Comparison of reliability models

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Abstract. Many stochastic models have been used in solving reliability problems, motivated by a high degree of variability or randomness of the studied phenomena. Therefore different types of stochastic laws derived from the basic distributions are proposed for modelling a hazard-rate function too. The main contribution of the present paper is to proposed new adaptive hazard rate functions, derived from classical models. A numerical example is provided for the introduced models and comparison is also discussed. It can be seen that these new adaptive functions are competitive models for describing the bathtub-shaped failure rate of the lifetime data.

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Key words: lifetime distributions, hazard-rate function, bathtub-shaped failure rate, lifetime data.

1 Introduction

A common method of analyzing the lifetime of the products, systems, etc. is to use distributional models, which describe the process under the study [10]. The paper reviews, proposes and compares stochastic laws capable of modelling the lifetime, of the items, denoted T. This study based on the concept from probability theory, hazard-rate function, $h(t) = f(t)/R(t)$, where $f(t)$ denotes the probability density function, and $R(t)$ the reliability function, which is also called the survivorship function. The quantity $h(t)$, named the hazard function, represents the probability that a device of age t will fail in the small interval of time $[t; t + dt]$, known that in the time interval $[0; t]$ it did not fail.

The bathtub curve does not depict the failure rate of a single item: it depicts the relative failure rate of the entire population of items over time. The name bathtub suggests the cross-sectional shape of the eponymous device and comprises three time periods, [11; 14].

The classical Weibull distribution is ubiquitous for describing dates arising from lifetime and fatigue. However, the history of the Weibull law goes back earlier. For the first time in 1928 Fisher and Tippet obtained this distribution function. Rosin and

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Rammler (1933) used a similar stochastic model for describing the seed separation process on the sieves too. The Swedish physicist Wallodi Weibull used this probability distribution for describing the lifetime of components with variable failure rate.

Modelling in the Reliability Engineering involves random variables that take values on the bounded intervals, within which the distributions are constrained to vary. A requirement for any model is the improvement, so that to make it more accurate. The paper presents only a few former stochastic models, and proposes adaptive distributions, because does not exist a general rule for all cases.

2 Some Stochastic Models Used in Reliability

2.1 Weibull Models

The weibullean distributions have been applied in the modelling of the lifetime or response time data from reliability experiments and survival studies of items. The density probability functions for some classical weibullean distributions are [5;6;7;8;9]:

$$f_1(t) = \frac{\beta}{\eta^\beta} t^{\beta-1} \exp\left[-\left(\frac{t}{\eta}\right)^\beta\right], \quad t \geq 0$$

(two-parameter Weibull distribution)

$$f_2(t) = \frac{\beta}{\eta^\beta} (t-\delta)^{\beta-1} \exp\left[-\left(\frac{t-\delta}{\eta}\right)^\beta\right], \quad t \geq \delta$$

(three-parameter Weibull distribution)

$$f_3(t) = \frac{1}{\gamma(p)} \frac{\beta}{\eta^{\beta p}} (t-\delta)^{\beta p-1} \exp\left[-\left(\frac{t-\delta}{\eta}\right)^\beta\right], \quad t \geq \delta$$

(four-parameter Weibull distribution)

$$f_4(t) = e^{-\lambda} \frac{\beta}{\eta^{\beta p}} (t-\delta)^{\beta p-1} \exp\left[-\left(\frac{t-\delta}{\eta}\right)^\beta\right] \sum_{k=0}^{\infty} \frac{\lambda^k}{k!\gamma(p+k)} \left(\frac{t-\delta}{\eta}\right)^{\beta k}, \quad t \geq \delta$$

(five-parameter Weibull distribution).

Weibull distributions are widely used in reliability analysis, because for many data sets they fit very well [5; 12]. Some lifetime laws, derived from the weibullean distributions, have a bathtub-shaped hazard-rate function. In order to achieve these behaviors many new models were developed as the poly-Weibull hazard model, the modified Weibull distribution, etc.

2.2 Poly-Weibull Distribution

A lifetime distribution, derived from the 2-parameter Weibull distribution, is the poly-Weibull model, capable of modelling many real-life data. For this brief presentation we based on the papers [1; 5].

Let a system composed of $m \geq 2$ elements; the lifetime distribution T_j of the j-th element has a Weibull distribution with the parameters β_j and μ_j, and the hazard function $h_j(t) = \dfrac{\beta_j t^{\beta_j - 1}}{\eta_j^{\beta_j}}$.

It is supposed, that the random variables $T_1, \ldots, T_m$ are independent. The random variable $T = \min\{T_1, \ldots, T_j, \ldots, T_m\}$ is said to have a poly-Weibull distribution, with the function:

$$h(t) = \sum_{j=1}^{m} h_j(t) = \sum_{j=1}^{m} \beta_j(t) \frac{t_j^{\beta-1}}{\eta^{\beta_j}} ,$$

where $\eta_j > 0$, $\beta_j > 0$ are unknown parameters. The poly-Weibull model allows decreasing, constant, increasing hazard functions, and also non-monotone ones.

2.3 A Modified Weibull Distribution

Recently Lai, Xie and Murthy [4] proposed a new modified Weibull distribution, obtained as a limiting case of the beta integrated model. The law can be considered as derived of the weibullean distribution. The probability density function and the hazard-rate function are:

$$f_5(t) = a(b + ct) * t^{b-1} * \exp(ct) * \exp[-at^b \exp(ct)]$$

$$h_5 = a(b + ct) * t^{b-1} * \exp(ct)$$

were the parameters $a > 0$, $b \geq 0$, $c > 0$. The shape of the hazard-rate function depends only on b. If $b \geq 1$ is, the hazard-rate function is an increasing one. If $0 < b < 1$, the curve initially decreases, and then increases in t, implying a bathtub shape for it.

2.4 The Hjorth Distribution

The Hjorth model is essentially a competing risk model involving two sub-populations, with one having an increasing hazard-rate function, and the other having a decreasing hazard-rate function [2;3]. The probability density function and the hazard-rate function are:

$$f_6(t) = \frac{at(1 + bt) + c}{(1 + bt)^{\frac{c}{b}+1}} \exp\left(-\frac{at^2}{2}\right), \quad t \geq 0, \quad h_6(t) = at + \frac{c}{1 + bt}.$$

If $a = 0$, the function $h(t)$ is decreasing, and if $0 < a < bc$, the function $h(t)$ has a bathtub-shaped.

2.5 New Models with Bathtub-Shaped Failure Rate

Burr (1992) constructed a system of distributions for the goal of fitting the probability cumulative distribution functions to a diversity of frequency data forms. Zimmer et al. [11; 13] used the Burr XII distribution in reliability analysis. The probability density function and the hazard-rate function are:

$$f_7(t) = \frac{kc}{s^c} \frac{t^{c-1}}{\left[1 + \left(\frac{t}{s}\right)^c\right]}; \quad s > 0,\ c > 0,\ k > 0,\ \text{for } t > 0,\ h_7(t) = \frac{kc}{s^c} \frac{t^{c-1}}{\left[1 + \left(\frac{t}{s}\right)^c\right]}.$$

Wang proposed in [11] the additive Burr XII model, which combines two Burr XII distributions. One distribution has a decreasing failure rate and another has an increasing failure rate. It results that the hazard function is given by:

$$h_8(t) = \frac{k_1 c_1}{s_1^{c_1}} \frac{t^{c_1-1}}{\left[1 + \left(\frac{t}{s_1}\right)^{c_1}\right]} + \frac{k_2 c_2}{s_2^{c_2}} \frac{t^{c_1-1}}{\left[1 + \left(\frac{t}{s_2}\right)^{c_2}\right]}, \quad t \geq 0,$$

where $0 < c_1 < 1$; $c_2 > 2$; $k_1,\ k_2,\ s_1, s_2 > 0$.

In the paper it was proposed an adaptive hazard functions of type beta (for the data from table 1):

$$h_9(t) = at^b(60 - t)^c, \quad h_{10} = a_1 t^{b_1}(10 - t)^{c_1} + k + a_2 t^{b_2}(60 - t)^{c_2}.$$

Further it was introduced two polynomial models too:

$$h_{11}(t) = B_0 + B_1 t + \ldots + B_4 t^4, \quad h_{12}(t) = C_0 + C_1 t + \ldots + C_6 t^6.$$

The following used model is an exponential one:

$$h_{13}(t) = \exp(a + bt + ct^2).$$

3 Modelling Data Set

In order to make a comparison of the analysed models one virtual (table 1) data are used.

Table 1 Failure-Time Data

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
24	22	21	20	17	15	16	15	14	10	8	9	8	9	8

16	17	18	13	19	20
7	8	9	6	7	9

22	23	24	25	26	27	28	29	30	31	32	33	34	35	36
10	8	7	9	6	10	12	8	6	8	4	6	8	10	14

37	38	39	40
6	8	10	9

41	42	43	44	45	46	47	48	49	50
7	6	5	8	4	8	7	6	5	8

51	52	53	54	55	56	57	58	59	60
10	12	13	16	18	19	20	21	22	24

The used adequate models can be easily adapted in the similar conditions. Given the lifetime data set (table 1), are applied a few rate-failure functions for modelling. For the eulerian distribution it results:

$$h_9(t) = 168.88t^{-0.64}(60 - t)^{-0.43},$$

reduced chi-square = 17.39, residual sum of squares = 991.28. For the polynomial model it obtains:

$$h_{11}(t) = 28.41 - 3.05t + 0.17t^2 - 0.004t^3 + 0.00003t^4,$$

residual sum of squares = 222.93. For the exponential law it results:

$$h_{13}(t) = \exp(3.28 - 0.09t + 0.00154t^2),$$

reduced Chi-square = 5.57, residual sum of squares = 317.34. It was used the regression curve, which has the property that the sum of the squared vertical distances from it to point on the scatter diagram is the minimum. The analysed models are convenient for the goodness of fit. From the analysis of values of residual sum of squares and reduced chi-square coefficients, it results that the proposed laws have the graphs closed to experimental data. The parameters of the choose functions (h_9, h_{11}, h_{13}) were estimated from the data set and the computations were carried out with software Microcal Origin Professional Software too.

4 Conclusions

In the paper it was reviewed known stochastic models, and proposed new adequate functions for describing the evolution of failure rate. Passing from Weibull distributions to poly-Weibull distribution, modified Weibull distribution, or Hjorth distribution, it allows the description of the hazard rate with only one function on a finite interval taking the form of the classic curve of bathtub, useful models for many practical cases. For example the Hjorth distribution can describe non-monotone phenomena too. In addition, in the present paper are introduced and other models with bathtub-shaped failure rates for which was calculated the parameters, residual sum of squares, reduced chi-square coefficients. The numerical results proved a goodness of fit. These adequate bounded distributions can be useful in the practice and in design work as well.

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