1 - x^n)(1 + x - x^2) - nx^n(1 - x)](1 + x)^2 \leq n(1 - x^2) - x^2(1 - x^{2n})e^{2(x-1)}$$

and equality holds if and only if $x = 1$.

Proof. The function $f(z) := z^p e^{z-1}$ is holomorphic in the whole complex plane. It applies the open unit disc into itself, and the closed unit disc into itself. In particular, conditions required in [38, Section 12.5, pp.245], are accomplished. At any point z with $|z| < 1$, we have:

$$|f'(z)| \leq \frac{1 - |f(z)|^2}{1 - |z|^2}.$$

Since f' is continuous on the closed unit disc, the last inequality remains valid for all z with $|z| \leq 1$. Writing the inequality for our f, the first conclusion follows. Obviously, at $z = 1$, equality holds. One can prove that $z = 1$ is the only point at which equality holds. For the second inequality, we use the relation proved above for all x in $[0, 1[$, and all natural p in $\{1, 2, \ldots, n\}$. Then we sum over p from 1 to n. Computing the geometric finite sums and the derivative of one of them, and multiplying by $(1 - x)^2$, the second inequality follows. $\square$

The last result of this section concerns some applications of complex analysis to some limits and sums of series. These sums appear as areas of the images of the unit disc U, through the exponential function and also through some polynomial functions converging to the exp - function.

Theorem 3.5 *Let* U *be the unit disc centered at* O, *and* $D = \exp(U)$. *Then we have*:

(a) Area $(D) = \pi \cdot \{1 + 1/[1! \cdot 2!] + \ldots + 1/[(n-1)! \cdot n!] + \ldots\} =$

$$\pi \cdot \lim[1 + 2(C_n^2/n^2)^2 + \ldots + p(C_n^p/n^p)^2 + \ldots + (n-1)(C_n^{n-1}/n^{n-1})^2 + 1/n^{2n-1}]$$

where the limit is over n *going to infinity.*

(b) D *is a part of the annulus defined by* $D_2 = \{w = u + iv;\ 1/e < (u^2 + v^2)^{(1/2)} < e,$ $-1 < Arg\,(w) < 1\}$, *hence* Area $(D) <$ Area $D_2 = e^2 - (1/e)^2$.

Proof. It is known that for any injective holomorphic function f on U, $f(z) = \sum\limits_{n=0}^{\infty} c_n z^n$, we have Area $f(U) = \pi \cdot \left(\sum\limits_{n=1}^{\infty} n|c_n|^2 \right)$ (see [38, Chapter 11, exercise 6]): the Jacobian of f is exactly $|f'|^2$. Applying this formula to $f(z) = \exp(z)$, for which $c_n = 1/n!$, the first equality follows (it is easy to see that exp is injective in U). On the other hand, we have $\text{Exp}(z) = \lim(1 + z/n)^n = \lim P_n(z)$, and the absolute values of the polynomials P_n are uniformly bonded on U by e. Vitaly's Theorem allow to deduce that $\exp = \lim P_n$ uniformly in U (see [17, pp. 415 - 416]). This implies $\exp' = \lim P_n'$ uniformly in U, which leads to $\text{Area}(\exp(U)) = \lim \text{Area}(P_n(U))$. One can prove also that P_n are injective in U, so that the above formula can be applied to compute Area $(P_n(U))$. Expanding the bynomials $P_n(z)$, the second relation from (a) follows too. To prove (b), observe that $|e^z| = e^x$ runs over $]1/e, e[$, when $x = \text{Re}\,(z)$ runs over $]-1, 1[$, z in U. On the other hand, $\text{Arg}(\exp(z)) = \text{Arg}(\exp(x))+ \text{Arg}(\exp(iy)) = 0 + y = y$, which runs over $]-1, 1[$, when z is in U. Hence $D \subset D_2$. This proves the first assertion at (b), i.e. Area $(D) <$ Area (D_2). The second one follows from elementary geometry, since the area of the whole annulus is $\pi[e^2 - (1/e)^2]$, while here we have only a sector of it, the corresponding angle being $= 2$. Thus, Area (D_2) is $2 \cdot \dfrac{\pi}{2} = \pi$ times smaller than the whole area. This proves the last assertion of (b). □

4 An operator - valued moment problem

Let H be a Hilbert space, A, B two positive commuting self-adjoint linear bounded operators from H into H. Let

$$Y_1(A, B) = \{S \text{ in } B(H);\ S \text{ seld-adjoint},\ AS = SA,\ BS = SB\},$$

$$Y = Y(A, B) = \{T \text{ in } Y_1;\ TS = ST \text{ for all } S \text{ in } Y_1\}.$$

Then Y is an order complete vector lattice, with respect to the natural order relation on self - adjoint operators, defined by the convex cone: $Y_+ = \{T \text{ in } Y; \langle T(h), h\rangle \geq 0 \text{ for all } h \in H\}$. Obviously, Y is also a commutative algebra (for details see [6] and [16]). The main technical tool in proving such-existence results is the spectral measure associated to a self-adjoint bounded operator (see also [26], [10]). The idea is the approximation by polynomials of some functions on $\mathbf{R}_+^2$, by using Lemma 1.3 or similar results.

Theorem 4.1 *Let $X = \{x; x$ real function, defined and Lebesgue measurable on $\mathbf{R}_+^2$, f dominated by a polynomial p on $\mathbf{R}_+^2\}$. Let X_+ be the cone of all nonnegative functions from X. Then for any permutable positive self - adjoint operators A, B on H, and for any sequence $(S_{(m,n)})_{(m,n)} \in Y(A, B)$, the following statements are equivalent:*

(a) *there exists a linear positive operator $F : X \to Y(A, B)$, such that $F(x_{(m,n)}) = S_{(m,n)}$, for all $(m, n) \in \mathbf{N}^2$, and*

$$0 \leq F(x\mathbf{1}_{s(A)\times s(B)}) \leq \iint_{s(A)\times s(B)} x(t_1, t_2)dE_A dE_B,$$

for all $x \in X_+$, x continuous; in particular, it follows that:

$$|F(x\mathbf{1}_{s(A)\times s(B)})| \leq \iint_{s(A)\times s(B)} |x(t_1, t_2)|dE_A dE_B,$$

and $||F(x\mathbf{1}_{s(A)\times s(B)})|| \leq ||x\mathbf{1}_{s(A)\times s(B)}||_\infty$, $\forall x \in X$.

(b) $\sum_{k,l} a_{(k,l)} t_1^k t_2^l \geq 0$ *for all* (t_1, t_2) *in* $\mathbf{R}_+^2 \Rightarrow 0 \leq \sum_{k,l} a_{(k,l)} S_{(k,l)} \leq \sum_{k,l} a_{(k,l)} A^k B^l$ (*finite sums, arbitrary real scalars* $a_{(k,l)}$).

Proof. (a) implies (b) is obvious. To prove (b) implies (a), define the linear operator F_0 from the subspace X_0 of all polynomials into Y, by:

$$F_0 \left[\sum_{k,l} a(k,l) x(k,l) \right] = \sum_{k,l} a(k,l) S(k,l), \quad x_{(k,l)}(t_1, t_2) = t_1^k t_2^l,$$

(t_1, t_2) in $\mathbf{R}_+^2$. From (b) we have:

$$0 \leq F_0(p) \leq \iint_{s(A)\times s(B)} p(t_1, t_2) dE_A dE_B,$$

for all polynomials p from X_+. If $x \in X_+$, then by [25, Lemma 1.3(d)], there exists a sequence $(p_m) \in P_+$, $p_m > x$ for all m, $\lim p_m = x\mathbf{1}_{s(A)\times s(B)}$ uniformly on the compact $s(A) \times s(B)$. Also using the preceding notations and remarks, we infer that:

i) there exists a linear positive extension F of F_0 to X (X_0 is a majorizing subspace of X, hence one applies the extension result from [6, pp. 160]);

ii) we have:

$$0 \leq F(x\mathbf{1}_{(s(A)\times s(B))}) = \lim F(p_m) \leq \lim \iint_{s(A)\times s(B)} p_m dE_A dE_B =$$

$$= \iint_{s(A)\times s(B)} x dE_A dE_B,$$

for all $x \in X_+$. Then, from the preceding inequality for functions from X_+, one obtains:

$$\begin{aligned} |F(x\mathbf{1}_{(s(A)\times s(B))})| &\leq |F((x_+)\mathbf{1}_{(s(A)\times s(B))})| + |F((x_-)\mathbf{1}_{(s(A)\times s(B))})| \\ &\leq \iint_{s(A)\times s(B)} |x| dE_A dE_B, \quad \forall x \in X. \end{aligned}$$

Since the norm on Y is solid, the last assertion follows from the preceding inequality. The proof is complete. □

Remark. Complementary results can be found in [14], [45], [44].

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