Minimum energy transfer and controllability in a class of multidimensional continuous discrete systems

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Abstract. The problem of the minimum energy transfer is analysed for a class of multidimensional time-varying continuous-discrete systems. A suitable separable multidimensional partial differential-difference equation is solved and a variation-of-parameters type formula is established.This formula is employed to derive the formula of the state and the input-output map of the system. The concepts of controllability and reachability are analyzed and suitable controllability and reachability Gramians are constructed to characterize the controllable and the reachable time-varying systems. The optimal control which realizes the minimum-energy transfer is determined and the formula of the minimum control energy is provided in the case of completely controllable systems.

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1 Introduction

In the last three decades the theory of two-dimensional (2D) or, more generally, of multidimensional (nD) systems become a distinct and important branch of the systems theory due its continually growing interest. The reasons for the interests in this domain are on one side the richness in potential application fields and on the other side the richness and significance of its theoretical approaches. The application fields include circuits, control and signal processing, image processing, computer tomography, gravity and magnetic field mapping, seismology, control of multipass processes, etc. The domain of nD systems needs a specific theoretical approach, since many aspects of the 1D systems do not generalize and there are many nD systems phenomena which have no 1D systems counterparts. Various state space 2D discrete-time models have been proposed in literature by Roesser [15], Fornasini-Marchesini [5] Attasi [1], Eising [4] and others.

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A quite new field of the nD systems theory is represented by the 2D continuous-discrete (hybrid) models, whose state equation is of differential-difference type [8], [12], [13]. These 2D continuous-discrete models have applications in various areas such as linear repetitive processes [7], [16], pollution modelling [6], long-wall coal cutting and metal rolling [17] or in iterative learning control synthesis [11].

This paper focusses on the concept of controllability, which is fundamental in control theory and which was introduced for 1D systems by Kalman in [9], being imposed by the engineering problems of time optimal control. This concept and its dual (observability) characterize the minimal systems. In the case of 2D systems some notions such as local and global controllability were introduced, but they are not satisfactory from the point of view of minimality. In [10] the concepts of modal controllability/observability were defined and it was shown that a system is minimal iff it is modally controllable and observable, but these notions do not allow the richness of the characterizations of the 1D notions. We notice that in the case of continuous-discrete 2D Fornasini-Marchesini type model the controllability was studied by Kaczorek [8] and in a geometric approach in [3] and [18].

In the present paper we consider a class of multidimensional continuous-discrete systems described by differential-difference state equations. These systems depend on q and r respectively continuous and discrete time variables and they represent an extension of the Attasi's 2D discrete-time time-invariant model. The considered systems are studied from the point of view of controllability and reachability. This class of systems was preferred since it allows the generalizations of many important results which characterize the 1D linear systems.

In Section 2 a suitable separable multidimensional partial differential-difference equation is solved and a variation-of-parameters type formula is established. The state space representation of the systems is derived from this formula and the formulas of the state as well as of the input-output map are obtained.

Section 3 introduces the notions of completely controllable and completely reachable systems. Suitable controllability and reachability Gramians are constructed and they are used to obtain necessary and sufficient conditions of controllability and reachability for time-varying systems.

In Section 4 the minimum-energy transfer is analyzed and the optimal control as well as the minimum control energy are provided in the case of completely controllable systems.

We shall use the following notations: $q \in \mathbf{N}$ and $r \in \mathbf{N}$ being the number of continuous and discrete variables respectively, a function $x(t_1, \ldots, t_q; k_1, \ldots, k_r)$, $t_i \in \mathbf{R}$, $k_i \in \mathbf{Z}$ will be sometimes denoted by $x(t; k)$, where $t = (t_1, \ldots, t_q)$, $k = (k_1, \ldots, k_r)$. By $s \leq t,\ s, t \in \mathbf{R}^q$ we mean $s_i \leq t_i\ \forall i \in \bar{q}$ and a similar signification has $l \leq k,\ l, k \in \mathbf{Z}^r$; $(s; l) < (t; k)$ means $s \leq t,\ (l \leq k)$ and $(s; l) \neq (t; k)$. For $t^0, t^1 \in \mathbf{R}^q$ and $k^0, k^1 \in \mathbf{Z}^r,\ t^0 < t^1,\ k^0 < k^1$ we denote by $[t^0, t^1]$ and $[k^0, k^1]$ respectively the sets $[t^0, t^1] = \prod_{i=1}^{q} [t_i^0, t_i^1]$ and $[k^0, k^1] = \prod_{i=1}^{r} \{k_j^0, k_j^0 + 1, \ldots, k_j^1\}$. By $\overline{m}$ with $m \in \mathbf{N}^*$ we denote the set $\{1, 2, \ldots, m\}$ and by $\mathcal{P}(\overline{m})$ the family of all subsets of $\overline{m}$.

If $\tau = \{i_1, \ldots, i_l\}$ is a subset of $\overline{m}$, $|\tau| := l$ and $\tilde{\tau} := \overline{m} \setminus \tau$; for $i \in \overline{m}$, $\tilde{i} := \overline{m} \setminus \{i\}$ and $\tilde{\tilde{i}} := \{i+1, \ldots, m\}$. The notation $(\tau, \delta) \subset (\overline{q}, \overline{r})$ means that τ and δ are subsets of $\overline{q}$ and $\overline{r}$ respectively and $(\tau, \delta) \neq (\overline{q}, \overline{r})$. For $\tau = \{i_1, \ldots, i_l\}$ and $\delta = \{j_1, \ldots, j_h\}$ the operators $\dfrac{\partial}{\partial \tau}$ and σ_δ are defined by

$$\frac{\partial}{\partial \tau}x(t;k) = \frac{\partial^l}{\partial t_{i_1} \ldots \partial t_{i_l}}x(t;k),\ \sigma_\delta x(t;k) = x(t;k+e_\delta)$$

where $e_\delta = e_{j_1} + \ldots + e_{j_h}$, $e_j = (\underbrace{0,\ldots,0}_{j-1},1,0,\ldots,0) \in \mathbf{R}^r$; when $\tau = \overline{q}$ and $\delta = \overline{r}$ we denote $\partial/\partial\tau = \partial/\partial t$ and $\sigma_\delta = \sigma$.

If A_i, $i \in \overline{m}$ is a family of matrices, $\sum\limits_{i\in\varnothing} A_i = 0$ and $\prod\limits_{i\in\varnothing} A_i = I$.

If P is a positive definite matrix, one writes $P > 0$.

2 State space representation

The time set of the Attasi-type multidimensional system is $T = \mathbf{R}^q \times \mathbf{Z}^r$, $q, r \in \mathbf{N}$.

Definition 2.1. A (q,r)-D *hybrid system* is a set $\Sigma = (\{A_{ci} | i \in \overline{q}\}, \{A_{dj} | j \in \overline{r}, B, C, D)$ with $A_{ci}(t;k)$, $i \in \overline{q}$ and $A_{dj}(t;k)$, $j \in \overline{r}$ commuting $n\times n$ matrices $\forall t \in \mathbf{R}^q$, $\forall k \in \mathbf{Z}^r$ and $B(t;k)$, $C(t;k)$, $D(t;k)$ respectively $n\times m$, $p\times n$ and $p\times m$ real matrices, all these matrices being continuous with respect to $t \in \mathbf{R}^q$ for any $k \in \mathbf{Z}^r$; the state equation is

$$\begin{aligned}\frac{\partial}{\partial t}\sigma x(t;k) = &\sum_{(\tau,\delta)\subset(\overline{q},\overline{r})} (-1)^{q+r-|\tau|-|\delta|-1}\times \\ &\times \left(\prod_{i\in\tilde{\tau}} A_{ci}(t;k)\right)\left(\prod_{j\in\tilde{\delta}} A_{dj}(t;k)\right)\frac{\partial}{\partial\tau}\sigma_\delta x(t;k) + B(t;k)u(t;k)\end{aligned} \tag{2.1}$$

and the output equation is

$$y(t;k) = C(t;k)x(t;k) + D(t;k)u(t;k) \tag{2.2}$$

where

$$x(t;k) = x(t_1,\ldots,t_q;k_1,\ldots,k_r) \in \mathbf{R}^n$$

is the *state*, $u(t;k) \in \mathbf{R}^m$ is the *input* and $y(t;k) \in \mathbf{R}^p$ is the *output.*

For $\tau = \{i_1,\ldots,i_l\} \subset \overline{q}$, $\delta = \{j_1,\ldots,j_h\} \subset \overline{r}$ and $t_i \in \mathbf{R}$, $i \in \tau$, $t_i^0 \in \mathbf{R}$, $i \in \overline{\tau}$, $k_j \in \mathbf{Z}$, $j \in \delta$, $k_j^0 \in \mathbf{Z}$, $j \in \tilde{\delta}$ we use the notation

$$\begin{aligned}x(t_\tau, t^0_{\tilde{\tau}}; k_\delta, k^0_{\tilde{\delta}}) := x(&t_1^0,\ldots,t^0_{i_1-1},t_{i_1},t^0_{i_1+1},\ldots,t^0_{i_l-1},t_{i_l},t^0_{i_l+1},\ldots,t_q^0;\\ &k_1^0,\ldots,k^0_{j_1-1},k_{j_1},k^0_{j_1+1},\ldots,k^0_{j_h-1},k_{j_h},k^0_{j_h+1},\ldots,k^0_{j_r}).\end{aligned}$$

Let $\Phi_i(t_i,t_i^0;t_{\tilde{i}};k)$ be the (continuous) fundamental matrix of $A_{ci}(t;k)$ with respect to the variables t_i, t_i^0, $i \in \overline{q}$, i.e. the unique matrix solution of the system $\frac{\partial Y}{\partial t_i}(t;k) = A_{ci}(t;k)Y(t;k), Y(t_{\tilde{i}},t_i^0;k) = I$ for any $t_l \in \mathbf{R}$, $l \in \tilde{i}$ and $k \in \mathbf{R}^r$. If A_{ci} is a constant matrix then $\Phi_i(t_i,t_i^0;t_{\tilde{i}},k) = e^{A_{ci}(t_i-t_i^0)}$.

The *discrete fundamental matrix* $F_j(t;k_j,k_j^0;k_{\tilde{j}})$ of the matrix $A_{dj}(t;k)$ is defined by

$$F_j(t;k_j,k_j^0;k_{\tilde{j}}) =$$
$$= \begin{cases} A_{dj}(t;k_j-1,k_{\tilde{j}})A_{dj}(t;k_j-2,k_{\tilde{j}})\dots A_{dj}(t;k_j^0,k_{\tilde{j}}) & \text{for} \quad k_j > k_j^0 \\ I_n & \text{for} \quad k = k_j^0, \end{cases}$$

for any $k_h \in \mathbf{Z}$, $h \in \tilde{j}$ and $t \in \mathbf{R}^q$.

$F_j(t;k_j,k_j^0;k_{\tilde{j}})$ is the unique matrix solution of the difference system

$$Y(t;k_j+1,k_{\tilde{j}}) = A_{dj}(t;k)Y(t;k_j,k_{\tilde{j}}),\ Y(t;k_j^0;k_{\tilde{j}}) = I.$$

If A_{dj} is a constant matrix then $F_j(t;k_j,k_j^0;k_{\tilde{j}}) = A_{dj}^{k_j-k_j^0}$.

Remark 2.2. Under the hypothesis: (H) "The matrices $A_{dj}(t;k)$ are nonsingular for any $t \in \mathbf{R}^q, k \in \mathbf{Z}^r$ ", the discrete fundamental matrix F_j can be defined for $k_j < k_j^0$ by $F_j(t;k_j,k_j^0;k_{\tilde{j}}) = F_j(t;k_j^0,k_j;k_{\tilde{j}})^{-1}$. In this case the semigroup property $F_j(t;k_j,k_j^1;k_{\tilde{j}})F_j(t;k_j^1,k_j^0;k_{\tilde{j}}) = F_j(t;k_j,k_j^0;k_{\tilde{j}})$ holds for any k_j^0, k_j^1, k_j.

Definition 2.3. The vector $x^0 \in \mathbf{R}^n$ is called an *initial state* of the system Σ if

$$x(t_\tau, t_{\tilde{\tau}}^0; k_\delta, k_{\tilde{\delta}}^0) = \left(\prod_{i\in\tau} \Phi_i(t_i,t_i^0;t_{\tilde{i}};k)\right)\left(\prod_{j\in\delta} F_j(t;k_j,k_j^0;k_{\tilde{j}})\right)x^0 \tag{2.3}$$

for any $(\tau,\delta) \subset (\overline{q},\overline{r})$; equalities (2.3) are called the *initial conditions of* Σ.

In [14] one proves

Proposition 2.4. *The solution of the initial value problem*

$$\begin{aligned} \frac{\partial}{\partial t}\sigma x(t;k) = &\sum_{(\tau,\delta)\subset(\overline{q},\overline{r})} (-1)^{q+r-|\tau|-|\delta|-1} \times \\ &\times \left(\prod_{i\in\tilde{\tau}} (\sigma_\delta A_{ci}(t;k))\right)\left(\prod_{j\in\tilde{\delta}} A_{dj}(t,k)\right)\frac{\partial}{\partial \tau}\sigma_\delta x(t;k) + f(t;k) \end{aligned} \tag{2.4}$$

with the initial conditions (2.3) *is given by the generalized variation-of-parameters formula*

$$\begin{aligned} x(t;k) = &\left(\prod_{i=1}^{q} \Phi_i(t_i,t_i^0;t_{\overline{i-1}}^0,t_{\tilde{\tilde{i}}};k)\right)\left(\prod_{j=1}^{r} F_j(t^0;k_j,k_j^0;k_{\overline{j-1}}^0,k_{\tilde{\tilde{j}}})\right)x^0 + \\ &+ \int_{t_1^0}^{t_1}\dots\int_{t_q^0}^{t_q}\sum_{l_1=k_1^0}^{k_1-1}\dots\sum_{l_r=k_r^0}^{k_r-1}\left(\prod_{i=1}^{q}\Phi_i(t_i,s_i;s_{\overline{i-1}},t_{\tilde{\tilde{i}}};k)\right)\times \\ &\times\left(\prod_{j=1}^{r} F_j(s;k_j,l_j+1;l_{\overline{j-1}},k_{\tilde{\tilde{j}}})\right)f(s;l)ds_1\dots ds_q; \end{aligned} \tag{2.5}$$

here $s = (s_1,\dots,s_q)$, $l = (l_1,\dots,l_r)$ *and if for instance* $i = 1$, *then the corresponding variable* $t_{\overline{i-1}} = t_{\varnothing}^0$ *lacks;* $f : \mathbf{R}^q \times \mathbf{Z}^r \to \mathbf{R}^n$ *is a continuous function.*

Theorem 2.5. *The state of the system Σ (2.1) determined by the initial state $x_0 \in \mathbf{R}^n$ and the control u is*

$$\begin{aligned} x(t;k) = {} & \left(\prod_{i=1}^{q} \Phi_i(t_i, t_i^0; t^0_{\overline{i-1}}, t_{\tilde{\bar{i}}}; k)\right)\left(\prod_{j=1}^{r} F_j(t^0; k_j, k_j^0; k^0_{\overline{j-1}}, k_{\tilde{\bar{j}}})\right) x^0 + \\ & + \int_{t_1^0}^{t_1} \dots \int_{t_q^0}^{t_q} \sum_{l_1=k_1^0}^{k_1-1} \dots \sum_{l_r=k_r^0}^{k_r-1} \left(\prod_{i=1}^{q} \Phi_i(t_i, s_i; s_{\overline{i-1}}, t_{\tilde{\bar{i}}}; k)\right) \times \\ & \times \left(\prod_{j=1}^{r} F_j(s; k_j, l_j+1; l_{\overline{j-1}}, k_{\tilde{\bar{j}}})\right) B(s;l)u(s;l)ds_1 \dots ds_q. \end{aligned} \tag{2.6}$$

Proof. Equation (2.1) has the form (2.4) with $f(t;k) = B(u;k)u(t;k)$ and (2.6) results from (2.5) by replacing $f(t;k)$.

Corollary 2.6. *If A_{ci}, A_{dj} and B are constant matrices, then the state of Σ determined by the control u and the initial state x^0 with initial moments $t_i^0 = 0$, $i \in \overline{q}$, $k_j^0 = 0$, $j \in \overline{r}$ is*

$$\begin{aligned} x(t;k) = {} & \left(\prod_{i=1}^{q} e^{A_{ci}t_i}\right)\left(\prod_{j=1}^{r} A_{dj}^{k_j}\right) x^0 + \int_0^{t_1} \dots \int_0^{t_q} \left(\prod_{i=1}^{q} e^{A_{ci}(t_i - s_i)}\right) \times \\ & \times \sum_{l_1=0}^{k_1-1} \dots \sum_{l_r=0}^{k_r-1} \left(\prod_{j=1}^{r} A_{dj}^{k_j - l_j - 1}\right) B(s;l)u(s,l)ds_1 \dots ds_q. \end{aligned} \tag{2.7}$$

By replacing the state $x(t;k)$ (2.6) in the output equation (2.2) we obtain

Theorem 2.7. *The input-output map of the (q,r)-*D *hybrid system Σ (2.1), (2.2) is*

$$\begin{aligned} y(t;k) = {} & C(t;k)\left(\prod_{i=1}^{q} \Phi_i(t_i, t_i^0; t^0_{\overline{i-1}}, t_{\tilde{\bar{i}}}, k)\right)\left(\prod_{j=1}^{r} F_j(t^0; k_j; k^0_{\overline{j-1}}, k_{\tilde{\bar{j}}})\right) x^0 + \\ & + \int_{t_1^0}^{t_1} \dots \int_{t_q^0}^{t_q} \sum_{l_1=k_1^0}^{k_1-1} \dots \sum_{l_r=k_r^0}^{k_r-1} C(t;k)\left(\prod_{i=1}^{q} \Phi_i(t_i, s_i; s_{\overline{i-1}}, t_{\tilde{\bar{i}}}; k)\right) \times \\ & \times \left(\prod_{j=1}^{r} F_j(s; k_j, l_j; l_{\overline{j-1}}, k_{\tilde{\bar{j}}})\right) B(s;l)u(s;l)ds_1 \dots ds_q + D(t;k)u(t;k). \end{aligned} \tag{2.8}$$

Similarly with Corollary 2.6 we can derive from (2.8) the input-output map of a time-invariant system Σ.

3 Controllability and reachability of multidimensional hybrid systems

The controllability topic uses only the state equation (2.1), hence a (q,r)-D hybrid system reduces to the ensemble $\sum = (\{A_{ci} | i \in \bar{q}\}, \{A_{dj} | j \in \bar{r}\}, B)$.

A triplet $(t,k,\tilde{x}) \in \mathbf{R}^q \times \mathbf{Z}^r \times \mathbf{R}^n$ is said to be a *phase* of $\sum$ if $\exists u : T \to \mathbf{R}^m$ and $x^0 \in \mathbf{R}^n$ such that $\tilde{x} = x(t;k)$ where $x(t;k)$ is given by (2.6). In this case one says that the control u transfers the phase (t^0,k^0,x^0) to the phase $(t,k,\tilde{x})$.

Definition 3.1. A phase (t,k,x) of $\sum$ is said to be *controllable* if there exist $(t^1,k^1) \in T$, $(t^1,k^1) > (t,k)$ and a control u which transfers the phase (t,k,x) to $(t^1,k^1,0)$.

A phase (t,k,x) is said to be *reachable* if there exist $(t^0,k^0) \in T$, $(t^0,k^0) < (t,k)$ and a control u which transfers the phase $(t^0,k^0,0)$ to (t,k,x).

If for some fixed $(t^0,k^0),(t^1,k^1) \in T$ with $(t^0,k^0) < (t^1,k^1)$ a phase (t^0,k^0,x) ((t^1,k^1,x)) is controllable (reachable) one says that the state x is *controllable* (*reachable*) *on the multiple interval* $P = [t^0,t^1] \times [k^0,k^1]$. The system $\sum$ is said to be *completely controllable* (*completely reachable*) on P if any state $x \in \mathbf{R}^n$ is controllable (reachable) on P.

In the sequel we shall denote by $\displaystyle\int_{t_0}^{t}$ the multiple integral $\displaystyle\int_{t_1^0}^{t_1} \cdots \int_{t_q^0}^{t_q}$, by $\displaystyle\sum_{l=k^0}^{k-1}$ the sum $\displaystyle\sum_{l_1=k_1^0}^{k_1-1} \cdots \sum_{l_r=k_r^0}^{k_r-1}$ and $ds = ds_1 \cdots ds_q$.

Definition 3.2. The reachability Gramian of $\sum$ on P is the matrix

$$
\begin{aligned}
\mathcal{R}(t^0,t;k^0,k) = & \int_{t^0}^{t} \sum_{l=k^0}^{k-1} \left(\prod_{i=1}^{q} \Phi_i(t_i,s_i;s_{\overline{i-1}},t_{\overset{\approx}{i}};k) \right) \times \\
& \times \left(\prod_{j=1}^{r} F_j(s;k_j,l_j+1;l_{\overline{j-1}},k_{\overset{\approx}{j}}) \right) B(s,l)B(s,l)^T \times \\
& \times \left(\prod_{j=1}^{r} F_j(s;k_j,l_j+1;l_{\overline{j-1}},k_{\overset{\approx}{j}}) \right)^T \left(\prod_{i=1}^{q} \Phi_i(t_i,s_i;s_{i-1},t_{\overset{\approx}{i}}) \right)^T ds.
\end{aligned} \tag{3.1}
$$

Obviously, $\mathcal{R} = \mathcal{R}(t^0,t;k^0,k)$ is a symmetrical non-negative definite $n \times n$ matrix.

Now we replace in the state formula (2.6) $x(t;k) = x$ and $x^0 = 0$ and we obtain:

Proposition 3.3. The phase (t,k,x) is reachable on P if and only if there exists some control $u : P \to \mathbf{R}^m$ such that

$$
x = \int_{t^0}^{t} \sum_{l=k^0}^{k-1} \left(\prod_{i=1}^{q} \Phi_i(t_i,s_i;s_{\overline{i-1}},t_{\overset{\approx}{i}};k) \right) \times \left(\prod_{j=1}^{r} F_j(s;k_j,l_j+1;l_{\overline{j-1}},k_{\overset{\approx}{j}}) \right) B(s;l)u(s;l)ds. \tag{3.2}
$$

Now we can state the main result of this section.

Theorem 3.4. It is possible to transfer the phase (t^0,k^0,x^0) to the phase (t,k,x) if and only if

$$\begin{aligned} & x-\left(\prod_{i=1}^{q}\Phi_i(t_i,t_i^0;t^0_{\overline{i-1}},t_{\tilde{\bar{i}}};k)\right)\times \\ & \times\left(\prod_{j=1}^{r}F_j(t^0;k_j,k_j^0,h^0_{\overline{j-1}},k_{\tilde{\bar{j}}})\right)x^0\in \operatorname{Im}\mathcal{R}(t^0,t;k^0,k) \end{aligned} \tag{3.3}$$

Proof. Sufficiency. If (3.3) holds, then $\exists v\in\mathbf{R}^n$ such that

$$\begin{aligned} & x-\left(\prod_{i=1}^{q}\Phi_i(t_i,t_i^0;t_{\overline{i-1}},t_{\tilde{\bar{i}}};k)\right)\times \\ & \times\left(\prod_{j=1}^{r}F_j(t^0;k_j,k_j^0,h^0_{\overline{j-1}},k_{\tilde{\bar{j}}})\right)x^0=\mathcal{R}(t^0,t;k^0,k)v. \end{aligned} \tag{3.4}$$

Let us consider the control

$$u(s;l)=B(s,l)^T\left(\prod_{j=1}^{r}F_j(s;k_j,l_j+1;l_{\overline{j-1}},k_{\tilde{\bar{j}}})\right)^T\times\left(\prod_{i=1}^{q}\Phi_i(t_i,s_i;s_{\overline{i-1}},t_{\tilde{\bar{i}}};k)\right)^T v. \tag{3.5}$$

By (3.1) and (3.4) we obtain an equality similar with (2.6) with $x(t;k)=x$, i.e. u given by (3.5) realizes the indicated transfer.

Necessity. Let us assume that (2.6) holds with $x(t;k)=x$ for some control u^1. Let us denote by x^1 the vector introduced in (3.3). Since the matrix $\mathcal{R}=\mathcal{R}(t^0,t;k^0,k)$ is symmetrical, the direct sum decomposition $\mathbf{R}^n=\operatorname{Im}\mathcal{R}\oplus\operatorname{Ker}\mathcal{R}$ holds, hence there exist $x^2\in\operatorname{Im}\mathcal{R}$ and $x^3\in\operatorname{Ker}\mathcal{R}$ such that $x^1=x^2+x^3$. Since $x^2\in\operatorname{Im}\mathcal{R}$, x^2 can be represented as $x(t;k)=x^2$ in (2.6) for a control u^2 having the form (3.5). We consider the control $u^3=u^1-u^2$ and we subtract the above mentioned expressions (of the form (2.6)) of x^1 and x^2. One obtains

$$\begin{aligned} & x^3=x^1-x^2=\int_{t^0}^{t}\sum_{l=k^0}^{k-1}\left(\prod_{i=1}^{q}\Phi_i(t_i,s_i;s_{\overline{i-1}},t_{\tilde{\bar{i}}};k)\right)\times \\ & \times\left(\prod_{j=1}^{r}F_j(s;k_j,l_j+1;l_{\overline{j-1}},k_{\tilde{\bar{j}}})\right)B(s;l)u^3(s;l)ds. \end{aligned} \tag{3.6}$$

From $x^3\in\operatorname{Ker}\mathcal{R}$ we have $(x^3)^T\mathcal{R}x^3=0$ and taking into account the expression (3.1) of the Gramian $\mathcal{R}$ we get

$$\begin{aligned} & \int_{t^0}^{t}\sum_{l=k^0}^{k-1}||B(s;l)^T\left(\prod_{j=1}^{r}F_j(s;k_j,l_j+1;l_{\overline{j-1}},k_{\tilde{\bar{j}}})\right)^T\times \\ & \times\left(\prod_{i=1}^{q}\Phi_i(t_i,s_i;s_{\overline{i-1}},t_{\tilde{\bar{i}}};k)\right)^T x^3||ds. \end{aligned}$$

The integral being non-negative, it results that a.e.

$$B(s,l)^T \left(\prod_{j=1}^{r} F_j(s;k_j,l_j+1;l_{\overline{j-1}},k_{\tilde{\bar{j}}}) \right)^T \times \left(\prod_{i=1}^{q} \Phi_i(t_i,s_i;s_{\overline{i-1}},t_{\tilde{\bar{i}}};k) \right)^T x^3 = 0. \tag{3.7}$$

By (3.6) and (3.7) one obtains

$$||x^3||^2 = (x^3)^T x^3 = \int_{t^0}^{t} \sum_{l=k^0}^{k-1} (x^3)^T \left(\prod_{i=1}^{q} \Phi_i(t_i,s_i;s_{\overline{i-1}},t_{\tilde{\bar{i}}};k) \right) \times$$
$$\times \left(\prod_{j=1}^{r} F_j(s;k_j,l_j+1;l_{\overline{j-1}},k_{\tilde{\bar{j}}}) \right) B(s;l)u^3(s;l)ds = 0,$$

hence $x_3 = 0$ and $x_1 = x_2 \in \operatorname{Im}\mathcal{R}$.

We obtain from Theorem 3.4, by replacing $x^0 = 0$ in (3.3) (see Definition 3.1) the following:

Corollary 3.5. *The state x is reachable on $P = [t^0,t] \times [k^0,k]$ if and only if*

$$x \in \operatorname{Im}\mathcal{R}(t^0,t;k^0,k).$$

This statement can be reformulated as

Corollary 3.6. *The set of the states which are reachable on P is the subspace $X_r = \operatorname{Im}\mathcal{R}(t^0,t;k^0,k)$.*

It results that the system $\sum$ is completely reachable on P if and only if $\operatorname{Im}\mathcal{R}(t^0,t;k^0,k) = \mathbf{R}^n$, condition which gives the following criterion:

Theorem 3.7. *The system $\sum$ is completely reachable on P if and only if*

$$\operatorname{rank}\mathcal{R}(t^0,t;k^0,k) = n \tag{3.8}$$

In the definition of the following controllability Gramian we shall use the discrete time fundamental matrices F_j with the initial variables greater than the final ones, hence we shall assume the hypothesis (H) (all the matrices $A_{dj}(t;k)$ are nonsingular).

Definition 3.8. The *controllability Gramian* of $\sum$ on $P = [t^0,t] \times [k^0,k]$ is the matrix

$$\mathcal{C}(t^0,t;k^0,k) = \int_{t^0}^{t} \sum_{l=k^0}^{k-1} \left(\prod_{i=1}^{q} \Phi_i(t_i,s_i;s_{\overline{i-1}},t_{\tilde{\bar{i}}};k) \right) \times$$
$$\times \left(\prod_{j=1}^{r} F_j(s;k_j^0,l_j+1;l_{\overline{j-1}},k_{\tilde{\bar{j}}}) \right) B(s,l)B(s,l)^T \times$$
$$\times \left(\prod_{j=1}^{r} F_j(s;k_j^0,l_j+1;l_{\overline{j-1}},k_{\tilde{\bar{j}}}) \right)^T \left(\prod_{i=1}^{q} \Phi_i(t_i^0,s_i;s_{\overline{i-1}},t_{\tilde{\bar{i}}};k) \right)^T ds.$$

Theorem 3.9. It is possible to transfer the phase (t^0,k^0,x^0) to the phase (t,k,x) if and only if

$$\left(\prod_{i=1}^{q}\Phi_i(t_i^0,t_i;t^0_{\overline{i-1}},t_{\bar{\bar{i}}};k)\right)\times \tag{3.9}$$
$$\times\left(\prod_{j=1}^{r}F_j(t^0;k_j^0,k_j,k^0_{\overline{j-1}},k_{\bar{\bar{j}}})\right)x-x^0\in \operatorname{Im}\mathcal{C}(t^0,t;k^0,k)$$

The proof follows the lines of the Theorem 3.4 and it will be omitted.

By replacing $x=0$ in (3.9) we get

Corollary 3.10. *A state x^0 is controllable on P if and only if $x^0\in \operatorname{Im}\mathcal{C}(t^0,t;k^0,k)$.*

Corollary 3.11. *The set of the states of $\sum$ which are controllable on P is the subspace $X_c=\operatorname{Im}\mathcal{C}(t^0,t;k^0,k)$.*

Theorem 3.12. *A system $\sum$ is completely controllable if and only if*

$$\operatorname{rank}\mathcal{C}(t^0,t;k^0,k)=n \tag{3.10}$$

Now, we replace $x=0$ in (3.3) and we obtain.

Corollary 3.13. *A state x^0 is controllable on P if and only if*

$$\left(\prod_{i=1}^{q}\Phi_i(t_i,t_i^0;t^0_{\overline{i-1}},t_{\bar{\bar{i}}};k)\right)\times \tag{3.11}$$
$$\times\left(\prod_{j=1}^{r}F_j(t^0;k_j,k_j^0,h^0_{\overline{j-1}},k_{\bar{\bar{j}}})\right)x^0\in \operatorname{Im}\mathcal{R}(t^0,t;k^0,k)$$

Corollary 3.14. *The system is completely controllable on P if and only if*

$$\operatorname{Im}\left(\prod_{i=1}^{q}\Phi_i(t_i,t_i^0;t^0_{\overline{i-1}},t_{\bar{\bar{i}}};k)\right)\times \tag{3.12}$$
$$\times\left(\prod_{j=1}^{r}F_j(t^0;k_j,k_j^0,h^0_{\overline{j-1}},k_{\bar{\bar{j}}})\right)\subset \operatorname{Im}\mathcal{R}(t^0,t;k^0,k)$$

Under the hypothesis (H) we can premultiply (3.4) by the nonsingular matrix

$$M=\left(\prod_{i=1}^{q}\Phi_i(t_i^0,t_i;t^0_{\overline{i-1}},t_{\bar{\bar{i}}};k)\right)\left(\prod_{j=1}^{r}F_j(t^0;k_j^0,k_j,k^0_{\overline{j-1}},k_{\bar{\bar{j}}})\right) \tag{3.13}$$

and we obtain

Corollary 3.15. If all the matrices $A_{dj}(t;k)$ are nonsingular, then a state x^0 is controllable on P if and only if

$$x^0\in \operatorname{Im}M\mathcal{R}(t^0,t;k^0,k).$$

It results that the system $\sum$ is completely controllable if and only if $\operatorname{Im}M\mathcal{R}(t^0,t;k^0,k)=\mathbf{R}^n$, condition which is equivalent to $\operatorname{rank}M\mathcal{R}(t^0,t;k^0,k)=n$. Taking into account that the matrix M (3.13) is nonsingular, we obtain

Theorem 3.16. *If all the matrices $A_{dj}(t;k)$ are nonsingular, then the system $\sum$ is completely controllable if and only if rank $\mathcal{R}(t^0,t;k^0,k)=n$.*

By Theorems 3.7 and 3.16 and Corollary (3.14) we can emphasize the relationship between the concepts of complete reachability and complete controllability, which is in accordance with the discrete character of $\sum$ with respect to the variables $k_1,\dots k_r$.

Theorem 3.17. *If the system $\sum$ is completely reachable then $\sum$ is completely controllable.*

If the matrices $A(t;k)$ are nonsingular and $\sum$ is completely controllable, then $\sum$ is completely reachable.

4 Minimum-energy transfer and the optimal control law

Let us consider the multiple interval $P=[t^0,t]\times[k^0,k]\subset\mathbf{R}^q\times\mathbf{Z}^r$.

Definition 4.1. The *energy* of a control $u:P\to\mathbf{R}^m$ is

$$E_u=\int_{t_1^0}^{t_1}\dots\int_{t_q^0}^{t_q}\sum_{l=k_1}^{k_1-1}\dots\sum_{l=k_r^0}^{k_r-1}||u(s_1,\dots,s_q;l_1,\dots,l_r)||^2ds_1\dots ds_q \tag{4.1}$$

As we discussed previously, we can use for (3.1) the notation:

$$E_u=\int_{t^0}^{t}\sum_{l=k^0}^{k-1}||u(s;l)||^2ds.$$

Theorem 4.2. *Assume that the system $\sum$ is completely controllable. Then the control*

$$\begin{aligned}\tilde{u}(s;l)={}&B(s,l)^T\left(\prod_{i=1}^{q}\Phi_i(t_i,s_i,s_{\overline{i-1}},t_{\tilde{\bar{i}}};k)\right)^T\times\\ &\times\left(\prod_{j=1}^{r}F_j(s;k_j^0,l_j+1;l_{\overline{j-1}},k_{\tilde{\bar{j}}})\right)\mathcal{C}(t^0,t;k^0,k)^{-1}\times\\ &\times\left(\left(\prod_{i=1}^{q}\Phi_i(t_i^0,t_i;t^0_{\overline{i-1}},t_{\tilde{\bar{i}}};k)\right)\left(\prod_{j=1}^{r}F_j(s^0;k_j^0,k_j,k^0_{\overline{j-1}},k_{\tilde{\bar{j}}})\right)^T x^1-x^0\right)\end{aligned}\tag{4.2}$$

transfers the phase (t^0,k^0,x^0) to the phase (t,k,x^1) with the minimum expenditure of energy, i.e. $E_{\tilde{u}}\le E_u$ for any control u which realizes the same transfer.

Proof. Since the system $\sum$ is completely controllable, by Theorem 3.12 the matrix $\mathcal{C}(t^0,t;k^0,k)$ is nonsingular, hence the control $\tilde{u}$ (4.2) is well defined. We replace the control u in the equation (2.6) by $\tilde{u}$ (4.2) and we obtain (by using the semigroup properties the state of $\sum$ at the moment $(t;k)$):

$$
\begin{aligned}
x(t,k) = {} & \left(\prod_{i=1}^{q}\Phi_i(t_i,t_i^0,t^0_{\overline{i-1}},t_{\tilde{\tilde{i}}};k)\right)\times \\
& \times\left(\prod_{j=1}^{r}F_j(t^0;k_j,k_j^0;k^0_{\overline{j-1}},k_{\tilde{\tilde{j}}})\right)(x^0+\mathcal{C}(t^0,t;k^0,k)\mathcal{C}(t^0,t;k^0,k)^{-1}\times \\
& \times\left(\prod_{i=1}^{q}\Phi_i(t_i^0,t_i;t^0_{\overline{i-1}},t_{\tilde{\tilde{i}}};k)\right)\left(\prod_{j=1}^{r}F_j(t^0;k_j^0,k_j,k^0_{\overline{j-1}},k_{\tilde{\tilde{j}}})\right)x^1-x^0)=x^1,
\end{aligned}
$$

hence $\tilde{u}$ realizes the transfer of the phase (t^0,k^0,x^0) to the phase (t,k,x^1).

Let u be another control which realizes the same transfer, and let u^1 be the control $u^1=u-\tilde{u}$. By replacing in (2.6) the control by u and $\tilde{u}$ respectively and by the semigroup property we get two equalities:

$$
\begin{aligned}
& \left(\prod_{i=1}^{q}\Phi_i(t_i^0,t_i,t^0_{\overline{i-1}},t_{\tilde{\tilde{i}}};k)\right)\left(\prod_{j=1}^{r}F_j(t^0;k_j^0,k_j;k^0_{\overline{j-1}},k_{\tilde{\tilde{j}}})\right)x^1-x^0= \\
& =\int_{t^0}^{t}\sum_{l=k^0}^{k-1}\left(\prod_{i=1}^{q}\Phi_i(t_i^0,s_i;s_{\overline{i-1}},t_{\tilde{\tilde{i}}};k)\right)\left(\prod_{j=1}^{r}F_j(s;k_j^0,l_j+1,l_{\overline{j-1}},k_{\tilde{\tilde{j}}})\right)u(s,l)ds= \\
& =\int_{t^0}^{t}\sum_{l=k^0}^{k-1}\left(\prod_{i=1}^{q}\Phi_i(t_i^0,s_i;s_{\overline{i-1}},t_{\tilde{\tilde{i}}};k)\right)\left(\prod_{j=1}^{r}F_j(s;k_j^0,l_j+1,l_{\overline{j-1}},k_{\tilde{\tilde{j}}})\right)\tilde{u}(s,l)ds
\end{aligned}
$$

hence

$$
\int_{t^0}^{t}\sum_{l=k^0}^{k-1}\left(\prod_{i=1}^{q}\Phi_i(t_i^0,s_i;s_{\overline{i-1}},t_{\tilde{\tilde{i}}};k)\right)\times\left(\prod_{j=1}^{r}F_j(s;k_j^0,l_j+1,l_{\overline{j-1}},k_{\tilde{\tilde{j}}})\right)u^1(s,l)ds=0 \tag{4.3}
$$

By (4.2) and (4.3) we get $\int_{t^0}^{t}\sum_{l=k^0}^{k-1}\tilde{u}(s;l)^Tu^1(s;l)ds=0$, hence (4.1) gives

$$
\begin{aligned}
E_u &= \int_{t^0}^{t}\sum_{l=k^0}^{k-1}u(s;t)^Tu(s,l)ds= \\
&= \int_{t^0}^{t}\sum_{l=k^0}^{k-1}[u^1(s,l)^T+\tilde{u}(s,l)^T][u^1(s,l)+\tilde{u}(s,l)]ds= \\
&= \int_{t^0}^{t}\sum_{l=k^0}^{k-1}||u^1(s,l)||^2ds+\int_{t^0}^{t}\sum_{l=k^0}^{k-1}||\tilde{u}(s,l)||^2ds\geq E_{\tilde{u}}.
\end{aligned}
$$

Therefore $\tilde{u}$ (4.1) is the optimal control for the desired transfer.

Corollary 4.3. *If the system $\sum$ is completely controllable, the minimum "control energy" required by the transfer of (t^0,k^0,x^0) to (t,k,x^1) is*

$$
E_{\min}=(x^2)^T\mathcal{C}(t^0,t;k^0,k)^{-1}x^2 \tag{4.4}
$$

where

$$x^2 = \left(\prod_{i=1}^{q} \Phi_i(t_i^0, t_i; t^0_{\overline{i-1}}, t_{\tilde{\bar{i}}}; k) \right) \left(\prod_{j=1}^{r} F_j(t^0; k_j^0, k_j, k^0_{\overline{j-1}}, k_{\tilde{\bar{j}}}) \right) x^1 - x^0.$$

Proof. By (4.1) and (4.2) since $\mathcal{C}(t^0, t; k^0, k)^{-1}$ is symmetrical we have:

$$\begin{aligned} E_{\min} &= E_{\tilde{u}} = \int_{t^0}^{t} \sum_{l=k^0}^{k-1} ||\tilde{u}(s,l)||^2 ds = \int_{t^0}^{t} \sum_{l=k^0}^{k-1} \tilde{u}(s,l)^T \tilde{u}(s,l) ds = \\ &= (x^2)^T \mathcal{C}(t^0, t; k^0, k)^{-1} \mathcal{C}(t^0, t; k^0, k) \mathcal{C}(t^0, t; k^0, k)^{-1} x^2 = \\ &= (x^2)^T \mathcal{C}(t^0, t; k^0, k)^{-1} x^2. \end{aligned}$$

Conclusion. This paper studies a class of multidimensional continuous-discrete systems from the point of view of the concepts of controllability and reachability. Necessary and sufficient conditions of complete controllability and complete reachability are expressed by introducing two suitable controllability and reachability Gramians. The connection between the two concepts is emphasized. The optimal control which realizes the minimum-energy transfer is determined and the formula of the minimum control energy is provided. This study can be continued for other concepts as observability, stability, stabilizability (for instance as in [2]) and detectability of multidimensional continuous-discrete systems and it can be applied to the problems of optimal control.

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