

Geodesible vector fields and adapted invariant Riemannian metrics

Gabriel-Teodor Pripoae and Cristina-Liliana Pripoae

Abstract. In [8], we studied the existence of geodesible left invariant vector fields on Lie groups, with respect to left invariant Riemannian metrics. Accordingly, seven distinct types of specific behaviours led to a classification of Lie groups, through linear algebraic conditions. We prove now that three of these types do not exist, so the respective classification has exactly four classes. The type I Lie groups (such that *all* their left invariant vector fields are geodesible) are completely characterized as semi-direct products of solvable groups satisfying the property (Im), with compact Lie groups. We introduce some new linear algebraic invariants (the geodesic height and the geodesic dimensions), which provide geometrically inspired classifications of the Lie groups.

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1 Introduction

Given a "field of forces" on a differentiable manifold, do there exist Riemannian metrics such that the "particles" move free falling with respect to them? In the affirmative case, how many such metrics exist? To what extent the ("moduli") space of all these metrics characterizes the initial manifold?

These difficult problems are more tractable - for the beginning - in the particular case of Lie groups, where the algebraic tools lead easier to interesting and (some quite unexpected) results.

The starting point of this study was the geometrization of the trajectories of left invariant vector fields on Lie groups. A paper of V. Arnold from 1966 ([1]), describing the dynamics of the Euler equation in terms of geodesics of a left invariant Riemannian metric on $SO(3)$, opened a new promising area of investigations (see [4] for further details).

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There are also different but related approaches, as [3], where, on four-dimensional Lie groups, necessary and sufficient conditions were found in order that a system of second order ordinary differential equations be the Euler-Lagrange equations of a regular Lagrangian function (for other related theories, see [6], [13] and the references therein).

In [8] we proved that for every left invariant vector field ξ on a Lie group G , there exists a Riemannian metric g such that the trajectories of ξ be geodesics of g . The necessary and sufficient condition to exist a *left invariant* Riemannian metric with the respective property is

$$(1.1) \quad \xi \notin L_\xi(L(G))$$

where L_ξ the Lie derivative operator associated to ξ and $L(G)$ is the Lie algebra of G . If this does not happen, there exists however a left invariant *Lorentzian* metric g such that the trajectories of ξ are geodesics of g ([8]). Despite its simple form, the linear algebraic condition (1.1) catches deep geometric properties of the Lie groups.

For a given Lie group G , the fact that some (or all the) left invariant vector fields satisfy (1.1) classifies it into seven (potential) families (according to the elementary use of logical quantifiers). In §2 we prove that four and only four such families may exist, and give some examples of each type.

The Lie groups admitting bi-invariant Riemannian metrics were classified in [5] (they are exactly the products of compact groups with abelian ones); in such a Lie group, every left invariant vector field has the property (1.1). In Theorem 3.1 we extend this result, proving that the Lie groups, in which every left invariant vector field ξ has the property (1.1), are exactly the semi-direct products of compact Lie groups with some special solvable groups (such that the adjoint operators have only null or non-real eigenvalues).

For a given vector field ξ on the manifold M , the (total) moduli space $Riem(M, \xi)$ contains *all* the Riemannian metrics g on M such that the trajectories of ξ are geodesics of g . In §4, we determine the (restricted) moduli space of *left invariant* Riemannian metrics adapted for some left invariant vector fields ξ on a Lie group G . In Proposition 4.1, we establish an upper and a lower bound for the "dimension" of $Riem(G, \xi)$ (the former is attained, in particular, for the -eventually- central elements ξ).

Finally, we refine the classification of §2, using some new linear algebraic invariants of geometric inspiration: the *geodesic height* and the *geodesic dimensions* of a Lie group (§5). Case studies in dimensions 2 and 3 show some "fingerprints" of the new invariants.

2 The Lie groups classification: improvement

Let ξ be a non-vanishing left invariant vector field on an n -dimensional Lie group G . (The case of vanishing ones is trivial: any trajectory is constant and degenerate geodesic for any metric).

Remark. In [8] we proved the following results:

(i) For every left invariant vector field ξ on G , there exists a Riemannian metric g such that the trajectories of ξ be geodesics of g .

(ii) Let ξ be a non-vanishing left invariant vector field on G . Then there exists a left invariant Riemannian metric g on G , such that the trajectories of ξ be geodesics of g , if and only if the relation (1.1) holds.

(iii) Let ξ be a non-vanishing left invariant vector field on G .

(iii₁) If

$$(2.1) \quad \xi \in L_{\xi}(L(G))$$

then there exists a left invariant Lorentzian metric g on G , such that ξ be lightlike and the trajectories of ξ be geodesics of g .

(iii₂) Suppose $\dim L_{\xi}(L(G)) = \dim G - 1$. If there exists a left invariant Lorentzian metric g on G , such that the trajectories of ξ be geodesics of g and ξ be lightlike, then the relation (2.1) holds.

Remark. In the following, all the considered left invariant vector fields are supposed to be non-null. Elementary combinations of logical quantifiers in the relations (1.1) and (2.1) allowed the "classification" of the Lie groups in 7 types (see [8] for details):

I. the ones in which every left invariant vector field has the property (1.1).

Examples: the Lie groups admitting a bi-invariant Riemannian metric (in particular, the compact ones and the abelian ones); the Heisenberg group; the oscillator groups G_{λ} ([2]); (see also the Proposition 3.1).

II. the ones in which every $\xi \in L(G)$ has the property (2.1) and

$$\dim L_{\xi}(L(G)) = n - 1$$

III. the ones in which every $\xi \in L(G)$ has the property (2.1) and

$$\dim L_{\xi}(L(G)) \neq n - 1$$

IV. the ones in which every $\xi \in L(G)$ has the property (2.1), there exist $\alpha, \beta \in L(G)$, with $\dim L_{\alpha}(L(G)) = n - 1$ and $\dim L_{\beta}(L(G)) \neq n - 1$;

V. the ones in which there exist left invariant vector fields ξ having the property (1.1), $\eta, \theta \in L(G)$ with the property (2.1) and $\dim L_{\eta}(L(G)) = n - 1$, $\dim L_{\theta}(L(G)) \neq n - 1$;

Example: a "generic" 3-dimensional non-unimodular Lie group.

VI. the ones in which there exist left invariant vector fields ξ having the property (1.1), $\eta \in L(G)$ with the property (2.1) and $\dim L_{\eta}(L(G)) = n - 1$, and there does not exist $\theta \in L(G)$ with the property (2.1) and $\dim L_{\theta}(L(G)) \neq n - 1$;

Examples: the nonabelian 2-dimensional Lie group; $SL(2, \mathbb{R})$.

VII. the ones in which there exist left invariant vector fields ξ having the property (1.1), $\eta \in L(G)$ with the property (2.1) and $\dim L_{\eta}(L(G)) \neq n - 1$, and there does not exist $\theta \in L(G)$ with the property (2.1) and $\dim L_{\theta}(L(G)) = n - 1$.

Examples: the "exceptional" solvable Lie groups of class σ (i.e. those with the property that for every left invariant vector fields X, Y , the Lie bracket $[X, Y]$ belongs to the space spanned by X and Y).

The lack of examples in [8] for the families II, III, IV was not fortuite. In fact, we prove now the

Theorem 2.1. *There is no Lie group G such that relation (2.1) holds, for every $\xi \in L(G)$.*

Proof. Suppose ad absurdum that there exists a Lie group G such that (2.1) holds, for any $\xi \in L(G)$. Then $L(G) = [L(G), L(G)]$, hence the radical and the nilradical coincide. By the Levi-Malcev theorem ([10]), the algebra $L(G)$ splits in a semidirect sum of its nilradical N and a semisimple subalgebra S . Due to (2.1), we have

$$[N, S] = N \quad , \quad [S, S] = S \quad , \quad [N, N] \subset N$$

Suppose S is trivial; then $L(G)$ is nilpotent. Consider a non-vanishing $\xi \in L(G)$ and $Z \in L(G)$ such that $\xi = [Z, \xi]$. Because Z is nilpotent, there exists a natural k such that $(ad_Z)^k = 0$; so $\xi = (ad_Z)^k(\xi) = 0$, contradiction. So S cannot be trivial.

From the Cartan decomposition of S ([10]), we know there are two subalgebras l and p such that

$$S = l \oplus p \quad , \quad [l, l] \subset l \quad , \quad [l, p] \subset p \quad , \quad [p, p] \subset l$$

Moreover, the Killing form of S is negatively definite on l and positively definite on p .

If l is trivial, then $[p, p] = l = \{0\}$, so S is abelian, contradiction.

We may thus consider a non-vanishing $\xi \in l$. Let $Z \in l$ such that $\xi = [Z, \xi]$. But ad_Z is skew-symmetric, hence all its eigenvalues must be null or complex with null real part; we obtained again a contradiction.

So the initial hypothesis ad absurdum leads always to a contradiction; the theorem is thus proved. \square

Remark. We conclude that on each Lie group, there exist left invariant vector fields satisfying the relation (1.1). Thus, the families II, III, IV in the previous classification are void and the study must go on with the types I, V, VI and VII only.

We do not renumber the types, because we shall refine this classification in §5, using additional invariants.

3 Characterization of the Lie groups of type I

By contrast with the previous situation (as in Theorem 2.1), the Lie groups of type I are quite numerous, as may be seen from Theorem 3.1. In order to prove it, we need the

Definition. A Lie group *has the property (Im)* if the eigenvalues of all the adjoint operators are null or non-real.

The property (Im) is somewhat opposed to the property (R), when the eigenvalues of all the adjoint operators are real. (A solvable group with the property (R) is called also completely solvable). Any Lie group of type I must have the property (Im) and conversely. More precisely, we have the

Theorem 3.1. *The Lie groups of type I are exactly the semi-direct products of solvable groups satisfying the property (Im), with compact Lie groups (where the action is without non-vanishing real eigenvalues).*

Proof. First step. Consider G a solvable Lie group with the property (Im) and suppose ad absurdum that there exists a non-null $\xi \in L(G)$ which satisfies the relation (2.1). Hence, there exists a $Z \in L(G)$ such that $\xi = ad_Z(\xi)$, so ad_Z has at least one real non-null eigenvalue.

Consider G a compact group and denote by b_K the Killing form on the Lie algebra $L(G)$. When G is compact, b_K generates a negatively definite bi-invariant metric g on G . With respect to g , any adjoint operator ad_X is skew-symmetric, hence its eigenvalues are null or purely imaginary (i.e. with null real component). This property implies relation (1.1), for any non-null left invariant vector field on G .

On another hand, the type I property is closed under semi-direct product (acting without non-vanishing real eigenvalues), so such a semi-direct product of a solvable group satisfying the property (Im) with a compact group is also of type I.

Second step. Now, we shall prove the converse. Consider a Lie group G of type I. By the Levi-Malcev decomposition theorem, the G may be expressed as a semidirect product of a solvable group N and a semisimple one S . As in the first step, we verify easily that N must have the property (Im).

We show now that the semisimple factor must be compact. By the Cartan decomposition, we have $L(S) = l \oplus m$, where l and m are subalgebras, such that $[l, l] \subset l$, $[l, m] \subset m$, $[m, m] \subset l$; l is negatively definite and m is positively definite with respect to the Killing form. Suppose there exists a non-null $\xi \in m$; then the operator ad_ξ is symmetric (cf. [10], p.145), so it has real eigenvalues. As ξ cannot be central, at least one eigenvalue must be non-null, thus contradicting (1.1). Thus, $m = \{0\}$, so the Killing form is negatively definite on $L(S)$; that means that $L(S)$ (hence S) is compact.

A similar argument imposes that the action in the semi-direct product cannot have non-null real eigenvalues. \square

4 The moduli space associated to a left invariant vector field

Remarks. (i) Choose a basis $\{E_1, E_2\}$ in the Lie algebra of the non-commutative 2-dimensional Lie group G , such that $[E_1, E_2] = E_2$. Then $Riem(G, E_1)$ is the set of all left invariant Riemannian metrics spanned (via left translations) by the symmetric, positively definite bilinear forms on $L(G)$, with diagonal form in the respective basis.

(ii) Consider now the group G of rigid motions in the Euclidean space \mathbb{R}^2 . Its Lie algebra admits a basis $\{E_1, E_2, E_3\}$ such that

$$[E_1, E_2] = E_3 \quad , \quad [E_2, E_3] = E_1$$

A short calculation, based on the idea of proof of Theorem 2.1, shows that $Riem(G, E_2)$ is the set of all left invariant Riemannian metrics spanned by the symmetric, positively definite bilinear forms $B \in L(G)$, satisfying $B(E_1, E_3) = 0$.

In general, we have the

Proposition 4.1. *Let ξ be a left invariant vector field on an n -dimensional Lie group G , satisfying (1.1). Then:*

(i) the set $Riem(G, \xi)$ is parametrized by a convex open subset of a real vector space, of dimension at most $n(n+1)/2$ and at least $n(n-1)/2 + 1$.

(ii) The maximal dimension is "reached" for, and only for, the (eventually) central vectors $\xi \in L(G)$.

Proof. (i) As any left invariant Riemannian metric is generated (through left translations) by its value in the unit element e of G , it is sufficient to consider the symmetric, positively definite bilinear forms on $B \in L(G)$, given by $B = g_e$, where $g \in Riem(G, \xi)$.

Let $\{E_1, \dots, E_n\}$ be a basis in $L(G)$, such that $E_1 = \xi$ and

$$L_\xi(L(G)) \subset \text{span}\{E_2, \dots, E_n\}$$

Denote by c_{js}^k the structure constants of the given basis and $B_{ij} = B(E_i, E_j)$.

The Koszul's formula applied to B gives $B(E_1, [E_1, E_i]) = 0$, for any $i = \overline{2, n}$. This equation may be expressed as the system

$$(4.1) \quad \sum_{k=2}^n C_{1i}^k B_{1k} = 0 \quad , \quad i = \overline{2, n}$$

The solutions B belong to a vector space, whose maximal dimension is $n(n+1)/2$.

At least one of the coefficients $B_{11}, B_{12}, \dots, B_{1n}$ must be non-null. The other coefficients of B are not involved in the system (4.1), so the lower possible dimension for the previous vector space is $n(n-1)/2 + 1$.

The positiveness condition on B is an open one.

If B and \tilde{B} are solutions of (4.1) and $a \in [0, 1]$, then $aB + (1-a)\tilde{B}$ is again solution of (4.1), which proves the convexity requirement.

(ii) For any central element ξ , the system (4.1) is identically satisfied. Conversely, suppose $\xi \in L(G)$ satisfies (1.1) and (4.1), for any B ; then, a linear algebraic argument imposes that ξ must be central. \square

The examples in the remark preceding Proposition 4.1 show that the respective convex open sets have codimension 1, with respect to the maximal (potential) dimension. However, in general, one cannot improve the lower and the upper bounds in Proposition 4.1, as it may be seen on the non-abelian two-dimensional Lie group and on some three-dimensional Lie groups (cf. § 5).

Example 4.2. Consider $G = SO(3)$ and $\{E_1, E_2, E_3\}$ a basis in the Lie algebra $so(3)$. We have

$$[E_1, E_2] = E_3 \quad , \quad [E_2, E_3] = E_1 \quad , \quad [E_3, E_1] = E_2$$

Let $\xi = a^1 E_1 + a^2 E_2 + a^3 E_3$ be an arbitrary non-vanishing left invariant vector field and let $g \in Riem(G, \xi)$. Then $g(\xi, [\xi, Z]) = 0$, for every $Z \in so(3)$. Taking Z to be

E_1, E_2 and E_3 respectively, we obtain that the components of g in the given basis are subject to the following system of linear and homogeneous equations

$$\begin{aligned} a^1 a^3 g_{12} - a^1 a^2 g_{13} - (a^2)^2 g_{23} + (a^3)^2 g_{23} + a^2 a^3 g_{22} - a^2 a^3 g_{33} &= 0 \\ a^1 a^2 g_{23} - a^2 a^3 g_{12} - (a^3)^2 g_{13} + (a^1)^2 g_{13} + a^1 a^3 g_{33} - a^1 a^3 g_{11} &= 0 \\ a^2 a^3 g_{13} - a^1 a^3 g_{23} - (a^1)^2 g_{12} + (a^2)^2 g_{12} + a^1 a^2 g_{11} - a^1 a^2 g_{22} &= 0 \end{aligned}$$

A short computation shows that $Riem(G, \xi)$ is an open subset of a 4-dimensional real vector space.

As $SO(3)$ is compact, it admits bi-invariant Riemannian metrics. Of course, any such metric g must belong to $Riem(SO(3), \xi)$, for every $\xi \in so(3)$: indeed, due to the bi-invariance of g , we have $g(X, [X, Z]) = 0$, for every $X, Z \in so(3)$ (or, equivalently, the geodesics are one-parameter subgroups).

We may ask if in $Riem(SO(3), \xi)$ there exist left invariant Riemannian metrics which are not bi-invariant. For some ξ 's, the answer is affirmative: take the left invariant Riemannian metric whose components in the previous basis are $g_{11} = g_{22} = g_{33} = 2g_{13} = 1$ and $g_{12} = g_{23} = 0$. Then $g \in Riem(SO(3), E_1 + E_3)$ and is not bi-invariant, due to the fact that $2g(E_3, [E_3, E_2]) = -1$.

The next two propositions depict the geodesic behaviour of the left invariant vector fields, with respect to some left invariant *indefinite* metric; (we recall that a non-vanishing vector v is called timelike, spacelike or lightlike with respect to an indefinite metric g if $g(v, v)$ is negative, positive or null, respectively). The results are outside the main line of the paper, and are introduced only to suggest how further refinements may be made in the indefinite setting.

Proposition 4.3. *Let ξ be a left invariant vector field on a Lie group G , satisfying (2.1) and $\nu = \overline{1, n-1}$. Then:*

- (i) *there exists a left invariant semi-Riemannian metric g of index ν , such that ξ be lightlike and geodesic in (G, g) .*
- (ii) *there does not exist a left invariant semi-Riemannian metric g of index ν , such that ξ be timelike and geodesic in (G, g) .*
- (iii) *there does not exist a left invariant semi-Riemannian metric g of index ν , such that ξ be spacelike and geodesic in (G, g) .*

Proof. (i) Denote $n = \dim G$. We have $\dim L_\xi(L(G)) \leq n-1$. We may choose a basis $\{e_1, \dots, e_n\}$ for $L(G)$, such that

$$L_\xi(L(G)) \subset \text{span}\{\xi, e_3, e_4, \dots, e_n\}$$

and $\xi = e_1 - e_2$. We choose an indefinite left invariant metric on $L(G)$, such that $g(e_2, e_2) = -g(e_1, e_1) = 1$, $g(e_i, e_j) = \epsilon_i \delta_{ij}$ for the other indices, where $\epsilon_i = \pm 1$ such that the index of g be ν . It follows that ξ is lightlike in (G, g) .

(ii)+(iii) Suppose ad absurdum there exists an indefinite left invariant metric g such that ξ is geodesic and spacelike (or timelike) in (G, g) . Then $g(\xi, [\xi, Z]) = 0$, for every $Z \in L(G)$. On another hand, from (2.1) there exists $\eta \in L(G)$ such that $\xi = [\xi, \eta]$. It follows that $g(\xi, \xi) = 0$, false ! \square

Proposition 4.4. *Let ξ be a left invariant vector field on a Lie group G , satisfying (1.1) and $\nu = \overline{1, n-1}$. Then:*

(i) *there exists a left invariant semi-Riemannian metric g of index ν , such that ξ be spacelike and geodesic in (G, g) ;*

(ii) *there exists a left invariant semi-Riemannian metric g of index ν , such that ξ be timelike and geodesic in (G, g) ;*

(iii) *there exists a left invariant semi-Riemannian metric g of index ν , such that ξ be lightlike and geodesic in (G, g) , if and only if*

$$\dim L_\xi(L(G)) \leq n - 2$$

Proof. (i) Denote $n = \dim G$. We choose a basis $\{e_1, e_2, \dots, e_n\}$ of $L(G)$, such that $e_1 = \xi$ and $L_\xi(L(G)) \subset \text{span}\{e_2, \dots, e_n\}$. We construct an indefinite left invariant metric on G , such that $g(e_i, e_j) = \epsilon_i \delta_{ij}$, ($i, j = \overline{1, n}$, with $\epsilon_{ij} = \pm 1$). We may choose $\epsilon_1 = 1$ and the remaining ϵ_i 's such that the index of g be ν . Then ξ is orthogonal on $L_\xi(L(G))$, so $g(\xi, [\xi, Z]) = 0$, for all $Z \in L(G)$. This proves that ξ is geodesic and spacelike in (G, g) .

A similar proof works for (ii).

(iii) Due to (1.1), we have $\dim L_\xi(L(G)) \leq n - 1$. Suppose $\dim L_\xi(L(G)) = n - 1$ and suppose ad absurdum there exists an indefinite left invariant metric g such that $g(\xi, \xi) = 0$; from the geodesicity of ξ , it follows that $g(\xi, L_\xi(L(G))) = 0$, so ξ is orthogonal on $L(G)$. This contradicts the non-degeneracy of g .

Conversely, suppose $\dim L_\xi(L(G)) \leq n - 2$; we split $L(G) = L_\xi(L(G)) \oplus \text{span}\xi \oplus V$, where V is a subspace of $L(G)$ of dimension at least 1. We choose a basis $\{e_1, \dots, e_n\}$ of $L(G)$, such that for some k , $\{e_1, \dots, e_k\}$ be a basis of $L_\xi(L(G))$, $\{e_{k+1}, \dots, e_n\}$ be a basis of $\text{span}\xi \oplus V$, and $\xi = e_{n-1} - e_n$. Now, an indefinite left invariant metric g of index ν may be constructed, such that $g(e_{n-1}, e_{n-1}) = -g(e_n, e_n) = 1$, $g(e_i, e_j) = \epsilon_i \delta_{ij}$ (for the other indices i, j and suitable $\epsilon_i = \pm 1$). Then ξ is lightlike and geodesic in (G, g) . \square

5 The Lie groups classification: extension

Remark 5.1. For the Lie groups of type I, the the maximal number of linearly independent left invariant vector fields, satisfying the relation (1.1), equals the dimension. The same property arrises for a Lie group G , with the derived ideal $[L(G), L(G)]$ strictly included in $L(G)$: indeed, denote $n = \dim(G)$ and let $\{e_1, e_2, \dots, e_n\}$ be a basis of $L(G)$, such that e_1 does not belong to $[L(G), L(G)]$ and for some integer k , $\text{span}\{e_2, \dots, e_k\} = [L(G), L(G)]$. Then, for any $i = \overline{2, n}$, the left invariant vector field $e_1 + e_i$ has the property (1.1).

The previous remark, as well as the case studies in dimensions 2 and 3 suggest the following

Proposition 5.2. *On any Lie group G , the set of all left invariant vector fields, satisfying (1.1), is an open subset in $L(G)$.*

As a consequence, the maximal number of linearly independent left invariant vector fields, satisfying the relation (1.1), equals the dimension of G .

Proof. Let ξ be a left-invariant vector field, satisfying (1.1). Consider a basis $E_1 := \xi, E_2, \dots, E_n$ of $L(G)$. The function $f : L(G) \rightarrow \mathbb{R}, f(v) = v \wedge [v, E_2] \wedge \dots \wedge [v, E_n]$ is continuous. By (1.1), we obtain $f(\xi) \neq 0$. It follows there exists an open neighborhood U of ξ in $L(G)$, such that for every $\eta \in U$, the set $\{\eta, E_2, \dots, E_n\}$ is a basis of $L(G)$, and $f(\eta) \neq 0$. We conclude that all $\eta \in U$ have the property (1.1). \square

Remark. A direct consequence of Proposition 5.2. is the existence, on any n -dimensional Lie group, of a special basis $\{E_1, \dots, E_n\} \subset L(G)$, such that every E_i be a geodesic vector field, with respect to some (proper) left invariant Riemannian metric $g^{(i)} \in \text{Riem}(G, E_i)$. The respective metrics may differ from one vector field to another. So, we shall call the *geodesic height* of a basis $\{E_1, \dots, E_n\} \in L(G)$ the maximal integer k , such that there exists a left invariant Riemannian metric g on G such that $g \in \bigcap_{i=1}^k \text{Riem}(G, E_i)$ (i.e. such that k of the vectors of the basis be geodesic with respect to the same left invariant Riemannian metric).

We say G has *geodesic height* k if there exists a basis of left invariant vector fields with geodesic height k . The geodesic height classifies the n -dimensional Lie groups in (at most) $(n - 1)$ classes, for $1 \leq k \leq n$.

Consider a Lie group G , admitting a bi-invariant Riemannian metric g and an arbitrary basis $\{E_1, \dots, E_n\} \subset L(G)$. We have $g(X, [X, Z]) = 0$, for any $X, Z \in L(G)$; in particular, this holds for $X := E_i$ and $Z := E_j$, for any indices i and j . Hence, every vector field E_i is geodesic with respect to g . It follows that G has the geodesic height n .

The Lie groups admitting bi-invariant Riemannian metrics are not the only ones with maximal geodesic height. The Heisenberg Lie group H is another example: consider $\{E_1, E_2, E_3\} \subset L(H)$ a basis with $[E_3, E_1] = E_2$ (see also the Example 5.7,iv) and the left-invariant Riemannian metric g , such that $g(E_i, E_j) = \delta_{ij}$. Then $g \in \bigcap_{i=1}^3 \text{Riem}(H, E_i)$ so H has geodesic height 3. On another hand, g is not bi-invariant, because $g(E_1 + E_2, [E_1 + E_2, E_3]) \neq 0$.

The Lie groups admitting bi-invariant Riemannian metrics and the Heisenberg group have type I. It may be conjectured that type I Lie groups are characterized by the maximal geodesic height.

Definition 5.5. For a Lie group G , we define:

- the *first geodesic dimension* of order i , denoted by $geodim_1(G)(i)$, as the maximal number of linearly independent left invariant vector fields ξ , satisfying (1.1) and such that $dim L_\xi(L(G)) = i$ (here $0 \leq i \leq n - 1$).
- the *second geodesic dimension* of order j , denoted by $geodim_2(G)(j)$, as the maximal number of linearly independent left invariant vector fields θ , satisfying (2.1) and such that $dim L_\theta(L(G)) = j$ (here $1 \leq j \leq n - 1$).
- the "*geodesic fingerprint*" of G , as the vector with $(2n - 1)$ non-negative integer components, given by

$$[geodim_1(0), \dots, geodim_1(n - 1); geodim_2(1), \dots, geodim_2(n - 1)]$$

(we omitted the G 's). In the sequel, we shall determine these invariants for the Lie groups of dimensions 2 and 3. (Even if some properly choosed vector basis will be used, the final results are coordinate-free).

Examples 5.6. (The 2-dimensional case) (i) For the abelian 2-dimensional Lie group, the "geodesic fingerprint" is $(2,0;0)$.

(ii) Choose a basis $\{E_1, E_2\}$ in the Lie algebra of the non-commutative 2-dimensional Lie group G , such that $[E_1, E_2] = E_2$. A vector field $\xi = aE_1 + bE_2 \in L(G)$ has the property (1.1) if and only if $a \neq 0$. It follows that $geodim_1(0) = 0$, $geodim_1(1) = 2$, $geodim_2(1) = 1$ and the "geodesic fingerprint" is $(0,2;1)$.

(iii) The geodesics for an arbitrary ("generic") left invariant Riemannian metric g , associated to an arbitrary left invariant field of forces ξ , will be studied in detail, in a forthcoming paper [9].

Examples 5.7. (The 3-dimensional case) We use the classification of the 3-dimensional Lie groups from [5].

(i) For the abelian 3-dimensional Lie algebra, the "geodesic fingerprint" is $(3,0,0;0,0)$.

(ii) For $G = SO(3)$ (cf. the Example 4.2), the "geodesic fingerprint" is $(0,0,3;0,0)$.

(iii) Consider the Lie algebra of rigid motions in the 2-dimensional Euclidean space, and a basis $\{E_1, E_2, E_3\}$ with $[E_1, E_2] = 0$, $[E_2, E_3] = E_1$, $[E_3, E_1] = E_2$. Then, the "geodesic fingerprint" is $(0,2,3;0,0)$.

(iv) Consider a basis $\{E_1, E_2, E_3\}$ in the Heisenberg algebra, with $[E_1, E_2] = [E_2, E_3] = 0$ and $[E_3, E_1] = E_2$; we calculate the "geodesic fingerprint" $(1,3,0;0,0)$.

(v) Consider the Lie algebra of rigid motions in the 2-dimensional Minkowski space, and a basis $\{E_1, E_2, E_3\}$ with $[E_1, E_2] = 0$, $[E_2, E_3] = E_1$, $[E_3, E_1] = -E_2$. Then, the "geodesic fingerprint" is $(0,2,3;2,0)$.

(vi) Denote by G the exceptional non-unimodular 3-dimensional Lie group (belonging to the σ family), and consider a basis $\{E_1, E_2, E_3\}$ in $L(G)$, with $[E_1, E_2] = E_2$, $[E_1, E_3] = E_3$, $[E_3, E_2] = 0$; the "geodesic fingerprint" is $(0,0,3;2,0)$.

(vii) Consider the Lie group $SL(2)$, and a basis $\{E_1, E_2, E_3\}$ with $[E_1, E_2] = -E_3$, $[E_2, E_3] = E_1$, $[E_3, E_1] = E_2$. Then, the "geodesic fingerprint" is $(0,0,3;0,3)$.

(viii) Denote by G a generic non-unimodular 3-dimensional Lie group, and consider a basis $\{E_1, E_2, E_3\}$ in $L(G)$, with $[E_1, E_2] = \alpha E_2 + \beta E_3$, $[E_1, E_3] = \gamma E_2 + \delta E_3$, $[E_3, E_2] = 0$, where $\alpha, \beta, \gamma, \delta$ are real parameters with $\alpha + \delta = 2$ and $(\alpha - 1)^2 + \beta^2 + \gamma^2 + (\delta - 1)^2 > 0$; the invariant $D = \alpha\delta - \beta\gamma$ classifies these groups, up to an isomorphism.

If $D = 0$, then the "geodesic fingerprint" is $(0,3,0;1,0)$. If $D \neq 0$, all (of the infinite family of isomorphism classes of) the respective Lie groups have the "geodesic fingerprint" $(0,3,0;2,0)$.

6 Conclusions

As it follows from the above examples, the *geodesic fingerprint* completely determines the isomorphism classes in dimension 2 and 3, (with the exception of the generic non-unimodular families in dimension 3), and constitutes a valuable candidate for future classification results. These possible new classifications may involve also the maximal and the minimal "dimensions" of the moduli spaces of metrics, as in §4,

including refined versions for the indefinite cases. All these new invariants have deep geometric interpretations, and contain surely relevant hidden information about the (geodesic) dynamics of the respective Lie groups. Recent interesting results concerning the interplay between vector fields and Riemannian metrics were obtained in [7] and [12], where applications and interpretations in Convex optimization theory may be found.

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Authors' addresses:

Gabriel-Teodor Pripoe
University of Bucharest, Faculty of Mathematics and Computer Science,
Str. Academiei nr. 14, RO-010014, Bucharest, Romania.
E-mail adress: gpripoae@fmi.unibuc.ro

Cristina-Liliana Pripoe
Academy of Economic Studies, Department of Mathematics,
Piata Romana nr.6, R0-010374, Bucharest, Romania.
E-mail adress: cristinapripoe@yahoo.com